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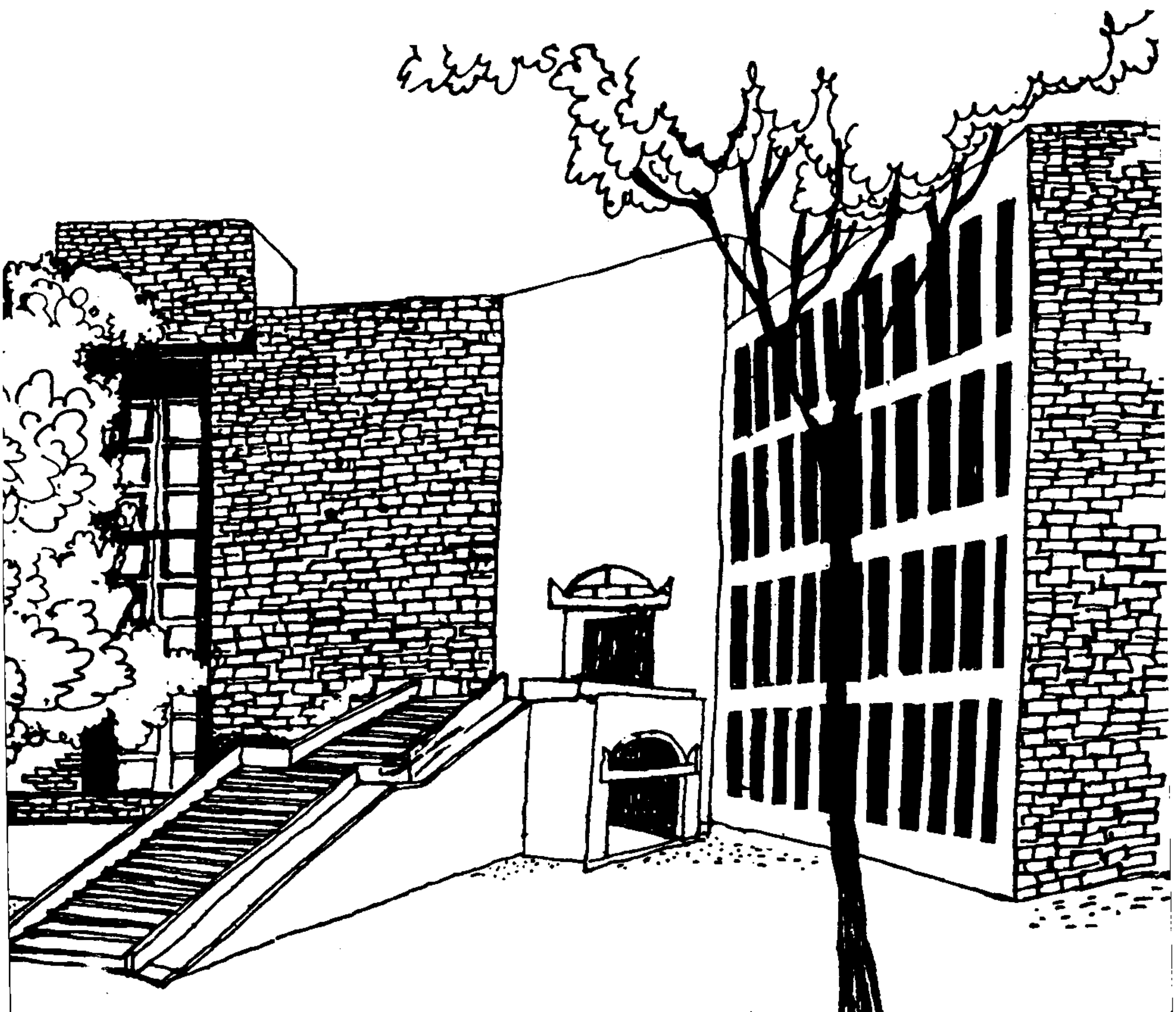

EXISTENCE OF EFFICIENT AND EGALITARIAN EFFICIENT TAX ALLOCATIONS

By

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ABSTRACT

In this paper, we study an optimal tax allocation problem where the utility of each agent depends on his own income as well as on the income of the other tax payers. The objective of the government is seen as maximizing tax incidence without subjecting the tax payers to too much hardship (sacrifice). In this framework we prove the existence of efficient tax allocations and subsequently we go on to establish the existence of egalitarian-efficient (i.e. equal sacrifice and efficient) tax allocations, under mild assumptions.

Introduction :- A classical criterion for apportioning taxes is that all should sacrifice equally in loss of utility. The study of tax-allocations which meets the criteria of distributive justice has usually been restricted to a framework where the utility of each tax-payer depends only on his/her own income. This for instance has been the framework in which Richter (1983), Young (1987, 1988) study the equal sacrifice principle. This has also been the framework in which Brunner (1989) studies the equal proportional sacrifice principle and the minimization of the sum of sacrifices principle of tax allocation.

There has been another approach to the optimal tax allocation problem, extensively surveyed in Mirrlees (1986), which also avoids the problems associated with externalities in preference structures.

In this paper, we study an optimal tax allocation problem where the utility of each agent depends on his own income as well as on the income of the other tax payers. The objective of the government is seen as maximizing tax incidence without subjecting the tax payers to too much hardship (sacrifice). In this framework we prove the existence of efficient tax allocations and subsequently we go on to establish the existence of egalitarian-efficient (i.e. equal sacrifice and efficient) tax allocations, under mild assumptions.

2. **Framework** :- There are n tax payers, $N=\{1,\dots,i,\dots,n\}$ with incomes $y_i \geqq 0$ before tax. (We could say 'index of ability to pay', instead of 'income'.) Let $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ be the i^{th} agent's utility function which for each distribution of income $Z=(Z_1,\dots,Z_n) \in \mathbb{R}^n_+$ gives the utility agent i derives. Here we allow, agent i's utility function to depend on the income of other agents as well as his own.

A tax allocation is a vector $t=(t_1,\dots,t_n) \in \mathbb{R}^n$.

A feasible tax allocation is a vector $t=(t_1,\dots,t_n)$ such that $t_i \leqq y_i$ for all $i \in N$. We thus allow for income redistribution as well.

Let $u(Z)=[u_1(Z),\dots,u_i(Z),\dots,u_n(Z)] \in \mathbb{R}^n$ denote the utility values at a particular income distribution $Z \in \mathbb{R}^n_+$.

Concerning vector comparisons we shall use the following convention : $x \geqq Z$ means $x_i \geqq Z_i \; \forall i \in N$; $x > Z$ means $x \geqq Z$ and $x \neq Z$; $x >> Z$ means $x_i > Z_i$ for all $i \in N$.

Given a utility profile $u \equiv (u_1,\dots,u_n) : \mathbb{R}^n_+ \to \mathbb{R}^n$ and $y=(y_1,\dots,y_n) \in \mathbb{R}^n_+$ the subjective sacrifice of player $i \in N$ will be a function of the tax allocation $t=(t_1,\dots,t_n)$. Define $S_i : \prod_{i=1}^{n} (-\infty, y_i] \to \mathbb{R}$ as :

$$S_i(t) = u_i(y) - u_i(y-t).$$

Consider now the following assumptions :

(1) $u_i : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous, for all $i \in N$.

(2) Let $x, Z \in \mathbb{R}^n_+$ be such that $x > Z$. If $x_i = Z_i$, we have $u_i(x) \leqq u_i(Z)$, $i = 1, 2,\dots,n$.

(3) For any $v \in \mathbb{R}^n$ there exists $t \in \prod_{i=1}^{n} (-\infty, y_i]$ such that $u_i(y-t) \geqq u_i(y) + v$.

Assumption ① is standard; assumption ② says that when an income distribution changes to a situation where some agents get more income whilst others get the same, the latter will not be happier. Assumption ③ says that any predetermined vector of utility increments can be reached via a suitable tax-subsidy scheme.

Let $v \in \mathbb{R}^n$ be a given vector of subjective sacrifice values. The problem of finding the total tax-incidence and their corresponding distribution so that those sacrifice levels are actually reached, can be formalized as the search for a solution to the following system :

$$\begin{aligned} &(i)\ \ u_i(y) - u_i(y-t) \leqq v_i \\ &(ii)\ u_i(y) - u_i(y-t) < v_i \text{ implies } t_i = y_i \\ &(iii)\ t_i \leqq y_i \quad \forall i \in N \end{aligned} \qquad (1)$$

A solution to system (1) gives us a tax-allocation such that all tax payers sacrifice at the predetermined levels, with one proviso : if some agent ends up with a lower sacrifice than his component, then his after-tax income is 0.

We shall say that a solution to system (1), $t^*=(t_1^*,\dots,t_n^*)$, is efficient if there is no $t \in \prod_{i=1}^{n}(-\infty, y_i]$ such that $\sum_{i=1}^{n} t_i > \sum_{i=1}^{n} t_i^*$ and $S_i(t) \leqq v_i \ \forall i \in N$.

3. Existence of Efficient Solutions :- The following theorem ensures the existence of efficient tax allocations.

Theorem 1:- Under assumptions ①, ② and ③, system (1) has an efficient solution, $t^* \in \prod_{i=1}^{n} (-\infty, y_i]$, for any $v \in \mathbb{R}^n$.

Proof :- Denote by $S(\nu) = \left\{t \in \prod_{i=1}^{n}(-\infty, y_i] \,/\, S(t) \leqq \nu\right\}$, and consider the following program

$$\max \sum_{i=1}^{n} t_i \qquad \text{s.t. } t \in S(\nu) \qquad (P)$$

Since $S(\nu)$ is non-empty (by assumption ③), closed and bounded from above, Weirstrass' Theorem ensures that program (P) will have a solution $t^* \in \mathbb{R}^n$. By construction this solution satisfies (i) and (iii) in (1); let us show that it verifies (ii) as well.

Suppose $S_i(t^*) < \nu_i$ and $t_i^* < y_i$ for some i; without loss of generality let this happen for i=1,2,...,h, whilst for i=h+1,...,n either $S_i(t^*) = \nu_i$ or $S_i(t^*) < \nu_i$ and $t_i^* = y_i$. Then define a vector $\bar{t} \in \prod_{i=1}^{n}(-\infty, y_i]$, as follows :

$$\bar{t} = t_i^* \text{ for } i=2,3,\ldots,n$$

$$\bar{t}_1 > t_1^*, \text{ with } S_1(\bar{t}) \leqq \nu_1$$

(we can always do this since S_i is continuous and $t_i^* < y_i$). As a result we have $\bar{t} > t^*$ and, by assumption ②,

$$S_i(\bar{t}) \equiv u_i(y) - u_i(y-\bar{t}) \leqq u_i(y) - u_i(y-t^*) \equiv S_i(t^*) \leqq \nu_i, i=2,3,\ldots,n$$

Furthermore, $S_1(\bar{t}) \leqq \nu_1$ by constuction.

Hence, $\bar{t} \in S(\nu)$. Also, $\sum_{i=1}^{n} \bar{t}_i > \sum_{i=1}^{n} t^*_i$, contradicting the claim that $t^* = (t_1^*, \ldots, t_n^*)$ solves problem (P).

Q.E.D.

The following corollary provides an interesting insight into Theorem 1:

Corollary 1 :- Under Assumptions ①, ② and ③, let $S^\circ = S(y)$ and let $\nu \in \mathbb{R}^n$ be a vector of sacrifices such that $\nu << S^\circ$. Then the equation system

$$S(t) = \nu$$

has an efficient solution $t^* \in \prod_{i=1}^{n}(-\infty, y_i]$.

Proof :- First notice that by assumption (2), for all $t \in \prod_{i=1}^{n}(-\infty, y_i]$ such that $t_i = y_i$, we have $S_i(t) \geq S_i^o$. Then $S_i(t) < S_i^o$ implies $t_i < y_i$. Theorem 1 ensures the existence of an efficient solution to system (1) for any $V \in \mathbb{R}^n$. Since $V << S^o$, we have $t_i < y_i$ for all i, and therefore $S_i(t) = V_i$ for all i.

Q.E.D.

Theorem 1 ensures the existence of efficient solutions for any predetermined vector of sacrifice levels. Corollary 1 states this result in a simpler way, provided the target sacrifice vector is below S^o.

4. The Equal Sacrifice Tax Allocation :-

Now suppose sacrifices can be compared interpersonally in ordinal terms. Consider the following definitions :

Definition 1 :- We shall say that a feasible tax allocation t*(i.e. $t^* \in \prod_{i=1}^{n}(-\infty, y_i]$), is egalitarian if, for all i, j = 1,2,...,n,

$$S_i(t^*) < S_j(t^*) \Rightarrow t_i^* = y_i.$$

Thus, we say that a feasible tax allocation is egalitarian when all tax payers shoulder the same sacrifice, or else those with lower sacrifice levels receive nothing.

Definition 2 :- We shall say that a feasible tax allocation is egalitarian-efficient if it is both egalitarian and efficient.

The following result is a consequence of the above definitions :

Theorem 2:- Let $g \leq \sum_{i=1}^{n} y_i$ be a given tax incidence. Under assumptions (1), (2) and (3) there exists an egalitarian-efficient tax allocation $t^* \in \prod_{i=1}^{n} (-\infty, y_i]$ such that $\sum_{i=1}^{n} t_i^* \geq g$.

Proof :- Let

$$T(g) = \left\{t \in \prod_{i=1}^{n}(-\infty, y_i] \;/\; \sum_{i=1}^{n} t_i \geq g\right\},$$

and

$$V(g) = \left\{\nu \in \mathbb{R}^n / \nu = S(t),\ t \in T(g)\right\}.$$

Consider a function $F : V(g) \to \mathbb{R}$, defined by $f(\nu) = \max_i \nu_i$.

Since $T(g)$ is a compact set and $S : \prod_{i=1}^{n} (-\infty, y_i] \to \mathbb{R}^n$ is a continuous function, $V(g)$ will also be a compact set. Let ν^* be the minimizer of f over $V(g)$ (which exists since f is continuous) ;

$$\therefore\ f(\nu^*) = \max_i \nu_i^* \leq f(\nu) = \max_i \nu_i \quad \forall \nu \in V(g).$$

$$\text{Let } \bar{\nu} = (f(\nu^*), \dots, f(\nu^*)) \in \mathbb{R}^n.$$

Then applying Theorem 1 to $\bar{\nu}$ we get the desired result.

Q.E.D.

Note :- $\bar{\nu}$ may not be in $V(g)$. However some components of $\bar{\nu}$ will agree with some components of ν^* and $\nu^* \in V(g)$. For coordinates of $\bar{\nu}$ which do not agree with those of ν^* i.e. $\forall j \in \{1, \dots, n\}$ such that $\bar{\nu}_j > \nu_j^*$, we set $t_j^* = y_j$ and the solution will belong to $T(g)$ anyway.

Remark :- In the case $\bar{\nu} \ll S^0$ (as defined in Corollary 1) we shall have $S_i(t^*) = S_j(t^*)$ for all $i, j = 1, 2, \dots, n$, according to Corollary 1.

We may view egalitarian-efficient solutions as the result of minimizing a particular function of individual sacrifices, i.e. the minmax rule. This equivalence between egalitarian-efficient tax allocations and the minimizer of a minimax function of sacrifices is an immediate Corollary of Theorem 2.

Corollary 2 :- Let $g \leq \sum_{i=1}^{n} y_i$ be a given tax incidence, and suppose assumptions ①, ② and ③ hold. Then the program

$$\min f\left[S(t)\right]$$

$$\text{s.t. } t \in \prod_{i=1}^{n} (-\infty, y_i]$$

$$\sum_{i=1}^{n} t_i \geq g$$

has a solution and this solution is egalitarian-efficient, when f is the minmax rule.

<u>Proof</u> :- Immediate.

Thus egalitarian-efficient solutions may be thought of as the result of minimizing a particular function of the individual sacrifices.

5. <u>Conclusion</u> :- In this paper we have analyzed the existence of efficient tax allocation on n tax payers, when each agent's utility of income depenos on the income of other agents (in a sense that assumption (2) makes precise). We have shown that for any predetermined vector of sacrifice values we can find a feasible tax allocation, which is efficient and no one sacrifices more than the specified levels. Then we have established that in the case where sacrifices are interpersonally comparable, there exists an efficient tax allocation equalizing all agents' sacrifice levels, for any given tax incidence.

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