A PERIODIC REVIEW INVENTORY MODEL FOR STOCHASTIC DEMAND WITH TWO STORAGE FACILITIES

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A PERIODIC REVIEW INVENTORY MODEL FOR STOCHASTIC DEMAND WITH TWO STROAGE FACILITIES

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ABSTRACT

In this paper a periodic review probabilistic inventory model for a single item with two storage facilities is developed; one warehouse is owned by the system under considerations (which is referred to as OW) and the other is a rented warehouse (RW). The capacity of OW is W units. Any quantity larger than W is to be kept in RW and are gradually withdrawn in batches of K units. The model determines optimum values of lot-size q and K. An example is given to illustrate the results obtained.

INTRODUCTION

Inventory models with two stroage facilities have attracted attention of research workers in past few years. When the inventory system has no sufficient space to store the optimal order quantity in their own warehouse (OW), excess inventory is required to be kept in rented warehouse(RW), which may be far away from OW. Demands are satisfied at OW only, and not from RW. Generally the inventory holding cost at RW are higher as compared to that of OW. Hence, it is natural to expect that the stocks of RW are cleared as early as possible in order to bring down holding costs.

Hartley [1] considered a two-level storage model with infinite

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replenishment rate in which the cost of transporting the units from RW to OW was ignored. Sarma [4,5] considers the same model by including transportation costs from RW to OW, and has also determined optimum K-release rule. Murdeshwar and Sathe [2] have considered above model with finite replenishment rate.

In all the above models, demand is assumed to be deterministic. In this study periodic review lot-size model with probabilistic demand is developed when shortages are not allowed. OW has a capacity to store W units only, and as soon as stocks at OW becomes (W-K), K units are transferred from RW to OW. Expressions for optimal values of lot-size q and for K are obtained. Finally, the model is illustrated with an example by taking an appropriate probability distribution of demand.

ASSUMPTIONS AND NOTATIONS

The mathematical model for the system under consideration is developed under the following assumption:

1. Inventory position of the system is reviewed at a regular interval w_p time units. Whenever it is found that the on-hand inventory is less than or equal to re-order point s, a lot-size q is scheduled for replenishment. Lead time is assumed to be zero, w_p and s are assumed to be prescribed, and q is the decision variable.

2. The demand x during any period w_p is a random variable (rv) with probability density function (pdf) g(x), and cumulative distribution function G(x), $\emptyset \leqslant x \leqslant M$, with

$$\mu = \int_{\emptyset}^{M} x\, g(x)\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

as the mean demand during w_{ρ} , where M denotes the maximum demand during a review period.

3. Shortages are not allowed.
4. The replenishment cost is Rs. A per order. The holding cost is Rs. F per unit per time unit at RW, and Rs. H per unit per time unit at OW, where F > H.
5. The cost of transportation of K units from RW to OW is C_t ; which is fixed.
6. OW can stock upto a maximum of W units only, where we assume that W $\geqslant$ s. Let S denote the on-hand inventory in the system at the begining of a review period after a replenishment, if any, then S is a rv with pdf

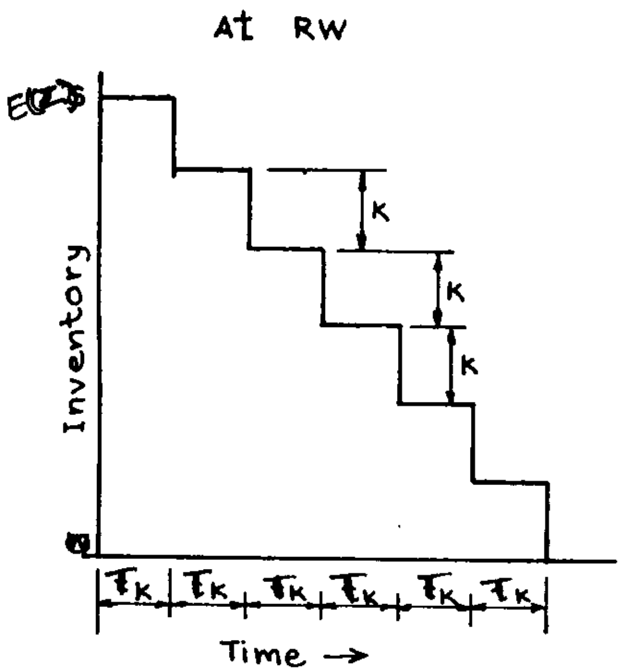
$$h(S) = 1 / q , \qquad s \leqslant S \leqslant s+q \quad \ldots\ldots(2)$$
$$= \emptyset \quad , \qquad \text{otherwise}$$

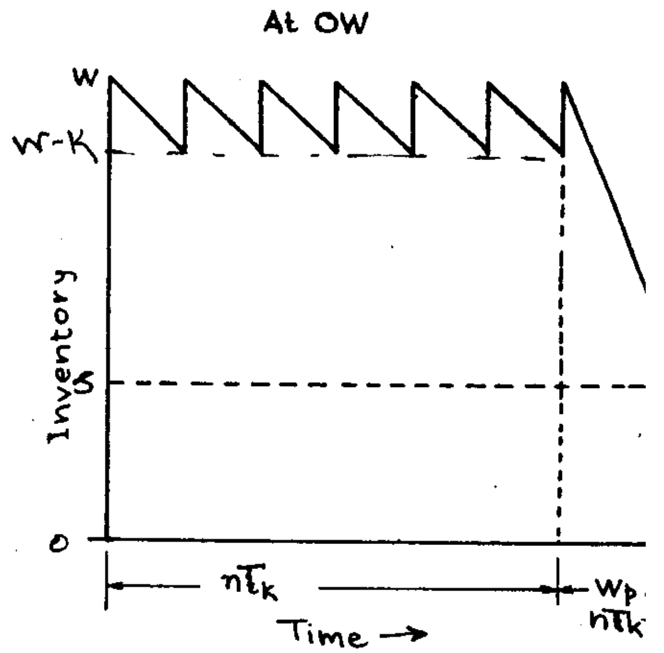
$$E(S) = \int_{s}^{s+q} h(S)\, ds = s + q/2$$

Hence if Z denotes the quantity stored at RW, then Z is a rv, with

$$E(Z) = E(S) - W = s + q/2 - W \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Initially, demands are satisfied from OW, until the stock level falls to (W-K). At this stage K units are transfered from RW to OW; this process is repeated till the stock at RW is completely exhausted. The remaining W units at OW are used till the inventory level is less than or equal to s. At this stage a lot-size of q units is ordered. Since, shortages are not allowed, we take s = M. The inventory fluctuations at OW and RW can be shown as under.





We now obtain total average expected cost of the system per time unit, which is composed of average expected inventory at RW and that at OW.

AVERAGE EXPECTED COST AT RW

Since average demand per w_p time units is μ , expected duration for the demand of K units is

$$T_K = E(t_K) = K.w_p / \mu \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

Where

t_K = time interval during which demand of K units occur; a random variable.

Also, expected number of transhipments from RW to Ow is

$$n = E(Z)/K \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

Total expected inventory at RW is

$$A_1(q,k) = T_K \sum_{i=1}^{n} (E(Z) - (i-1)K)$$
$$= (w_p/2\mu)\{E(Z)\}^2 + K\, w_p\, E(Z) / 2\mu \ldots\ldots\ldots (6)$$

and average cost of transportation for E(Z) units from RW to OW is

$$n\, C_t = E(Z).C_t /K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

From (6) and (7), total expected cost per time unit, due to RW is

$$T_1(q,K) = (FA_1(q,K) + n\, C_t)/ w_p$$
$$= (F/2\mu)\{E(Z)\}^2 + \{FK/2\mu + C_t/K\, w_p\}E(Z) \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

AVERAGE EXPECTED COST AT OW

During each period of duration $T_K = E(t_K)$ time units, average inventory at OW is $(W - K/2)$. There are n such durations. In the final period of $(w_p - n\, T_K)$ time units, it can be seen that if $Q(t/x)$ denotes on-hand inventory at time t ($n.T_K \leq t \leq w_p$), when x is the demand during w_p , then

$$Q(t/x) = (W - x/w_p)(t - n\, T_K) \qquad n\, T_K \leq t \leq w_p \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and hence, avearge inventory at OW during the period $(n\, T_K, w_p)$ is:

$$\frac{1}{w_p - n\, T_K} \int_{n.T_K}^{w_p} Q(t/x)\, dt = (W - x)(w_p - n\, T_K)/2\, w_p \ldots\ldots (10)$$

Thus, the average expected inventory at OW during this final period of $(w_p - n\, T_K)$ time units is

$$(W - \mu/2\, w_p)(w_p - n\, T_K) \mp (W - \mu/2) + E(Z)/2 \qquad \ldots\ldots (11)$$

So that total expected inventory at OW during a review period is

$$A_2(q,K) = n.\ T_K(W - K/2) + (w_p - n.T_K)(W - \mu/2 + E(Z)/2)$$
$$= w_p\,(W - \mu/2) + w_p\,(1 - K/2\mu)E(Z)$$
$$- (w_p/2\mu)\{E(Z)\}^2 \quad \ldots\ldots (12)$$

Following Naddor [3, p238], probability of replenishment at the end of a review period is

$$I_3(q) = Pr\,(S - x \leq s) = 1 - V(q)/q \quad \ldots\ldots (13)$$

Where

$$V(q) = \int_0^q G(x)\,dx \quad \ldots\ldots (14)$$

From (12) and (13), total expected average cost per time unit at OW is

$$T_2(q,K) = [\,H.A_2(q,K) + A\,I_3(q)]\,/w_p$$
$$= H\,(W - \mu/2) + H(1 - K/2\mu)E(Z) - (H/2\mu)\{E(Z)\}^2$$
$$- (A\,V(q)/\,q\,w_p) + A/w_p \ldots\ldots (15)$$

From (8) and (15), total average expected inventory cost of entire system per time unit is

$$T\,(q,K) = T_1(q,K) + T_2(q,K)$$
$$=((F-H)/\,2\mu)\{E(Z)\}^2 + \{(F-H)K/2\mu + (C_t/\,K\,w_p)E(Z)$$
$$+H(W - \mu/2 + E(Z)) - A\,V(q)/q\,w_p + A/w_p$$
$$= \{(F-H)/2\mu\}(s + q/2 - W)^2$$
$$+ [((F-H)K/2\mu) + C_t/K\,w_p](s + q/2 - W)$$
$$+ (H/2)(2s + q - \mu) - A\,V(q)/q\,w_p + A/w_p \ldots\ldots (16)$$

For optimum value of q_0 and K_0 from (16)

$$\frac{\partial T(q,K)}{\partial K} = [(F/2\mu) - (C_t / w_p K^2) - H/2\mu](s + q/2 - W) = 0$$

and

$$\frac{\partial T(q,K)}{\partial q} = ((F-H)/2\mu)(s + q/2 - W) + [(F-H)K/2\mu) + (C_t / K\, w_p)]/2 + H/2 - (A/w_p q^2)\int_0^q x\, g(x)\, dx = 0$$

gives

$$K_0 = \sqrt{2\, C_t\, \mu / w_p (F-H)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

and then we find that optimum $q = q_0$ can be obtained by solving

$$(A/w_p\, q_0^2)\int_0^{q_0} x\, g(x) dx - ((F-H)/4\mu)\, q_0 = H/2 + \sqrt{C_t(F-H)/2\mu}) - ((F-H)/2\mu)\,(W - s) \quad \ldots\ldots\ldots\ldots (18)$$

In general equation (18) may have more than one solution. A necessary condition for a solution q_0 to have minima at q_0 is

$$[\partial^2 T(q,K)/\partial q^2]_{q_0} > 0$$

that is

$$A\, g(q_0)/w_p - 3(F-H)q / 4\mu < H + \sqrt{2C_t(F-H)/\mu\, w_p} - (F-H)(W-s)/\mu \quad \ldots\ldots\ldots\ldots (19)$$

The optimum lot-size $q = q_0$ can be obtained from (19) by any numerical method.

Note that when $F = H$ and $C_t \to 0$, the total cost the system is

$$T(q) = H(s + q/2 - \mu/2) - AV(q)/q\, w_p + A/w_p \quad \ldots (20)$$

Optimum $q = q_0$ is solution of

$$(1/q_0^2)\int_0^{q_0} x\, g(x)\, dx = H\, w_p / 2A \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

and condition for optimality at q is

$$g(q_0) < H\, w_p / A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Equation (20) to (22) are the same as those given by Naddor [3] for a single storage model.

Comparing (16) and (20), we find that the first two terms on RHS of (16) represents additional cost due to RW.

COST REDUCTION DUE TO K-RELEASE RULE

We note that the unit cost of transportation with K-release rule is $C_t' = C_t/K$. Suppose the unit cost of transportation is C_t^* without bulk transportation. The bulk transportation will be economical only if $C_t^* > C_t'$. Hence without K-release rule, the cost function becomes,

$$T^*(q) = ((F-H)/2\mu)(s + q/2 - W)^2 + (H/2)(2s + q - \mu) + A\,V(q)/w_p q + A/w_p + C_t^*(s + q/2 - W)/w_p \quad \ldots\ldots(23)$$

In this case, for optimal value of $q = q^*$

$$\frac{\partial T^*(q)}{\partial q} = (F-H)/2\ (s + q/2 - W) + H/2 - (A/w_p q^2)\int_0^q x\, g(x)\, dx + C_t^*/2w_p = 0$$

Gives q_0 is a solution of

$$(A/w_p q^{*2})\int_0^{q^*} x\, g(x)\, dx - (F-H)q^*/4\mu = H/2 - (F-H)(W-s)/2\mu \quad \ldots\ldots(24)$$

and hence, the K-release will be economical if $(T^*(q) - T(q,K)) > 0$.

But

$$T^*(q) - T(q,K) = \{(C_t^* - C_t')/w_p - (F-H)K/2\mu\}(s + q/2 - W) > 0$$

Which implies

$$C_t^* - C_t' > w_p K\,(F-H)/2\mu \quad \ldots\ldots(25)$$

Example: Suppose that the pdf of demand x during a review period w_p is

$$g(x) = 1/M, \quad 0 \leq x \leq M$$
$$= 0, \quad \text{otherwise} \quad \ldots\ldots(1)$$

Where M is a constant; then

$$G(x) = x/M \qquad 0 \leq x \leq M$$
$$= 1 \qquad \text{otherwise} \qquad \ldots\ldots (2)$$

and $$\mu = E(x) = \int_0^q x\, g(x)\, dx = M/2 \qquad \ldots\ldots (3)$$

$$V(q) = \int_0^q G(x)\, dx = q^2/2M, \qquad \text{if } q \leq M$$
$$= q - M/2 \qquad \text{if } q > M \qquad \ldots\ldots (4)$$

Noting that $s = M$

$$T(q,K) = (F-H)/M\,(q/2 - (W-M))^2 + [(F-H)K/M + C_t/K\,w_p](q/2 - (W-M)) + H/4(3M + 2q) - A\,q/\,2M\,w_p + A/w_p \qquad \text{if } q \leq M \qquad \ldots\ldots (5)$$

and

$$= (F-H)/M\,(q/2 - (W-M))^2 + [(F-H)K/M + C_t/K\,w_p](q/2 - (W-M)) + H/4(3M + 2q) - (A/w_p)(1 - M/2q) + A/w_p, \qquad \text{if } q > M \qquad \ldots (6)$$

Now optimal value of

$$K_0 = \sqrt{C_t M/w_p(F-H)} \qquad \ldots\ldots (7)$$

and for q can be determined in two cases:

<u>Case</u> = <u>I</u> : for $q \leq M$

$$\frac{\partial T(q,K)}{\partial q_0} = (F-H)/M\,(q_0/2 - (W-M)) + [(F-H)K/M + C_t/K\,w_p + H]/2 - A/w_p\,M = 0$$

$$q_0 = 2(W - M - K_0) + ((A/w_p) - HM)/(F-H) \qquad \text{for } q < M$$

<u>Case</u> = <u>II</u>: for $q > M$

$$\frac{\partial T(q_0,K)}{\partial q_0} = ((F-H)/M))(q_0/2 - (W - M)) + [(F-H)K/M + (C_t/K\,w_p) + H]/2 - AM/\,2q\,w_p = 0$$

This can be solved by some numerical method.

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