EGALITARIAN DEPARTURES FROM THE IDEAL POINT

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ABSTRACT

In this paper we propose a new solution to bargaining or group decision problems, which requires an arbitrator to choose that point on the Pareto frontier of the feasible set where losses in utility from the ideal point are equal for all the agents. The solution is motivated by ones already existing in the literature. We then present two modes of characterizing this solution which uses familiar axioms in bargaining theory and also some axioms which are intuitively plausible variants of those existing in the literature. The two key axioms are Independence of Common Monotone Transformations modulo the Ideal Point and Redundancy of Additional Alternatives Other Than a Reference Point.

1. Introduction :

The bargaining problem, has since its origin in the work of Nash (1950), been central to the study of cooperative game theory. New axiomatic solutions have emerged and conditions which test the performance of these solutions have been suggested. The problem consists in reconciling the conflicting preferences of a group of people in a cooperative environment, to arrive at a unique feasible outcome given that the agents (or players) have the choice to disagree and therefore refuse to accept the compromise solution. Hence an important criteria which a solution to a bargaining problem must fulfill is acceptibility to the group as a whole.

A large portion of the literature dealing with bargaining problems concentrates on the case in which the group of participants consists of only two people. However as subsequent and recent research has shown, the relevance of this theory for groups larger than two both with and without a fixed number of participants is of paramount importance. We shall here be mainly concerned with the two-person case though.

A 2-player bargaining game is a pair (S,d) where S is a compact subset of R^2, $d \in S$ and there exists $x \in S$ with $d < x$. Let B^* be the set of all bargaining games defined above. The intuitive interpretation of a game $(S,d) \in B^*$ is the following : A point in S, the feasible set, is a utility pair that the players can reach under cooperation and if there is unanimous agreement. The <u>disagreement point</u> d (sometimes referred to as threat point or status quo point) is the utility pair that the players have for the state in which they do not agree on another point in the feasible set. An alternative interpretation is that S consists of all the compromises than an arbitrator may choose where d stands for the outcome of the situation if the arbitrator was not involved.

A <u>bargaining solution</u> on $B' \subseteq B^*$ is a mapping $f : B' \to R^2$ such that $f(S,d) \in S$ for all $(S,d) \in B'$. A solution associates to each bargaining game (S,d) a pair of utility levels, represented by a point in S, and interpreted as the utility levels corresponding to the outcome of the bargaining situation.

Given a bargaining game $(S,d) \in B^*$ and a point $x \in R^2$ we say that x is <u>individually</u> rational if $x \geq d (x_i \geq d_i$ for $i = 1,2)$. x is <u>strictly individually rational</u> if $x > d$. We say that x is <u>Pareto optimal</u> if $x \in S$ and for every $y \in S$ if $y \geq x$ then $y = x$. We say that x is <u>weakly Pareto optimal</u> if for every $y \in S$, $y \not> x$. Let P(S) denote the Pareto optimal set of S and W(S) the weakly Pareto optimal set of S. Let $B^{**} \equiv \{(S,d) \in B^* / P(S)$ is closed and connected$\}$.

We now turn to a formal presentation of the more significant solutions discussed in the literature. We restrict ourselves to an important subclass of games B called convex games, defined as follows $B \equiv \{(S,d) \in B^{**} / S$ is convex$\}$.

On B, the Nash (1950) solution N(S,d) is the unique maximizer of $(x_1 - d_1)(x_2 - d_2)$ for x in S satisfying $x \geqq d$; the Kalai-Smorodinsky (1975) solution K(S,d) is the maximal feasible point on the segment connecting the status-quo point to the "ideal point" a (S,d) where for each i, $a_i(S,d) \equiv \max\{x_i / x \in S, \ x \geqq d\}$; the egalitarian solution E(S,d) of Kalai (1977) is the maximal feasible point with equal increments from the disagreement point; the utilitarian solution U(S,d) which belongs to the set of maximizers of $(x_1 - d_1) + (x_2 - d_2)$, $x = (x_1, x_2) \in S$, $x \geqq d$; the lexicographic maximum solution L(S,d) of Imai (1983) which is the point that maximizes the function mix$\{x_1 - d_1, x_2 - d_2\}$ on P(S) the Pareto optimal set of S. Axiomatic characterizations of the above solutions are available in the literature.

In this paper we provide and axiomatically characterize a new solution

$$Q(S,d) = \underset{(x_1,x_2)\in P(S)}{\text{argmax}} \left[\min\{x_1 - a_1(S,d),\ x_2 - a_2(S,d)\}\right]$$

$$= a(S,d) + e \max\{t \in R / a(S,d) + t.e \in S\} \text{ where } e = (1,1) \in R^2.$$

This solution is similar on the one hand to the Kalai-Smorodinsky (1975) solution in that it depends on the ideal point and on the other to the Kalai (1977) egalitarian solution in that equal deviations from a reference point is the object of our study. Some properties of this solution have been studied by Yu (1973) and Freimer and Yu (1976). The Kalai - Smorodinsky (1975) solution arose out of a necessity to dispense with the Independence of Irrelevant Alternatives assumption in the Nash (1950) solution, and depends on three points. Our solution also depends on the same three points and does not require or satisfy Independence of Irrelevant alternatives in its characterization. Like the egalitarian solution our solution admits interpersonal comparisons of utility.

Roth (1979) uses an axiom of "Independence of Ordinal Transformations Preserving Interpersonal Comparisons" to characterize the bargaining solution that maximizes minimal gains from the disagreement point on the Pareto optimal set. The idea behind the axiom is that the players are able to make ordinal interpersonal comparisons of gains relative to the disagreement outcome. Nielsen (1983) simplifies this axiom. The solution is the same as before and is invariant under monotone transformations that preserve interpersonal comparisons of gains. The same monotone transformation is applied independently to each player's utility gains from the status quo point. Our first characterization relies on a similar technique.

Our second characterization of the same solution defined on the subclass of convex and comprehensive games (to be discussed later), relies heavily on an axiom called Redundancy of Additional Alternatives other than a Reference point. This axiom introduced and discussed at great length in Lahiri (1989 a), has been used in particular instances to characterize the Kalai-Smorodinsky (1975) and the egalitarian solution of Kalai (1977). It simplifies the proofs of the characterization theorems and greatly enhances our understanding of the solutions. What this axiom says is that if we add alternatives to an existing game, without affecting the Pareto-optimality (or weak Pareto optimality) property of the existing solution, then the solution to the expanded game remains the same as before.

Our paper thus consists mainly of these two modes of characterizing the solution we propose.

2. A first characterization :

Consider the following four axioms for a solution $f: B^{**} \to R^2$

Axiom 1:- Strict Individual Rationality (SIR): For all $(S,d) \in B^{**}$, $f(S,d) > d$

Axiom 2:- Pareto Optimality (PO) : For all $(S,d) \in B^{**}$, $f(S,d) \in P(S)$

Axiom 3:- Independence of Irrelevant Alternatives Other Than The Ideal Point (IIIA) : If $(S,d) \in B^{**}$, $(T,d') \in B^{**}$, $S \subset T$, $a(S,d) = a(T,d')$ and $f(T,d') \in S$, then $f(S,d) = f(T,d')$

Axiom 4:- Independence of Common Monotone Transformations modulo the Ideal Point (ICMTI) : Let $(S,d) \in B^{**}$, $a' \in R^2$, and let $t : \{x_1 - a_1(S,d) : x \in S\} \cup \{x_2 - a_2(S,d) : x \in S\} \to R$ be a strictly increasing continuous function with $t(o) = o$. Put $d' = a' + (t(d_1 - a_1(S,d)), t(d_2 - a_2(S,d))$ and $S' = \{(a'_1 + t(x_1 - a_1(S,d)), a'_2 + t(x_2 - a_2(S,d)) : x \in S\}$. Then $f_i(S',d') = a'_i + t(f_i(S,d) - a_i(S,d))$ for $i = 1,2$.

It should be noted that the bargaining game (S',d') appearing in the description of ICMTI is actually in B^*. Axiom SIR and PO are standard in the literature of bargaining games are valid even for the Nash (1950) solution. Axiom IIIA can be found in Roth (1979) where this axiom in conjunction with PO, a symmetry axiom and an axiom on independence of equivalent utility representations yields a nonexistence result. Our solution satisfies PO and Symmetry but does not satisfy Independence of Equivalent Utility Representations. This turns out to be the reason why we can characterize our solution using IIIA. For completeness we state below the Symmetry Axiom as well as the Independence of Equivalent Utility Representations Axiom.

<u>Axiom 5</u> :- <u>Symmetry</u> (S) : Suppose $(S,d) \in B^{**}$ is a symmetric bargaining game i.e. suppose that $d_1 = d_2$, and that if x is contained in S, then is every permutation of x. Then

$$f_1(S,d) = f_2(S,d)$$

<u>Axiom 6</u> :- <u>Independence of Equivalent Utility Representations</u> (IEUR) : For any bargaining game $(S,d) \in B^{**}$ and real numbers a_i and b_i for $i = 1,2$ such that each $a_i > 0$, let the bargaining game (S', d') be defined by $S' = \{ y \in R^2 /$ there exists an x in S such that $y_i = a_i x_i + b_i$ for $i = 1,2 \}$ and $d'_i = a_i d_i + b_i$ for $i = 1,2$. Then $f_i(S',d') = a_i f_i(S,d) + b_i$ for $i = 1,2$.

The Axiom ICMTI can be interpreted as follows: The players do not only have preferences among the possible outcomes of the bargaining situation. Viewing a (S,d) as the demands of the players, they are also able to determine, for any particular outcome, who has lost more relative to the demands; and they agree on these comparisons. Axiom ICMTI says that the solution is invariant under all changes in the utility representation which preserve the information about interpersonal comparisons of losses as well as the information about individual preferences. The equal rationing scheme that our solution recommends, satisfies ICMTI. Before we proceed to the main theorem of this section, we introduce a few notations.

$P(S,d) \equiv \{x \in P(S): x \geq d\}$ denotes the Pareto-optimal and individually rational subset of S.

$S^+ \equiv \{x \in S: x \geq d\}$ denotes the individually rational subset of S.

Given a game $(T,d) \in B^{**}$ its comprehensive hull is the game $(\text{Com}(T),d)$ where $\text{com}(T) = \{x \in R^2 / d \leq x \leq y$ for some $y \in T\}$.

Given a game $(S,d) \in B^{**}$, let $\underline{x}_1 = \max\{x_1 : (x_1, a_2(S,d)) \in S\}$, $\underline{x}_2 = \max\{x_2 : (x_2, a_1(S,d)) \in S\}$ and $\underline{x} = (\underline{x}_1, \underline{x}_2)$

Now we proceed with our theorem.

<u>Theorem</u> 1:- Let f be a bargaining solution on B^{**}. Then f = Q if and only if f satisfies PO, IIIA, and ICMT.

<u>Proof</u> :- It is clear that Q has the four properties. Assume that f has them. f is invariant under translations of the ideal point, and so it is enough to prove that $f(S,d) = Q(S,d)$ for games (S,d) with $a(S,d) = (1,1)$.

If $(S,d) \in B^{**}$, define a mapping $\varphi : [\underline{x}_1, a_1(S,d)] \to R$ by $\varphi(x_1) = \max\{x_2 : (x_1\ x_2) \in S\}$. Then φ is continuous and strictly decreasing, and $P(S,d) = \{(x_1, \varphi(x_1)) : x_1 \in [\underline{x}_1, a_1(S,d)]\}$.

By SIR and PO, $f(S,d) \in P(S,d)$; and by IIIA, $f(S,d) = f(P(S,d) \cup \{d\}, d)$. Similarly, $Q(S,d) = Q(P(S,d) \cup \{d\}, d)$. By IIIA, $f(P(S,d) \cup \{d\}, d) = f(\text{Com}(P(S,d)),d)$ and once again by IIIA, $f(\text{com}(P(S,d)),d) = f(\text{com}(P(S,d)),\underline{x}) = f(\text{com}(P(S,\underline{x})), \underline{x})$ (refer to the definition of $(\text{com}(T),d)$). Hence, $f(S,d) = f(\text{com}(P(S,\underline{x})),\underline{x})$. Similarly, $Q(S,d) = Q(\text{com}(P(S,\underline{x})),\underline{x})$. But $f(\text{com}(P(S,\underline{x})),\underline{x}) = f(P(S,\underline{x}) \cup \{x\}, \underline{x})$ by SIR, PO and IIIA. Similarly $Q(\text{com}(P(S,\underline{x})),\underline{x}) =$

$Q(P(S,\underline{x}) \cup \{\underline{x}\}, \underline{x})$.

Hence, it is enough to show that $f(P(S,\underline{x}) \cup \{\underline{x}\}, \underline{x}) = Q(P(S,\underline{x}) \cup \{\underline{x}\}, \underline{x})$.

There are two cases.

Case 1 :- $P(S,\underline{x}) \cup \{\underline{x}\}$ is symmetric i.e. $(x_2,x_1) \in P(S,\underline{x}) \cup \{x\}$ for all $x \in P(S,\underline{x}) \cup \{\underline{x}\}$. In this case $\underline{x}_1 = \underline{x}_2$, $\underline{x}_2 = \varphi(1) = \underline{x}_1 < 1 = \varphi(\underline{x}_1)$, $\varphi([\underline{x}_1, 1]) = [\underline{x}_1, 1]$, and $\varphi = \varphi^{-1}$.

There is $c \in (\underline{x}_1, 1)$ with $\varphi(c) = e$; and this time we can construct a transformation such that $P(S,\underline{x}) \cup \{\underline{x}\}$ is mapped onto itself, but the only fixed point in $P(S,\underline{x}) \cup \{x\}$ which is $> x$ is (e,c). Choose a strictly increasing continuous function $k: [c,1] \to [e,1]$ with $k(e) = e$, $k(1)=1$, and $k(v) < v$ for $v \in (e,1)$. Define $t: R \to R$ by

$$t(v) = \begin{cases} v \text{ if } v \leq \underline{x}_1 - 1, \\ \varphi k \varphi(v+1) - 1, \text{ if } \underline{x}_1 - 1 \leq v \leq e - 1, \\ k(v+1) - 1, \text{ if } e - 1 \leq v \leq 0 \\ v \quad , \text{ if } 0 \leq v \end{cases}$$

Then t is continuous and strictly increasing and

$P(S,\underline{x}) \cup \{\underline{x}\} = \{(1 + t(x_1 - 1), 1 + t(x_2 - 1)) : x \in P(S,\underline{x}) \cup \{x\}\}$; and the only point $x \in P(S,\underline{x}) \cup \{\underline{x}\}$ with $(1+t(x_1-1),\ 1+t(x_2-1)) = (x_1,x_2) > (\underline{x}_1,\underline{x}_2)$ is (c,c). By ICMTI and SIR, $f(P(S,\underline{x}) \cup \{\underline{x}\}, \underline{x}) = (c,c)$

Hence, $f(S,d) = (c,c)$. A similar argument shows that $Q(S,d) = (c,c)$.

Case 2 :- $P(S,\underline{x}) \cup \{\underline{x}\}$ is not necessarily symmetric. In this case, $1 > \varphi(1) = \underline{x}_2$ and $\varphi(\underline{x}_1) > \underline{x}_1$, and again there is $c \in (\underline{x}_1, 1)$ with $\varphi(c) = c$. The idea here is to extend $(\mathrm{Com}(P(S,\underline{x})),\underline{x})$ to a symmetric game. Assume without loss of generality that $\underline{x}_1 \lneq \underline{x}_2$.

Define $\psi : [\underline{x}_1, 1] \rightarrow R$ by

$$\psi(v) = \max\{\varphi(v),\ \varphi^{-1}(v)\}.$$

It is easy to see that ψ is continuous and strictly decreasing and that $\psi(c) = c$. Put $T = \{x : \underline{x}_1 \leq x_1 \lneq 1,\ \underline{x}_1 \lneq x_2 \lneq \varphi(x_1)\}$.

Then $(T, (\underline{x}_1,\underline{x}_1)) \in B^{**}$. Moreover, $(T, (\underline{x}_1,\underline{x}_1))$ as well as $(P(T,(\underline{x}_1,\underline{x}_1)) \cup \{(\underline{x}_1,\underline{x}_1)\}, (\underline{x}_1,\underline{x}_1))$ are symmetric games. By a proof analogou to that of Case 1, $f(T,(\underline{x}_1,\underline{x}_1)) = (c,c)$. Now $P(S,\underline{x}) \cup \{\underline{x}\} \subset T$ and $a(P(S,\underline{x}) \cup \{\underline{x}\}, \underline{x}) = a(T,(\underline{x}_1,\underline{x}_1)) = (1,1)$. Further $(c,c) \in P(S,\underline{x}) \cup \{\underline{x}\}$. Hence by IIIA, $(c,c) = f(P(S,\underline{x}) \cup \{\underline{x}\}, \underline{x})$. Thus, $(c,c) = f(S,d)$. Similarly $Q(S,d) = (c,c)$.

Q.E.D.

3. A Second Characterization :- A second characterization of our solution Q but now restricted to a subclass $\bar{B}$ of B (which consists of convex comprehensive games) is obtained using what has been defined in Lahiri (1989 a) to be redundancy of additional alternatives other than a reference point. A game $(S,d) \in B$ (the class of convex games) is said to be convex and comprehensive

if $\{x \in R^2 / d \lneq x \lneq y$ for some $y \in S\} \subseteq S$. Intuitively, such games allow for free disposal of utility. Hence $\bar{B} = \{(S,d) \in B/(S,d)$ is convex and comprehensive$\}$.

The redundancy of additional alternatives other than a reference point assumption is similar to both the IIA axiom as well as the IIIA axiom. What this assumption says, is that if we add alternatives to an existing game so that the reference point in question remains the same and the Pareto optimality property of the existing solution remains unaffected, then the solution to the expanded game is the erstwhile solution. As shown in Lahiri (1989 a), by an appropriate definition of the reference point, the Kalai-Smorodinsky (1975) satisfies this property and this property can be used to characterize the Kalai-Smorodinsky Solution.

Given, $(S,d) \in \bar{B}$, define $g(S,d) = (0, a_2(S,d) - a_1(S,d))$. Then $g : \bar{B} \to R^2$. $g(S,d)$ gives the excess of player 2's demand over the demand of player 1. The concept of a reference function was introduced for the first time by Thomson (1981). We now formulate an axiom which characterizes our solution $Q : \bar{B} \to R^2$.

Axiom 7 :- <u>Redundancy of Additional Alternatives other than the reference point</u>. Let (S,d) and (S',d') belong to $\bar{B}$ and let $g: \bar{B} \to R^2$ be as defined above. Suppose $S \subset S'$ and $g(S,d) = g(S',d')$ and $f(S,d) \in P(S')$. Then $f(S',d') = f(S,d)$.

Our second characterization theorem is as follows:

Theorem 2 :- If f satisfies PO and RAg, then $f = Q$ on $\bar{B}$

Pf:- That Q satisfies PO and RAg is immediate. Conversely suppose, f satisfies PO and RAg and let $(S,d) \in \bar{B}$. We need to show that $Q(S,d) = f(S,d)$.

Consider the game (T,d) where $T = \{(x_1,x_2) \in R^2 / d_i \leq x_i \leq Q_i(S,d), i = 1,2\}$. Since $(S,d) \in \bar{B}$, $(T,d) \in \bar{B}$ also and $T \subset S$. Further $g(S,d) = (0, a_2(S,d) - a_1(S,d) = (0, Q_2(S,d) - Q_1(S,d) = g(T,d)$. Also f satisfies PO implies $f(T,d) = Q(S,d)$ which belongs to $P(S)$. Hence by RAg, $f(S,d) = Q(S,d)$.

In Lahiri (1989 a) we strengthen this axiom to what has been called there, as the redundancy of additional alternatives (RAA) axiom, and use that to characterize the family of proportional solutions due to Kalai (1977). This axiom as used above dispenses with the IIIA axiom which was used in our first characterization.

4. Conclusion :- We have thus been able to obtain two appealing characterizations of a solution to group decision problems, which both existed and grew out of familiar solutions already existing in the literature. The characterizations use assumptions which have either themselves been used or have strong resemblances to other axioms which were used to describe significant solutions to bargaining problems. Extensions of the solution to games with variable population as well as to threat bargaining games (see Lahiri 1989 b, 1989 c, 1989 d) is immediate. However its properties in such contexts is the subject of further and future research.

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