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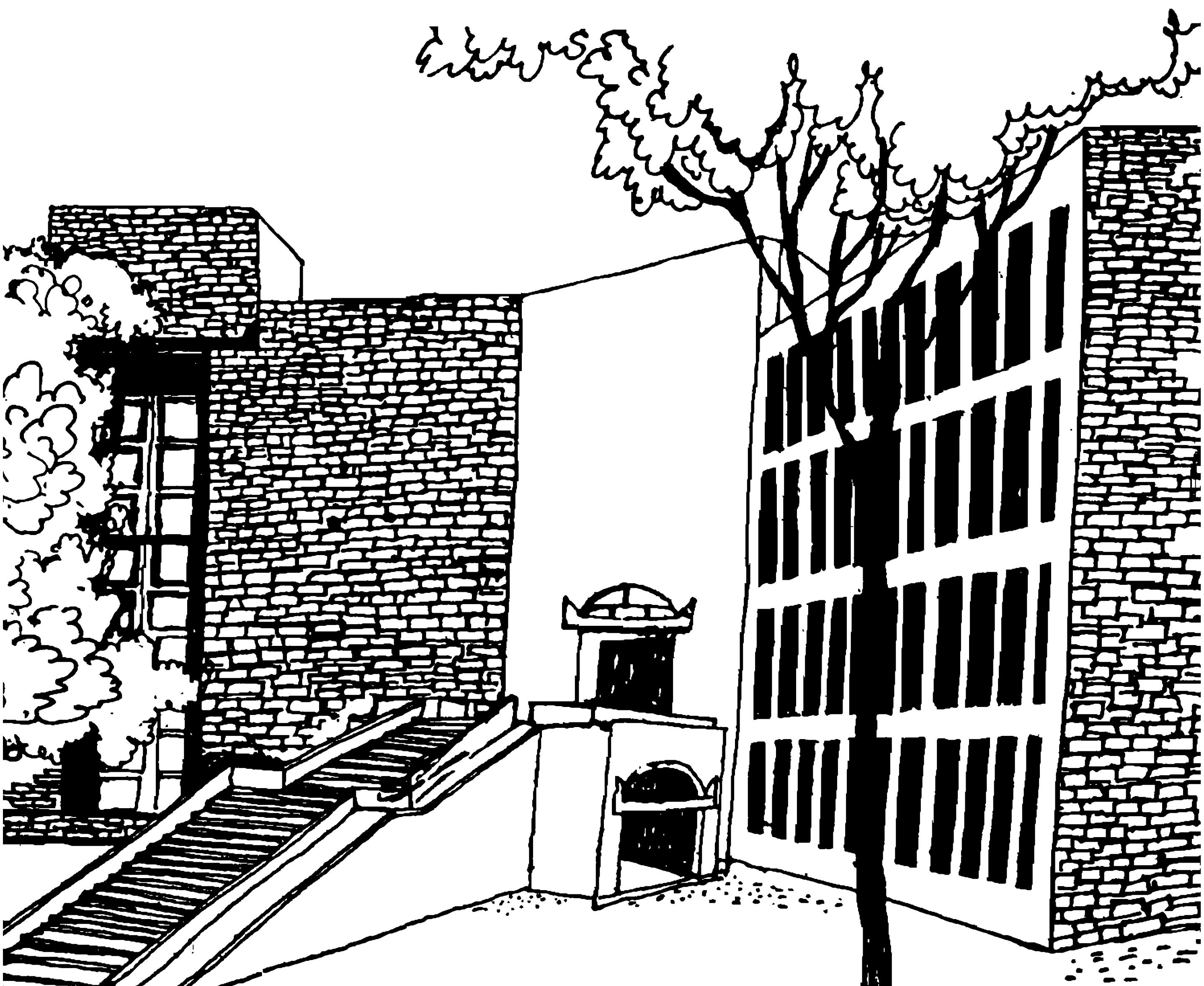

PATH INDEPENDENCE AND CHOICE ACYCLICITY PROPERTY

By

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Abstract

In this paper we study lower and upper approximations of choice functions and propose a necessary and sufficient condition for a choice function to have a path independent lower approximation. In an appendix to this paper we provide a simple proof of Theorem 2.10 in Deb(1983) in the case when the universal set is finite. In a final section of this paper, we characterize all batch choice functions using a choice acyclicity property.

1 Introduction

In choice theory, a decision maker is assumed to be equipped with a decision rule or a choice function which associates with each non-empty finite subset of a universal set, the set of all chosen points from the given set of feasible alternatives. The purpose of choice theory as to characterize choice functions satisfying desirable properties and also to establish interrelations between properties.

One such property is Path Independence, due to Plott [1973]. Along with a property called Concordance (which basically says that if a point is chosen from two sets, then it would also be chosen from their union), Path Independence is necessary and sufficient for a choice function to be rationalized by a quasi-transitive, reflexive and complete binary relation. Path Independence implies the Superset Property of Blair, Bordes, Kelly and Suzumura (1976) and has been shown to be equivalent to the simultaneous satisfaction of this latter property and Chernoff's Axioms.

A slight strengthening the Superset Property is the celebrated Outcasting axiom due to Nash [1950]. The superset property says that if two sets are given with the first contained in the second and if the chosen points of the second are contained in the set of chosen points of the first, then the two sets of chosen points coincide. Outcasting on the other hand requires that if the chosen points of the second set are contained in the first set, then the two sets of chosen points coincide. Theorem 9 of Aizerman (1985) [Theorem 4.6 of Aizerman and Aleskerov (1995)] asserts that Path Independence is equivalent to the simultaneous satisfaction of Chernoff's Axiom and Outcasting.

In this paper we study lower and upper approximations of choice functions and propose a necessary and sufficient condition for a choice function to have a path independent lower approximation. However, for upper approximations, the scene is a lot more dismal. Satisfaction of well known and reasonable conditions, fail to guarantee the existence of path independent upper approximations. The concept of lower and upper approximations is due to Litvakov [1981]. Given a choice function, the lower approximation of the choice function satisfying a set of properties is the union of all choice functions contained in the given choice function and satisfying the stated properties. The upper approximation on the other hand is the intersection of all choice functions containing the given choice function and satisfying the properties. The natural question is : whether the lower approximations and the upper approximation continue to satisfy the stated properties? To prove our results we make significant use of Theorem 9 of

Aizerman (1985). An alternative, proof of Theorem 9 is also provided in the paper (see Theorem 4.6 of Aizerman and Alaskerov [1995]).

In an appendix to this paper we provide a simple proof of Theorem 2.10 in Deb [1983] in the case when the universal set is finite. We use this theorem to establish the existence of Path Independent lower approximations. Our proof does not require the use of Zorn's Lemma, which has been used in Deb [1983] to prove Theorem 2.10. Our method of proof is entirely constructive.

In a final section of this paper, we characterize all choice functions which satisfy a new property called, choice acyclicity property. This property is stronger than outcasting. It turns out that choice functions satisfy the choice acyclicity property if and only if they are batch choice functions [see Aizerman and Aleskerov (1995)]. A batch choice function assigns to each set the unique subset which maximizes a real valued function (defined on all non-empty subsets of the universal set) amongst all subsets of the given set. This interesting result gives additional appeal to choice theory.

It is interesting to note that the choice acyclicity property is implied by a property called functional acyclicity [see, Aizerman and Aleskerov (1995), Lahiri (1998)]. In Aizerman and Aleskerove (1995) it is asserted that a choice function satisfies functional acyclicity if and only if there is a real valued function defined on the universal set and a real valued function defined on all non-empty subsets of the universal set, such that the chosen points corresponding to each feasible set of alternative coincides exactly with those feasible points whose value according to the first function is greater than or equal to the real number assigned by the second function to the given feasible set. Such choice functions are called threshold rationalizable in Lahiri [1998]. In Lahiri [1998] a correct proof of this equivalence is available.

2 The Model

Let X be a non-empty universal set of alternatives and let $\sum$ denote the set of all non-empty finite subsets of X. A choice function is a function $C : \sum \rightarrow \sum$ such that $C(S) \subset S \forall S \in \sum$.

A choice function C is said to satisfy Cheroff's Axiom (CA) if $S, T \in \sum, S \subset T \rightarrow C(T) \cap S \subset C(S)$.

A choice function C is said to satisfy Outcasting (0) if $C(T) \subset S \subset T \in \sum \rightarrow C(S) = C(T)$.

A choice function C is said to satisfy Superset Property (Su) if $C(T) \subset C(S) \subset S \subset T \in \sum \rightarrow C(S) = C(T)$.

It is easy to see that $(0) \rightarrow (Su)$.

A choice function C is said to satisfy Path Independence (PI) if $\forall S, T \in \sum, C(S \cup T) = C(C(S) \cup C(T))$.

Suzumura [1983] proves that a Choice function satisfies Path Independence if and only if it satisfies Chernoff's Axiom and the Superset Property.

We will proceed (for the sake of completeness and for the purpose of being self-contained) to provide a complete proof of Theorem 9 in Aizerman [1985], which is Theorem 1 in our paper.

Lemma 1 : C satisfies $PI \leftrightarrow C(S \cup T) = C(C(S) \cup T) \forall S, T \in \sum$.

Lemma 2: C satisfies $(CA) \leftrightarrow C(S \cup T) \subset C(S) \cup C(T) \forall S, T \in \sum$.

Proof : Let C satisfy CA. Thus for $S, T \in \sum$,

$$C(S \cup T) \cap S \subset C(S)$$
$$C(S \cup T) \cap T \subset C(T)$$

Hence $C(S \cup T) \subset C(S) \cup C(T)$.

Conversely suppose, $C(S \cup T) \subset C(S) \cup C(T) \ \forall \ S, T \in \sum$.

Let $S, T \in \sum$ with $S \subset\subset T$.

$$C(T) \subset C(S) \cup C(T \setminus S).$$

If $x \in C(T) \cap S$, then $x \notin C(T \setminus S)$.

Thus $x \in C(S)$.

Thus $C(T) \cap S \subset C(S)$..

Lemma 3 : C satisfies $PI \rightarrow C(C(S) = C(S)) \ \forall \ S \in \sum$.

Proof : Simply put $S = T$ in the definition of PI.

Q.E.D.

Theorem 1 : C satisfies $PI \leftrightarrow$ C satisfies CA and 0.

Alternative Proof : Suppose C satisfies CA and 0.

By Lemma 2, $\forall S, T \in \sum$.

$$C(S \cup T) \subset C(S) \cup C(T) \subset S \cup T.$$

By '0', $C(S \cup T) = C(C(S) \subset C(T))$.

Thus C satisfies satisfies PI.

Now Suppose C satisfies PI.

Thus for all $S, T \in \sum$,

$$C(S \cup T) = C(C(S) \cup C(T)) \subset C(S) \cup C(T).$$

By Lemma 2, C satisfies CA.

Let $C(T) \subset S \subset T \in \sum$.

By CA (which is implied by PI),

$$C(T) = C(T) \cap S \subset C(S)$$

Now,

$$\begin{aligned} C(T) = C(T \cup S) &= C(C(T) \cup C(S)) \;\; by \;\; PI \\ &= C(C(S)) \\ &= C(S) \;\; by \;\; Lemma\ 3. \end{aligned}$$

Thus C satisfies 0.

Q.E.D.

3 Lower Approximations

Let $C : \sum \to \sum$ be a choice function and let $\phi \neq Q$ be a class of choice function $\sum$.

Say that a choice function C' is contained in C if $C'(S) \subset C(S) \forall S \in \sum$. In such a situation we write $C' \subset C$.

The lower approximation of C given Q is the choice function

$$C^L : \sum \to \sum \;\; such\ that\ C^L(S) = \bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(S)$$

Suppose X is finite. A choice function C is said to satisfy Property (a^*) [see Deb [1983]] if $\forall S \in \sum$, there exists $x_o \in C(S)$ such that $\phi \neq T \subset S, x_o \in T \rightarrow x_o \in C(T)$.

Lemma 4 : If C' satisfies CA and $C' \subset C$ then C satisfies Property (a^*).

Proof : Given $S \in \sum$, let $x_o \in C'(S) \subset C(S)$.

Now $T \subset S \rightarrow C'(S) \cap T \subset C'(T)$ *by* CA.

Therefore $x_o \in T \rightarrow x_o \in C'(T) \subset C(T)$.

Thus C satisfies Property (a^*).

Q.E.D.

Theorem 2 : Let Q be the set of all Path Independent choice function. A choice function C has a lower approximation in Q if and only if C satisfy Property (a^*).

Proof : Suppose C satisfies Property (a^*). Then by the main result in Deb [1983], and Theorem 1, there exists C' in Q such that $C' \subset C$. Let $C^L : \sum \rightarrow \sum$ be defined as the lower approximation of C from Q. We have to show C^L in Q.

$$\text{Let } \phi \neq T \subset S.\ C^L(S) \cap T = \bigcup_{\substack{C' \subset C \\ C' \in Q}} [C'(S) \cap T] \subset \bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(T) = C^L(T),$$

where we appeal to CA for C'.

Thus C^L satisfies CA.

Let $C^L(T) \subset S \subset T \in \sum$.

$$\text{Thus, } \bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(T) \subset S \subset T \in \sum .$$

Therefore $C'(T) \subset S \subset T \in \sum \quad \forall\, C' \subset C, C' \in Q$.

By '0' applied to $C', C'(T) = C'(S)$.

$$\text{Therefore, } \bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(T) = \bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(S)$$

Therefore $C^L(T) = C^L(S)$.

Thus C^L satisfies 0.

Thus C^L is in Q.

Conversely, suppose C has a lower approximation C^L in Q. Since C^L satisfies CA, by the previous lemma C satisfies Property (a*).

Q.E.D.

A choice function C on $\sum$ is said to satisfy Concordance (CON) if $\forall\ S, T \in \sum, C(S) \cap C(T) \subset C(S \cup T)$.

Actually, Deb [1983] proves that a choice function C on $\sum$ contains a choice function C' satisfying Arrow's Axiom (Arrow (1959) (which implies CA, CON and O) if and only if C satisfies Property (a^*). In view of this result we can prove the followng :

Theorem 3 : Let Q be the set of all choice functions satisfying Path Independence and CON. A choice function C has a lower approximation in Q if and only if C satisfies Property (a*).

Theorem 4 : Let Q be the set of all choice functions satisfying CA and CON. A choice function C has a lower approximation in Q if and only if (satisfies Property (a*).

Theorem 5 : Let Q be the set of all choice functions satisfying CA. A choice function C has a lower approximation in Q if and only if C satisfies Property (a*).

The only bit where the proofs of Theorems 3 and 4 essentially differ from the one provided for Theorem 2 is in exhibiting the fact that C^L satisfies CON, if each C' does. However,

$$C^L(S) \cap C^L(T) = \left[\bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(S) \right] \cap \left[\bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(T) \right]$$

$$\subset \left[\bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(S) \cap C'(T) \right] \subset \bigcup_{\substack{C' \subset C \\ C' \in Q}} C'(S \cap T) = C^L(S \cup T)$$

The above analysis is considerably different from a similar analysis reported in Litvakov (1981), Aizerman (1985) and Aizerman and Aleskerov (1995), because we require all our choice functions to be non-empty valued. This makes a lot of difference in the analysis.

A related question posed in Litvakov (1981), Aizerman (1985), and Aizerman and Aleskerove (1995) is about the existence of upper approximations.

Let $C : \sum \to \sum$ be a choice function and let $\phi \neq Q$ be a class of choice functions on $\sum$. Say that a choice function C' contains C if $C(S) \subset C'(S) \forall S \in \sum$. In such a situation we write $C \subset C'$. The upper approximation of C given Q is the choice function $C : \sum \to \sum$ such that

$$C^u(S) = \bigcap_{\substack{C' \subset C \\ C' \in Q}} C'$$

The question that naturally arises is if C satisfies CA (which is considerably stronger than Property (a*)) does it have an upper approximation which satisfies Path Independence? The answer, is in the negative.

Example 1: Let $X = \{x, y, z\}$, $C(X) = \{x\}$ and $C(S) = S$ for all nonempty proper subsets of X. Let $C'(X) = \{x, y\}$ and $C'(S) = S$ otherwise. Let $C''(X) = \{x, z\}$ and $C''(S) = S$ otherwise. Here C satisfies CA but not 0. Both C' and C'' contain C and satisfy Path Independence. However, $C'(S) \cap C''(S) = C(S) \ \forall \ S \in \sum$. Thus C has no upper approximation satisfying Path Independence.

In fact we can show that even if C satisfies CA and CON, it may fail to have an upper approximation which satisfies PI (and CON).

Example 2 : Let $X = \{x, y, z\}$. Let $C(X) = \{x\}, C(\{x, y\}) = \{x\}, C(\{x, z\}) = \{x, z\}, C(\{y, z\}) = \{y\}$ and $C(\{a\}) = \{a\} \forall a \in X$. C satisfies CA and CON but does not satisfy PI :

$$\begin{aligned}
\{x\} = C(X) &= C(\{x,y\}) \cup \{x,y\}) \neq \{x,z\} = C(C(\{x,y\}) \cup C(\{x,z\})). \\
Let\ C'(X) &= \{x,y\}, C'\{x,y\}) = \{x,y\}, C'(\{x,z\}) = \{x,z\}, \\
C'(\{y,z\}) &= \{y\}, C'(\{a\}) = \{a\} \forall a \in X. C'\ satisfies\ PI\ and\ CON. \\
Let\ C''(X) &= \{x,z\}, C''\{x,y\}) = \{x\}, C''(\{x,z\}) = \{x,z\}, \\
C''(\{y,z\}) &= \{y,z\}, C''(\{a\}) = \{a\} \forall a \in X. C''\ satisfies\ PI\ and\ CON.
\end{aligned}$$

Observe, $C \subset C'$ and $C \subset C''$. However, $C(S) = C'(S) \cap C''(S) \forall S \in \sum$. Thus C has no upper approximation satisfying PI (and CON).

4 Choice Functions Satisfying Choice Acyclicity :

Choice functions which satisfy outcasting are rather interesting. Aizerman and Aleskerov (1995) show that if $f : \sum \to \Re$ satisfy the property : [$\forall\ S \in \sum, \{T \subset S / f(T) \geq f(T')\ forall\ T' \subset S\}$ is a singleton], then the choice function $C_f : \sum \to \sum$ defined by

$$C_f(S) = argmax_{\substack{T \subset S \\ T \in \sum}} [f(T)],$$

satisfies outcasting. Such choice functions may be called batch choice functions. Batch choice functions satisfy an even stronger property :

Choice Acyclicity Property (CAP) : There does not exist a positive integer k and sets $S_1, \ldots S_k \in \sum$, all distinct such that $C(S_i) \subset S_{i+1}, i = 1, \ldots k-1, C(S_k) \subset S_1$ and $C(S_i) \neq C(S_j)$ for some $i, j (i \neq j)$.

If $k = 2$ and $S_1 \subset S_2$, we get outcasting.

Let $C_f : \sum \to \sum$ be a batch choice function. Suppose there exists $S_1, S_2, \ldots, S_k \in \sum$ such that $C(S_i) \subset S_{i+1}, i = 1, \ldots, k-1, C(S_k) \subset S_1$ and $C(S_i) \neq C(S_j)$ for some $i, j (i \neq j)$. Then $f(C(S_{i+1})) \geq f(C(S_i)), i = 1, \ldots, k-1$ and $f(C(S_1)) \geq f(C(S_k))$ with at least one strict inequality. But this is impossible.

We can now prove the converse result, that every choice function satisfying choice acyclicity property is a batch choice function.

Assume in what follows that X is a non-empty finite set.

Theorem 6 : A choice function is a batch choice function if and only if it satisfies CAP.

Proof : The fact that a batch choice function satisfies CAP has been established above. Hence, let us assume that C satisfies CAP.

Given $S \in \sum$, let $\wp(S)$ denote the set of all non-empty subsets of S and let $2^S = \wp(S) \cup \{\phi\}$. Given $\phi \neq Y \subset \sum, \wp(S) \subset Y$ will be said to be maximal for Y if there does not exist $\wp(T) \subset Y$ with $S \subset\subset T$ and $C(T) \subset S$. Given $\phi \neq Y \subset \sum$, let $\wp(T_i), i = 1, \ldots, m$ be maximal for Y (: provided there exist at least one $\wp(S)$ maximal for Y). Note : if $Y = \wp(S)$, for some $S \in \sum$, then $\wp(S)$ is the only one of its kind maximal for Y. Consider T_1. If $i \neq 1$ implies $[C(T_1)$ *not a subset of* T_i or $C(T_1) = C(T_i)]$, let $F(Y) = C(T_1)$. If there exists $T \in \{T_i\}_{i \neq 1}$ such that $C(T_1) \subset T, C(T_1) \neq C(T)$, then consider $C(T)$. Say $T = T_2$. By CAP, $C(T_2)$ *not a subset of* T_1. If $i \neq 2$ implies $[C(T_2)$ *not a subset of* T_i or $C(T_2) = C(T_i)]$, let $F(Y) = C(T_2)$. If not there exists T_3 say such that $C(T_2) \subset T_3, C(T_2) \neq C(T_3)$. Repeat the above procedure for $C(T_3)$. By CAP, and the finiteness of $\sum$, there exists $C(T_i)$ such that for $j \neq i$, either $C(T_i) = C(T_j)$ or $C(T_i)$ *not a subset of* T_j. Let $F(Y) = C(T_i)$.

If there does not exist any $S \in \sum$ such that $\wp(S) \subset Y$, then put $F(T) = \phi$.

Example : Let $X = \{x, y\}, C(X) = \{x\}, C(\{a\}) = \{a\}$ for all $a \in X. \sum = \{\{x, y\}, \{x\}, \{y\}\}$ Thus $F(\sum) = \{x\}, F(\{\{x, y\}, \{y\}\}) = \{y\}, F(\{\{x, y\}\}) = \phi$.

$$\begin{aligned} Now\ F(\textstyle\sum) = C(X) &= S_1 \\ F(\textstyle\sum \backslash \{S_1\}) &= S_2 \\ F(\textstyle\sum \backslash \{S_1, S_2\}) &= S_3 \\ F(\textstyle\sum \backslash \{S_1, S_2, \ldots, S_k\}) &= \phi \end{aligned}$$

where k is the first positive integer to satisfy the above property.

Since C satisfies outcasting, F satisfies the following property : $\wp(S) \subset Y \subset \sum, F(Y) \in \wp(S)$ implies $F(\wp(S)) = C(S) = F(Y)$.

Note : If $\wp(S)$ is not maximal in Y, then there exists $\wp(S')$ maximal in Y such that $S \subset\subset S'$ and $C(S') \subset S$. Thus by outcasting $F(\wp(S')) = C(S') = C(S)$. Since either $F(Y) \notin \wp(S')$ or $F(Y) = C(S')$ we get the above assertion.

$$\begin{aligned} Let\ f(S_i) &= k - i + 1 \ \ for\ all\ i = 1, \ldots k \\ and\ f(S) &= -1 \ \ if\ \ S \in \textstyle\sum \backslash \{S_1 \ldots, S_k\}. \end{aligned}$$

Thus $f : \sum \rightarrow \Re$ *is well defined.*

Let $S \in \sum$. Clearly $S \subset X$. If $S_1 \subset S$, then by 0,

$$C(S) = S_1 = argmax_{\substack{T \subset S \\ T \in \sum}} [f(T)].$$

If S_1, *not a subset of* S, then $\wp(S) \subset \sum \backslash \{S_1\}$.
If $F(\sum \backslash \{S_i\}) \in \wp(S)$, then $F(\sum \backslash \{S_1\}) = F(\wp(S)) = C(S)$.

$$Therefore\ C(S) = S_2 = argmax_{\substack{T \subset S \\ T \in \sum}} [f(T)].$$

If S_2 *not a subset of* S, then $\wp(S) \subset \sum \backslash \{S_1, S_2\}$ and we reepat the above process again. At each new step we need to undertake, because S_{i-1} *not a subset of* S, either $S_i \subset S$, so that

$$C(S) = S_2 = argmax_{\substack{T \subset S \\ T \in \sum}} [f(T)].$$

(since $\wp(S) \subset \sum \backslash \{S_1, S_2, \ldots, S_{i-1}\}$) or S_i *not a subset of* S.

Since $\sum$ is finite, there exists i such that S_{i-1} *not a subset of* S and $S_i \subset S$. Thus

$$C(S) = S_i = argmax_{\substack{T \subset S \\ T \in \sum}} [f(T)].$$

Q.E.D.

Note : In the above iterative construction of $S_1, S_2, \ldots, S_k$, where S_i is removed, then only those $\wp(S)$'s are affected for which $\wp(S) \subset \sum \backslash \{S_1, S_2, \ldots, S_{i-1}\}$ and $S_i \in \wp(S)$. But then, by the construction of $F, C(S) = S_i$.

Example : 0 does not imply CAP. Let $X = \{x, y, z\}$; let $C(X) = X, C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}, C(\{a\}) == \{a\}\ \forall\ a \in X$. C satisfies 0 vacuously. However let $S_1 = \{x, y\}, S_2 = \{x, z\}$ and $S_3 = \{y, z\}$.] Then $C(S_1) \subset S_2, C(S_2) \subset S_3, C(S_3) \subset S_1$. Further $C(S_i) \neq C(S_i)$ for $i \neq j$. Thus C violates CAP.

It is worthwhile to consider the analogue of the weak ordinality result due to Deb [1983] in the present context. Towards that goal we formulate the following property :

Weak Choice Acyclicity Property (W. CAP) : For all $S \subset X$, there exists a non-empty subset $C_S \subset C(S)$ for which the following holds : it is not possible to find a positive integer k and sets $S_1, \ldots, S_k$ in $\sum$ such that $C_{S_j} \neq C_{S_j}$ for some i and j and $C_{S_i} \subset S_{i+1}, i = 1, \ldots, k-1, C_{S_k} = S_1$..

The proof of the following theorem is analogous to the proof of Theorem 6.

Theorem : Suppose X is finite. Then there exists a function $f : \sum \rightarrow \Re$ satisfying

(a) $\{T \subset S / f(T) \geq f(T') \ \forall \ \phi T' \subset S\}$ is on sigleton for all $S \in \sum$;

(b) $\underset{\substack{T \subset S \\ T \in \sum}}{argmax} [f(T)] \subset C(S) \ for \ all \ S \in \sum,$

if and only if C satisfies W.CAP.

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Appendix

In this appendix we provide a simple proof of Theorem 2.10 in Deb [1983] in the case when the universal set is finite. Our proof does not require the use of Zorn's Lemma, which has been used in Deb [1983] to prove Theorem 2.10.

Let X be a finite set of alternatives. A choice function C is said to satisfy Property (a*) if $\forall\, S \in \sum$, there exists $x_o \in S$ such that $T \in \sum, T \subset S, x_o \in T$ implies $x_o \in C(T)$.

It is easy to see that this x_o (which depends on S) must belong to $C(S)$. [Simply take T = S in the definition].

Theorem : Given a choice function C, there exists a function $\Psi : X \rightarrow \Re$ such that $\forall\, S \in \sum$,

$$\{x \in S/\Psi(x) \geq \Psi(y)\ \forall\, y \in S\} \subset C(S) \qquad (*)$$

if and only if C satisfies Property (a*).

Proof : Let C be a choice function satisfying (*). Given $S \in \sum$, pick $x_0 \in \{x \in S/\Psi(x) \geq \Psi(y)\ \forall\, y \in S\}$. Then $T \in \sum, T \subset S, x_o \in T \rightarrow \Psi(x_0) \geq \Psi(y)\ \forall\, y \in T$.

Thus $x_o \in C(T)$.

Now suppose C satisfies Property (a*).

Let $x_1 \in C(X)$ satisfy the conditions of Property (a*) for X.

Let $x_2 \in C(X \setminus \{x_1\})$ satisfy the conditions of Property (a*) for $X \setminus \{x_1\}$.

Having selected $x_1, x_2, \ldots, x_{i-1}$ choose $x_i \in C(X \setminus \{x_1, \ldots, x_{i-1}\})$ satisfying the conditions of Property (a*) for $(X \setminus \{x_1, \ldots, x_{i-1}\})$.

Since X is finite, $X = \{x_1, x_2, \ldots, x_s\}$ for some positive integer 's'.

Define, $\Psi(x_i) = s - i + 1, i = 1, \ldots, s$.

Let $S = \{x_{i_1}, \ldots, x_{i_m}\} \in \sum$ with $i_1 < i_2 < \ldots, < i_m$.

Now $\{x \in S/\Psi(x) \geq \Psi(y)\ \forall\, y \in S\} = \{x_{i_1}\}$.

Further, $S \subset X \setminus \{x_1, x_2, \ldots, x_{i_1 - 1}\}$ and $x_{i_1} \in S$.

By Property (a*) and construction of x_{i_1}, $x_{i_1} \in C(S)$.

Therefore, $\{x \in S/\Psi(x) \geq \Psi(y)\ \forall\, y \in S\} \subset C(S)$.