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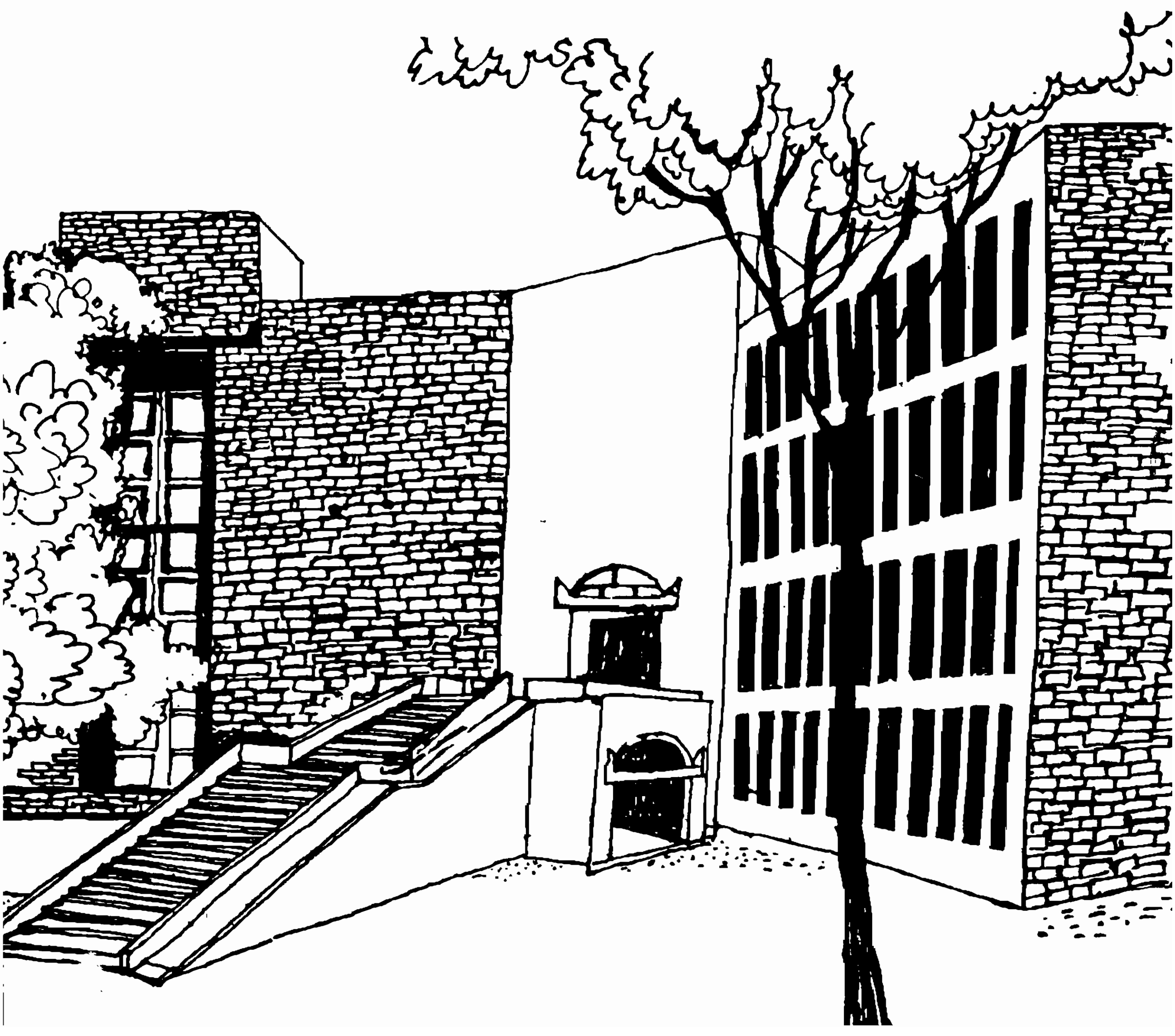

AXIOMATIC CHARACTERIZATION OF BUDGET CONSTRAINED PARETO EFFICIENT SOCIAL CHOICE FOR FAIR DIVISION PROBLEMS

By

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A B S T R A C T

In recent times, beginning with the work of Thomson [1988], attention in social choice theory has focused on problems of fair division. The problem is one of dividing a social endowment among a group of agents keeping in mind the issues of equity and efficiency.

There is a sizeable literature which has dealt with the problem of axiomatically characterizing the equal income market equilibrium solution. Notable among them are the works of Thomson [1988], Lahiri (1997, forthcoming).

In Thomson and Varian [1985], there is the concept of an income fair allocation. Basically, these are allocations which cost the same for all agents. However, the price vector at which the allocation is evaluated, is in the existing literature, endogenously determined by market forces, whence it coincides with equal income market equilibrium allocations. The income fairness concept derives appeal as an independent entity, only if the price at which the allocation is evaluated is specified exogenously. Along with efficiency, it then coincides with what Balasko [1979], calls budget constrained Pareto efficient allocations. To be relevant to the literature on fair division, it should be required that the monetary worth of the allocation at exogenously specified prices, be the same for all agents.

In this paper, we axiomatically characterize the social choice correspondence, which picks for each economy, the set of equal income budget constrained Pareto efficient allocations. We are able to characterize this social choice correspondence uniquely, with the help of the following assumptions : Consistency, Equal Budget Property, Pareto efficiency for two agent problems and Local Independence. The most extensive use of Consistency is due to Thomson, as surveyed in Thomson [1990]; the equal budget property is due to Varian [1976], Pareto efficiency for two agent problems is due to van der Nouweland, Peleg and Tijs [1996], Local Independence is due to Nagahisa [1991]. In a final section, we replace Consistency by Converse Consistency (Thomson [1990]) and Pareto Efficiency for two agent problems by Binary Efficiency (see Lahiri [1997]) to obtain yet another axiomatic characterization of the same social choice correspondence.

1. Introduction:

In recent times, beginning with the work of Thomson [1988], attention in social choice theory has focused on problems of fair division. The problem is one of dividing a social endowment among a group of agents keeping in mind the issues of equity and efficiency.

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In this paper, we axiomatically characterize the social choice correspondence, which picks for each economy, the set of equal income budget constrained Pareto efficient allocations. We are able to characterize this social choice correspondence uniquely, with the help of the following assumptions : Consistency, Equal Budget Property, Pareto efficiency for two agent problems and Local Independence. The most extensive use of Consistency is due to Thomson, as surveyed in Thomson [1990]; the equal budget property is due to Varian [1976], Pareto efficiency for two agent problems is due to van der Nouweland, Peleg and Tijs [1996], Local Independence is due to Nagahisa [1991]. In a final section, we replace Consistency by Converse Consistency (Thomson [1990]) and Pareto Efficiency for two agent problems by Binary Efficiency (see Lahiri [1997]) to obtain yet another axiomatic characterization of the same social choice correspondence.

2. The Model:-

Let $\phi \neq P (\subset \mathbf{N})$ denote a set of potential agents with cardinality of P greater than or equal to two. Agent sets are non-empty, finite subsets of P. Let $\mathbf{R}$ (*resp.* $\mathbf{R}_{+}, \mathbf{R}_{++}$) denote the set of real numbers (resp. non-negative real numbers, strictly positive real numbers). Given M $\in$ P, let $\mathbf{R}^{M}$(*resp.* $\mathbf{R}^{M}_{+}, \mathbf{R}^{M}_{++}$) denote the set of all functions from M to $\mathbf{R}$ (*resp.* $\mathbf{R}_{+}, \mathbf{R}_{++}$) .

Let there be k $\geq$ 2 (k $\in$ $\mathbf{N}$) infinitely divisible goods in the economy. A utility function is a continuous function u : $\mathbf{R}^{k}_{+} \to \mathbf{R}$.

Let U be the set of all utility functions satisfying:

1. Semi-strict quasi concavity :

$x \in \mathbf{R}^{k}_{+},\ y \in \mathbf{R}^{k}_{+},\ u(x) > u(y),\ t \in [0,1) \Rightarrow u(t\,y + (1-t)\,x) > u(y)$

2. $x \in \mathbf{R}^{k}_{++},\ y \in \mathbf{R}^{k}_{+},\ u(x) = u(y),\ \Rightarrow y \in \mathbf{R}^{k}_{++}$

3. Strict Monotonicity on $\mathbf{R}^{k}_{++}$:

   $x \in \mathbf{R}^{k}_{++},\ y \in \mathbf{R}^{k}_{+},\ x > y \Rightarrow u(x) > u(y).$

4. Smoothness: U is differentiable at each $x \in \mathbf{R}^{k}_{++}$ with all partial derivatives strictly positive.

For any realization of an agents set, $\omega \in \mathbf{R}^k_{++}$ denotes, the social endowment.

Given M ϵ P, let U^M denote the set of all functions from M to U.

Given M ϵ P, an M agent problem (of fair division) is an element of $U^M \times \mathbf{R}^k_{++}$. Let $\mathscr{E}^M$ denote the set of all M agent problems and let, $\mathscr{E} = \bigcup_{M \in P} \mathscr{E}^M$.

Given M ϵ P and $[(u^i)_{i \in M}, \omega] \equiv e$ belonging to $\mathscr{E}^M$, let

$A(e) = \{ (x^i)_{i \in M} \in (\mathbf{R}^k_+)^M / \sum_{i \in M} x^i = \omega \}$. A (e) denotes the set of all feasible allocations for e.

A social choice correspondence is a correspondence

$F : \mathscr{E} \to\to X$ such that $F(e) \subset A(e)\ \forall e \in \mathscr{E}$. Here $X = \bigcup_{M \in P} (\mathbf{R}^k_+)^M$.

A social choice correspondence is said to satisfy Pareto efficiency if $\forall$ M ϵ P, $\forall e \equiv [(u^i)_{i \in M}, \omega]$, $(x^i)_{i \in M} \in F(e)$ implies the non-existence of $(y^i)_{i \in M} \in A(e)$ with $u^i(y^i) \geq u^i(x^i)\ \forall i \in M$, with at least one strict inequality. For $e \in \mathscr{E}$, let P(e) denote the set of all Pareto efficient allocations.

Let $\hat{p} \in \mathbf{R}^k_{++}$ be a fixed vector of prices.

The following concept is due to Balasko (1979).

The equal income budget constrained Pareto Efficient social choice correspondence $\mathbf{B} : \mathscr{E} \rightarrow X$ is defined thus :

$\forall M \in P, \forall e \equiv [(u^i)_{i \in M}, \omega] \in \mathscr{E}^M, (x^i)_{i \in M} \in B(e)$, if and only if

i) $(x^i)_{i \in M} \in P(e)$

ii) $\hat{p}.x^i = \hat{p}. \frac{\omega}{|M|} \quad \forall i \in M.$

3. <u>A Preliminary Result about the Correspondence B:</u>

<u>Lemma 1:</u> Given $M \in P$ and $e \in \mathscr{E}^M$, let $(x^i)_{i \in M} \in P(e)$. Then $\forall i \in M$, $x^i \in \mathbf{R}^k_{++} \cup \{0\}$.

<u>Proof</u>: Let $e \equiv [(u^i)_{i \in M}, \omega]$

<u>Case</u> 1:- $\forall i \in M, x^i \in \mathbf{R}^k_+ \setminus (\mathbf{R}^k_{++} \cup \{0\})$.

Then by the second property of utility functions,

$u^i(\omega/|M|) > u^i(x^i) \; \forall i \in M$

Since the allocation

$(y^i)_{i\in M}$, $y^i = \omega/|M|\ \forall\ i\in M$ is in $A(e)$, $(x^i)_{i\in M} \notin P(e)$

Case 2: For some $i \in M$,

$x^i \in \mathbf{R}^k_{++}$ and for some $j\in M$, $x^j \in \mathbf{R}^k_+ \setminus (\mathbf{R}^k_{++} \cup \{0\})$

Consider $(y^h)_{h\in M}$:

$$y^h = \begin{cases} x^h, & \text{if } \notin \{i,j\} \\ x^i + x^j, & \text{if } h = i \\ 0, & \text{if } h = j. \end{cases}$$

$$(y^h)_{h\in M} \in A(e),\quad u^h(y^h) = u^h(x^h), \text{ if } h \neq i$$
$$u^i(y^i) > u^i(x^i).$$

Thus, $(x^h)_{h\in M} \notin P(e)$, which is a contradiction. Hence the Lemma.

Q.E.D.

Given $e \in \mathscr{E}$, let $M(e)$ denote the agent set for e.

Lemma 2: Given, $e \in \mathscr{E}$, if $(x^i)_{i\in M(e)} \in B(e)$, then

$x^i \in \mathbf{R}^k_{++}\ \forall\ i \in M(e)$

Proof: By Lemma, 1,

$$x^i \in \mathbf{R}^k_{++} \cup \{0\}\ \forall\ i \in M(e).$$

Since $\hat{p}.x^i > 0\ \forall\ i \in M(e)$, $x^i \in \mathbf{R}^k_{++}\ \forall\ i \in M(e)$

Q.E.D.

4. A Result for Cobb-Douglas economies: The following result is similar to a result occurring in Lahiri [1997].

Proposition 1: Let

$$p \in \mathbb{R}^k_{++}, M \in P. \text{ Let } (\overline{x}^i)_{i \in M} \in (\mathbb{R}^k_{++})^M \text{ with } \hat{p} \cdot \overline{x}^i = c > 0 \; \forall\, i \in M \text{ and } \omega = \sum_{i \in M} (\overline{x}^i)$$

Then there exists $(v^i)_{i \in M} \in U^M$ and

$$\lambda^i > 0 (i \in M): \;\; \forall\, i \in M, \forall\, j \in 1, \ldots, k, \; \frac{\partial v^i}{\partial x^i_j} (\overline{x}^i) = \lambda^i p_j \;.$$

Proof: Let $W^i = p \cdot \overline{x}^i > 0, i \in M$

and $\alpha^i_j = \dfrac{p_j \, \overline{x}^i_j}{W^i}$, $i \in M$, $j \in (1, \ldots, k)$.

Let $v^i (x^i) = \prod_{j=1}^{k} (x^i_j)^{\alpha^i_j}$, for $x_i \in \mathbb{R}^k_+$.

(Note: If in the above, we set

$\alpha^i_j = p_j \, \overline{x}^i_j \;\; \forall\, i \in M, \; j \in [1, \ldots, k]$, then $\lambda^i = 1 \; \forall\, i \in M$) . Q.E.D.

Proposition 2:

Let $p \in \mathbb{R}^k_{++}$ and $M \in P$. Let $(\overline{x}^i)_{i \in M} \in (\mathbb{R}^k_{++})^M$ with $\hat{p}.\overline{x}^i = \hat{p}.\overline{x}^j \; \forall\, i, j \in M$ and $\omega = \sum_{i \in M} \overline{x}^i$.

Then there exists $e \equiv [(v^i)_{i\in M}, \omega] \in \mathcal{E}^M$ *such that* $B(e) = \{(\overline{x}^i)_{i\in M}\}$ and

$$\frac{\partial v^i}{\partial x^i_j}(\overline{x}^i) = p_j \ \forall \ i\in M \text{ and } j\in\{1,\ldots,k\}.$$

Proof: As in proposition 1, construct v^i, $i\in M$, with

$$\alpha^i_j = p_j \, \overline{x}^i_j \ \forall \ i\in M \text{ and } j\in\{1,\ldots,k\}.$$

Clearly, $(\overline{x}^i)_{i\in M} \in B(e)$

Now suppose

$(x^i)_{i\in M} \neq (\overline{x}^i)_{i\in M}$ *with* $x^i \in \mathbf{R}^k_{++} \ \forall \ i\in M$ *and* $\hat{p}.x^i = \hat{p}.x^j \ \forall \ i, j\in M$.

(By Lemma 2, $\mathbf{R}^k_{++}$ *is sufficient*).

Suppose there exists $\overline{p} \in \mathbf{R}^k_{++}$ *and* $\lambda^i, i\in M$.

$$\alpha^i_j \frac{v^i(x^i)}{x^i_j} = \frac{\partial v^i}{\partial x^i_j}(x^i) = \lambda^i \, \overline{p}_j \ \forall \ i\in M, \ j \in \{1,\ldots,k\}$$

$$\therefore \frac{\alpha^i_j}{\alpha^i_1}\frac{x^i_1}{x^i_j} = \frac{\overline{p}_j}{\overline{p}_1} \ \forall \ i\in M, \ j \in \{1,\ldots,k\}$$

Clearly, there exists i and h ϵ M and j ϵ $\{1,\ldots,k\}$, such that

$x_j^i > \overline{x}_j^i$ *and* $x_j^h < \overline{x}_j^h$

and m ϵ $\{1,\ldots,k\}$ such that $x_m^i < \overline{x}_m^i$ *and* $x_m^h > \overline{x}_m^h$. .

Without loss of generality assume m = 1.

$$\therefore \frac{\overline{p}_j}{\overline{p}_1} = \frac{\alpha_j^i}{\alpha_1^i} \cdot \frac{x_1^i}{x_j^i} < \frac{\alpha_j^i}{\alpha_1^i} \frac{\overline{x}_1^i}{\overline{x}_j^i} = \frac{p_j}{p_1} :$$
$$\frac{\alpha_j^h}{\alpha_h^1} \frac{\overline{x}_1^h}{\overline{x}_j^h} < \frac{\alpha_j^h}{\alpha_1^h} \frac{x_1^h}{x_j^h} = \frac{\overline{p}_j}{\overline{p}_1}$$

which is a contradiction. Thus B(e) = $\{(\overline{x}^i)_{i\epsilon M}\}$. Q.E.D.

5. <u>Consistency and Efficiency</u>

A social choice correspondence is said to satisfy <u>consistency</u> if

$$\forall M \epsilon P,\ \forall L \epsilon P,\ L \subset M,\ \forall e = [(u)^i)_{i\epsilon M},\ \omega],\ \textit{if } (x^i)_{i\epsilon M} \in F(e),$$
$$\textit{then } (x^i)_{i\epsilon L} \in F(e'),\ \textit{where } e' = [(u^i)_{i\epsilon L}, \sum_{i\epsilon L} x^i]$$

A social choice correspondence is said to satisfy <u>Pareto efficiency for two agent problems</u> if $\forall$ e ϵ $\mathscr{E}$ with cardinality of M(e) equal to 2, F(e) $\subset$ P(e).

A social choice correspondence is said to satisfy the <u>equal budget property</u> if

$$\forall e \in \mathscr{E},\ \forall (x^i)_{i\epsilon M(e)} \in F(e), \hat{p} \cdot x^i = \hat{p} \cdot \overline{x}^j\ \forall i,j \in M(e).$$

It is easy to check that B satisfies all the properties mentioned above.

<u>Theorem 1:-</u> Let F be a social choice correspondence which satisfies consistency, Pareto efficiency for two agent problems and the equal budget property. Then $F(e) \subset B(e)\ \forall\ e \in \mathcal{E}$.

Proof: Let $e = [(u^i)_{i\epsilon M}, \omega]$ and let $(\overline{x}^i)_{i\epsilon M} \in F(e)$.

Let $i, j \in M$ and let $L = \{i,j\}$.

By Consistency, $(\overline{x}^i)_{i\epsilon L} \in F(\ [(u^i)_{i\epsilon L}, \overline{x}^i+\overline{x}^j]\ )$

By Pareto efficiency for two agent problems, there exists $p^L \in \mathbf{R}^k_{++}$ and $\lambda^i_L,\ \lambda^j_L > 0$.

$$\frac{\partial u^i}{\partial x^i_h}(\overline{x}^i) = \lambda^i_L\, p^L_h,\quad h \in \{1,\ldots,k)\ \}$$

Hence λ^i_L and p^L_h are independent of L.

Hence, there exists $p \in \mathbf{R}^k_{++}$ and $\lambda^i > 0\ \ (i \in M)$:

$$\frac{\partial u^i}{\partial x^i_h}(\overline{x}^i) = \lambda^i\, p_h\ \forall\ i \in M.$$

$\therefore (\overline{x}^i)_{i\epsilon M} \in P(e)$.

$\therefore (\bar{x}^i) \in B(e)$, since, we already know that it satisfies equal budget property.

Q.E.D.

6. Local Independence and the Main Result:

A social choice correspondence F is said to satisfy Local Independence if $\forall M \in \mathbf{P}$, $\forall e = [(u^i)_{i\epsilon M}, \omega]$,

$$e' = [(v^i)_{i\epsilon M}, \omega] \in \mathcal{E}^M, \forall (x^i)_{i=1}^M \in A(e) = A(e')$$

$$\text{with } x^i \in \mathbf{R}^k_{++} \forall i, [\frac{\partial u^i}{\partial x^i_j}(x^i) = \frac{\partial v^i}{\partial x^i_j}(x^i) \forall i, j]$$

and $(x^i)_{i\epsilon M} \in F(e)$ *implies* $(x^i)_{i\epsilon M} \in F(e')$. .

Theorem 2: Let F be a social choice correspondence which satisfies Consistency, Pareto Efficiency for two agent problems, Equal Budget Property and Local Independence. Then $F(e) = B(e) \; \forall e \in \mathcal{E}$. .

Proof: It is easy to check that B satisfies Local Independence and Theorem 1 established that $F(e) \subset B(e) \; \forall e \in \mathcal{E}$. .

Let $e = [(u^i)_{i\epsilon M}, \omega]$ *and* $(\bar{x}^i)_{i\epsilon M} \in B(e)$

Let $e' = [(v^i)_{i\epsilon M}, \omega]$ be as constructed in Proposition 2, with p_j

$\frac{\partial u^i}{\partial x^i_j}$ $(\bar{x}^i)$, $j = \{1,...k\}$. . Since $(\bar{x}^i)_{i \in M} \in P(e)$, p_j is independent of j.

Further, $B(e') = (\bar{x}^i)_{i \in M}$.

By Theorem 1, $F(e') = \{(\bar{x}^i)_{i \in M}\}$

By Local Independence of F, $(\bar{x}^i)_{i \in M} \in F(e)$. $\therefore B(e) \subset F(e)$.

Q.E.D.

It might be said, that a characterization of the budget constrained Pareto optimal social choice correspondence using Pareto Optimality for two agent problems and the equal budget property, is almost tautological. This is not quite correct as the proof of Theorem 2 (though not of Theorem 1) is far from immediate.

7. Characterization using Converse Consistency and Binary Efficiency:

A social choice correspondence is said to satisfy Converse Consistency if $\forall$ M ϵ P, $\forall$ $e = [(u^i)_{i \in M}, \omega]$ if $(x^i)_{i \in M} \in A(e)$ and $(x^i)_{i \in L} \in F(e')$ for all L ϵ P, L $\subset$ M with cardinality of L being two, then $(x^i)_{i \in M} \in F(e')$, where e' $\left[(u^i)_{i \in L}, \sum_{i \in L} x^i\right]$

It is routine to check that B satisfies Converse Consistency.

A social choice correspondence is said to satisfy Binary Efficiency if $M \in P$, $\forall\, e = \left[(u^i)_{i \in M},\ \omega\right]$ and for all $(x^i)_{i \in M} \in F(e)$ the following holds: it is not possible to find $L \in P$, $L \subset M$, cardinality of L being two and $(y^i)_{i \in L}$, with $y^i \in \mathbf{R}^k_+ \ \forall\, i \in L$ and $\sum_{i \in L} y^i = \sum_{i \in L} x^i$ such that $u^i(y^i) \geq u^i(x^i)\ \forall\, i \in L$ with at least one strict inequality.

Binary Efficiency implies Pareto efficiency for two agent problems and once again, it is routine to check that B satisfies Binary Efficiency. However, if we use Binary Efficiency in place of Pareto efficiency for two agent problems, we can replace Consistency by Converse Consistency in our axiomatic characterization. Thus, we have the following:

Theorem 3: The only social choice correspondence which is non-empty valued for all two agent problems with Cobb-Douglas preferences and satisfies Converse Consistency, Binary Efficiency, Equal Budget Property and Local Independence is B.

Proof: We already know that B satisfies the above properties. Hence, let F be a social choice correspondence satisfying the above properties. Let $e = \left[(u^i)_{i \in M},\ \omega\right] \in \mathscr{E}$ and let $(\bar{x}^i)_{i \in M} \in F(e)$. All that we need to show is that $(\bar{x}^i)_{i \in M} \in P(e)$. By Binary Efficiency, and the smoothness condition on preferences,
$\forall$ i, j $\in$ M if $L = \{1, j\}$, then there exists $p^L \in \mathbf{R}^k_{++}$, λ^L_i, $\lambda^L_j > 0$ s.t.

$$\frac{\partial u^h}{\partial x_j^h}(\overline{x}^h) = \lambda_h^L p_j^L \; \forall \; h \in L, \; j \in \{1,\ldots k,\}.$$

Thus, λ_h^L *and* p_j^L are independent of L. Thus, $(\overline{x}^i)_{i\epsilon M}$ ϵ P(e). Thus, $(\overline{x}^i)_{i\epsilon M}$ ϵ B(e).

Now suppose $(\overline{x}^i)_{i\epsilon M}$ ϵ B(e). We have to show that $(\overline{x}^i)_{i\epsilon M}$ ϵ F(e). By Binary Efficiency and equal budget property, $(\overline{x}^i)_{i\epsilon L}$ ϵ B(e') whenever, $e' = \left[(u^i)_{i\epsilon L} \sum_{i\epsilon L} \overline{x}^i\right]$, $L \in P$, $L \subset M$ and Cardinality of L is two.

Now as in Proposition 2, there exists

$$\overline{e} = \left[(v^i)_{i\epsilon L}, \sum_{i\epsilon L} \overline{x}^i\right] \text{ with } \frac{\partial v^i}{\partial x_j^i}(\overline{x}^i) = \frac{\partial u^i}{\partial x_j^i}(\overline{x}^i) \; \forall \; i\epsilon L \text{ and } j\epsilon\{1,\ldots,k\},$$

such that, $\{(\overline{x}^i)_{i\epsilon L}\} = B(\overline{e})$. . Since F has been assumed to be non-empty valued for all two agent problems with Cobb-Douglas preferences and since $F(\overline{e}) \subset B(\overline{e})$, $F(\overline{e}) = \{(\overline{x}^i)_{i\epsilon L}\}$.

By Local Independence,

$(\overline{x}^i)_{i\epsilon L} \in F(e')$. *By Converse Consistency,* $(\overline{x}^i)_{i\epsilon M} \in F(e)$.

Note: In the above, we make use of the fact that B is non-empty valued for all two agent problems with Cobb-Douglas Preferences. A proof of this statement is contained in the Appendix.

APPENDIX

Let $v^i(x^i) = \prod_{j=1}^{k} (x_j^i)^{\alpha_j^i}$, $i = 1, 2$, $k \geq 2$. Let m_1 and m_2 be positive real numbers and let $\hat{p} \in \mathbb{R}^k_{++}$. *Let* $W \in \mathbb{R}^k_{++}$. Suppose, $\hat{p}. W = m_1 + m_2$. *Assume,* $\alpha_j^i > 0 \; \forall \, i, j$ *and* $\sum_{j=1}^{k} \alpha_j^i = 1$.

Let $e = [(v^i)_{i \in L}, W]$ where $L = \{1, 2\}$.

<u>Claim 1:</u> Let $k = 2$. Then, there exists $(\bar{x}^i) \in P(e)$ *such that* $\hat{p} \,.\, \bar{x}^i = m_i \; \forall \, i \in L$.

<u>Proof</u>: We need to show that the following system of equations have a solution:

$$\frac{\alpha_1^1}{x_1^1} \cdot \frac{x_2^1}{1-\alpha_1^1} = \frac{\alpha_1^2}{W_1 - x_1^1} \cdot \frac{W_2 - x_2^1}{1-\alpha_1^2}$$

$$\hat{p}\, x_1^1 + \hat{p}_2\, x_2^1 = m_1 \,,$$

Consider the point $x_1^1 = \dfrac{m_1 - \epsilon\, \hat{p}_2}{\hat{p}_1}$, $x_2^1 = \epsilon$.

$$\frac{\alpha_1^1 . x_2^1}{x_1^1(1-\alpha_1^1)} - \frac{\alpha_1^2}{W_1 - x_1^1} \cdot \frac{W_2}{1-a_1^2} < 0 \; if \; x_1^1 = \frac{m_1 - \epsilon\, \hat{p}_2}{\hat{p}_1}$$

$\forall \, \epsilon > 0$ sufficiently small.

Now consider the point $x_1^1 = \epsilon > 0$, $x_2^1 = \dfrac{m_1 - \epsilon\,\hat{p}_1}{\hat{p}_2}$

Here $\dfrac{\alpha_1^1}{x_1^1} \cdot \dfrac{x_2^1}{1-\alpha_1^1} = \dfrac{\alpha_1^1}{\epsilon} \cdot \dfrac{m_1 - \epsilon\,\hat{p}_1}{\hat{p}_2(1-\alpha_1^1)}$ and

$$\frac{\alpha_1^2}{W_1-\epsilon} \cdot \frac{\hat{p}_2\ W_2 - m_1 + \epsilon\,\hat{p}_1}{\hat{p}_2(1-\alpha_1^2)} = \frac{\alpha_1^2}{W_1 - x_1^2} \cdot \frac{W_2 - x_2^1}{1-\alpha_1^2} .$$

As ϵ goes to zero, $\dfrac{\alpha_1^1}{\epsilon} \cdot \dfrac{m_1 - \epsilon\,\hat{p}_1}{\hat{p}_2} - \dfrac{\alpha_1^2}{W_1 - \epsilon} \cdot \dfrac{\hat{p}_2 W_2 - m_1 + \hat{p}_1}{\hat{p}_2(1-\alpha_1^2)}$

becomes very large positive. Hence by the intermediate value theorem, there exists $(\overline{x}^i)_{i \epsilon L}$ such that the system of equations have a solution. This proves the claim . Q.E.D.

<u>Theorem:</u> Let k ≥ 2. Then, there exists $(\overline{x}^i) \in P(e)$ *such that* $\hat{p} \,.\, \overline{x}^i = m_i \ \forall\ i \in L$.

<u>Proof</u>: We prove this theorem, by induction on "k". The previous claim establishes the theorem for k = 2. Hence assume the theorem is true for k = r Let k = r + 1. For each value of x_{r+1}^1 *and consequently that of* $W_{r+1} - x_{r+1}^1$, the induction hypothesis

establishes the existence of $(x_j^1)_{j=1}^{r}$:

$$\frac{\alpha^1_{r+1} \; / \; \overline{x}^1_{r+1}}{\alpha^2_{r+1} \; / \; \left(W_{r+1} - \overline{x}^1_{r+1}\right)} = \frac{\alpha^1_j / \overline{x}^1_j}{\alpha^2_j \; / \; \left(W_j - \overline{x}^1_j\right)} \quad for \; j = 1, \ldots, r. \; .$$

Thus, the theorem is true for k = r + 1 if it is true for k = r. Since it is true for k = 2, it holds for all k. Q.E.D.

Note: In the above theorem, no fixed point theorem was either required or used, since the set of budget constrained Pareto efficient equilibria for a given distribution of income is unique for an economy with Cobb-Douglas preferences, and further since the dependence of the solution on the relevant parameters of the model is continuous.

$$\frac{\alpha_j^1/x_j^1}{\alpha_j^2/(W_j-x_j^1)} = \frac{\alpha_h^1/x_h^1}{\alpha_h^1/W_h-x_h^1} \ \forall\, j,\ h \in \{1,\ldots,r\}$$

satisfying $\sum_{j=1}^{r} \hat{p}_j\, x_j^1 = m_1 - \hat{p}_{r+1}\, x_{r+1}^1$.

For $x_{r+1}^1 = \frac{m_1-\epsilon}{\hat{p}_{r+1}},\ W_{r+1} - x_{r+1}^1 = W_{r+1} - \frac{m_1-\epsilon}{\hat{p}_{r+1}} > 0$.

For $\epsilon > 0$ sufficiently small, x_r^1 is very small and $W_r - x_r^1$ very large,

$$\frac{\alpha_{r+1}^1/x_{r+1}^1}{\alpha_{r+1}^2/(W_{r+1} - x_{r+1}^1)} - \frac{\alpha_r^1/x_r^1}{\alpha_r^2/(W_r - x_r^1)} < 0\ .$$

On the other hand for

$x_{r+1}^1 = \epsilon\, /\hat{p}_{r+1}$, for $\epsilon > 0$ sufficiently small,.

$$\frac{\alpha_{r+1}^1\ /\ x_{r+1}^1}{\alpha_{r+1}^2\ /\ \left(W_{r+1} - x_{r+1}^1\right)} - \frac{\alpha_r^1/x_r^1}{\alpha_r^2\ /\ \left(W_r - x_r^1\right)} > 0\ .$$

Since $\left(x_j^1\right)_{j=1}^{r}$ depend continuously on x_{r+1}^1, by the intermediate value theorem, there exists $\bar{x}_{r+1}^1$ *in* $\left[\frac{\epsilon}{\hat{p}_{r+1}}, \frac{m_1 - \epsilon}{\hat{p}_{r+1}}\right]$ such that

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