A Multi-tier Linking Approach to Analyze Performance of Vehicle-based Warehouse Systems

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Abstract: To improve operational flexibility, throughput capacity, and responsiveness in order fulfillment operations, several distribution centers are implementing autonomous vehicle-based storage and retrieval (AVS/R) system in their high-density storage areas. In such systems, vehicles are self-powered to travel in horizontal directions (x- and y- axes), and use lifts or conveyors for vertical motion (z-axis). In this research, we propose a multi-tier queuing modeling framework for the performance analysis of such vehicle-based warehouse systems. We develop an embedded Markov chain based analysis approach to estimate the first and second moment of inter-departure times from the load-dependent station within a

semi-open queuing network. The linking solution approach uses traffic process approximations to analyze the performance of sub-models corresponding to individual tiers (semi-open queues) and the vertical transfer units (open queues). These sub-models are linked to form an integrated queuing network model, which is solved using an iterative algorithm. Performance estimates such as expected transaction cycle times and resource (vehicle and vertical transfer unit) utilization are determined using this algorithm, and can be used to evaluate a variety of design configurations during the conceptualization phase.

Keywords: vehicle-based warehouse systems, integrated queuing model, linking algorithm, embedded Markov chains, semi-open queues

1 Introduction

Autonomous vehicle-based technologies were introduced during the late 1990s to improve the flexibility and responsiveness in handling unit-loads within a warehouse. Savoye Logistics, a France-based equipment manufacturer, pioneered the development of the Autonomous Vehicle-based Storage and retrieval system (AVS/RS) (see www.sayove.com). The main components of an AVS/RS are autonomous vehicles, lifts, and a system of rails in the rack area. Autonomous vehicles provide horizontal movement (x-axis and y-axis) within a tier using rails, and lifts provide vertical movement (z-axis) between tiers. Several variants of AVS/RS have been introduced by Vanderlande Industries and Nedcon, and are practiced to handle both unit-load pallets as well as totes.

Although autonomous vehicle-based warehouse automation systems offer substantial throughput flexibility, they also involve additional operational complexities due to blocking and bottlenecks among the horizontal and vertical load transfer mechanisms. The objective

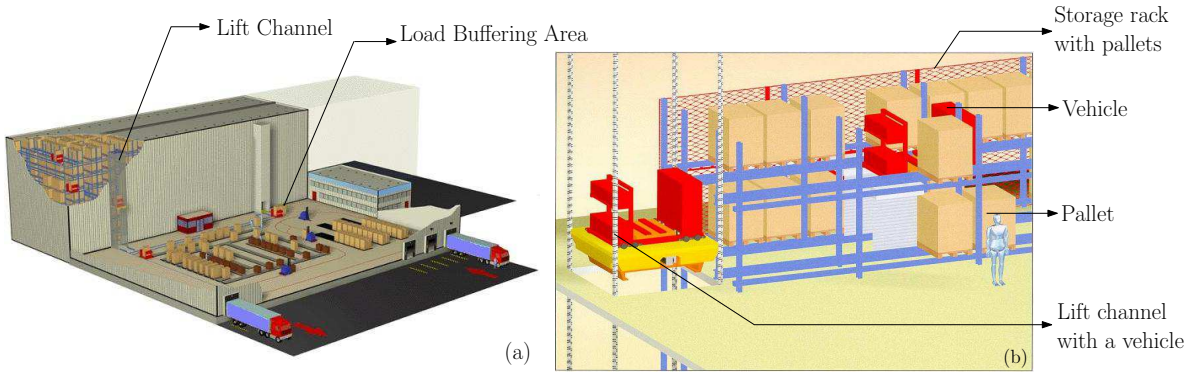

Figure 1: Illustration of: (a) a warehouse using AVS/RS - system view, and (b) vehicle-lift interface in AVS/RS (source: Savoye Logistics)

of this paper is to provide a modeling framework and solution methodology to evaluate the performance of AVS/R with alternate vertical transfer mechanisms. We describe this methodology and demonstrate its application by analyzing the design tradeoffs in tier-captive warehouse systems. However, the models can be used to analyze the performance of other variants of autonomous vehicle-based storage and retrieval systems.

In the literature, a few studies on AVS/RS use semi-open queuing network (SOQN) models to analyze the system performance (see Malmborg [2003], Roy et al. [2012], Roy et al. [2014], Ekren et al. [2013]). The existing models either analyze the performance of a single tier (with and without blocking) or they analyze the performance of multi-tier systems with pooled vehicles (by ignoring the blocking effects). A few studies are also available on the performance analysis of AVS/RS with tier-captive vehicles. Amongst them, Heragu et al. [2011] proposed an open queuing network (OQN) model where both lifts and tiers are modeled as shared FCFS servers. The OQN is solved using a parametric decomposition approach. Current literature has two main limitations. First, it does not provide the distribution of vehicles in the aisles and cross-aisle of the tiers. Distribution of vehicles in

a tier is of significant interest to design engineers because they provide information on the congestion effects at aisles, cross-aisles, and LU points. Second, they do not capture the vehicle interference in the cross-aisles and aisles, which results in additional delays. The SOQN model realistically captures the synchronization between transaction and vehicles because within a tier either a transaction could wait for a vehicle or a vehicle could wait for a transaction arrival. However, there are other challenges related to the analysis of these SOQNs. First, SOQNs do not have a product-form solution, which makes the analysis harder. Second, as the number of stations in the network grows, the analysis of SOQNs using Markov chains becomes infeasible because of the curse of state space dimensionality. Third, the resource travel times follow a general service time distribution, which makes the analysis more complex. Further, there are specific service protocols for the use of resources (vehicles, vertical transfer units) during service, which need to be analyzed carefully. The blocking delays introduced due to sharing of resources such as aisles and cross-aisles should also be captured in the analysis.

We propose a decomposition-based analysis approach, which addresses the above challenges. The individual tiers are modeled using a semi-open queuing network (SOQN) and the vertical transfer subsystem is modeled using an open queuing network (OQN). These subsystems are combined into an integrated queuing network model, which is composed of multiple SOQN models denoting the tiers and an OQN model denoting the conveyor. This network is complex to solve in its original form. This results in an integrated queuing network model consisting of multiple inter-connected SOQNs. In the integrated queuing network model, each single tier is replaced by an equivalent load-dependent station and the individual tiers and the vertical transfer unit are linked using an algorithm based on an embedded Markov chain analysis. Modeling each tier dynamics with a load-dependent station greatly reduces the number of components in the SOQN state-space description and

the number of states for describing each SOQN. The vehicle routing within a tier captures the service protocols and the blocking delays are measured using queues in the model for each tier. We conduct our analysis with the first two-moments of the relevant distributions, which keeps our analysis relatively simple; this seems sufficient for estimating the performance measures. This solution approach is validated against detailed simulations using practical data and also used to test the performance of alternate vertical transfer mechanisms and investigate its effect on system throughput capacity. Existing SOQN solution methods cannot efficiently solve multiple SOQNs that are interconnected with each other (Avi-Itzhak and Heyman [1973], Dallery [1990], Buitenhek et al. [2000], Jia and Heragu [2009]). Our approach provides a solution framework that addresses all of these challenges.

The rest of this paper is organized as follows. Section 2 describes the system operations and explains the system modeling approach. The queuing network model for horizontal movement within a single tier is described in Section 3 whereas the departure process analysis for a tier is discussed in Section 4. The queuing network model for the vertical transfer mechanism is illustrated in Section 5. In Section 6, the integrated queuing network model, which links the queuing models for tiers with the vertical transfer unit, is discussed and the approach to estimate the performance measures of the vertical transfer subsystem is presented in Section 7. Numerical results are presented in Section 8 and the conclusions of this study are discussed in Section 9.

2 System Description and Modeling Approach

We first describe two variations of AVS/RS and then present a *common* modeling approach for analyzing system performance. The first variation is a conveyor-based AVS/RS composed of a set of tiers and one vertical conveyor system that transfers pallets between the

tiers (Figure 2a). The second variation is an AVS/RS with a lift mechanism (Figure 2b), where a single lift is used to transfer pallets in the vertical direction. These two variations have been chosen for illustrative purposes, and the modeling approach can be applied to other variations of AVS/RS easily.

In either system, a tier of a storage area is composed of a set of aisles with storage racks on both the sides of each aisle. A cross-aisle is located at the end of the tier and it runs orthogonal to the aisles, and vehicles travel between aisles using the cross-aisle. A system of rails guides the rectilinear movement of vehicles along the x and y dimensions. The Load/Unload (LU) point is located at the middle of the cross-aisle on each tier. In other words, the LU point divides the cross-aisle into two equal segments (CA_R and CA_L: corresponding to the right and left segment of the cross-aisle). In the conveyor-based system, the conveyor is located along the LU points of each tier, and is composed of multiple bi-directional conveyor loops where each loop transfers pallets between consecutive tiers. Note that unlike the lift-based systems, conveyors enable multiple pallets to be transferred simultaneously.

2.1 Storage and Retrieval Operations

To retrieve a pallet in a conveyor-based system, the vehicle in tier $i+1$ retrieves the pallet and deposits the pallet at the tier $i+1$'s LU point. To move the pallet from tier $i+1$ to tier i, the conveyor loop i picks up the pallet from the LU point of tier $i+1$ and moves it to the LU point of tier i. From the LU point, the conveyor loop $i-1$ picks the pallet and transfers it to the successive loop. The conveyor transfer process is complete when the pallet reaches the LU point of tier 1. Storage operations can be described in a similar manner. The guide path of a conveyor loop is bi-directional, that is, the conveyor loop switches its direction of travel when the type of transaction changes. For instance, if the loop rotates in a clock-wise

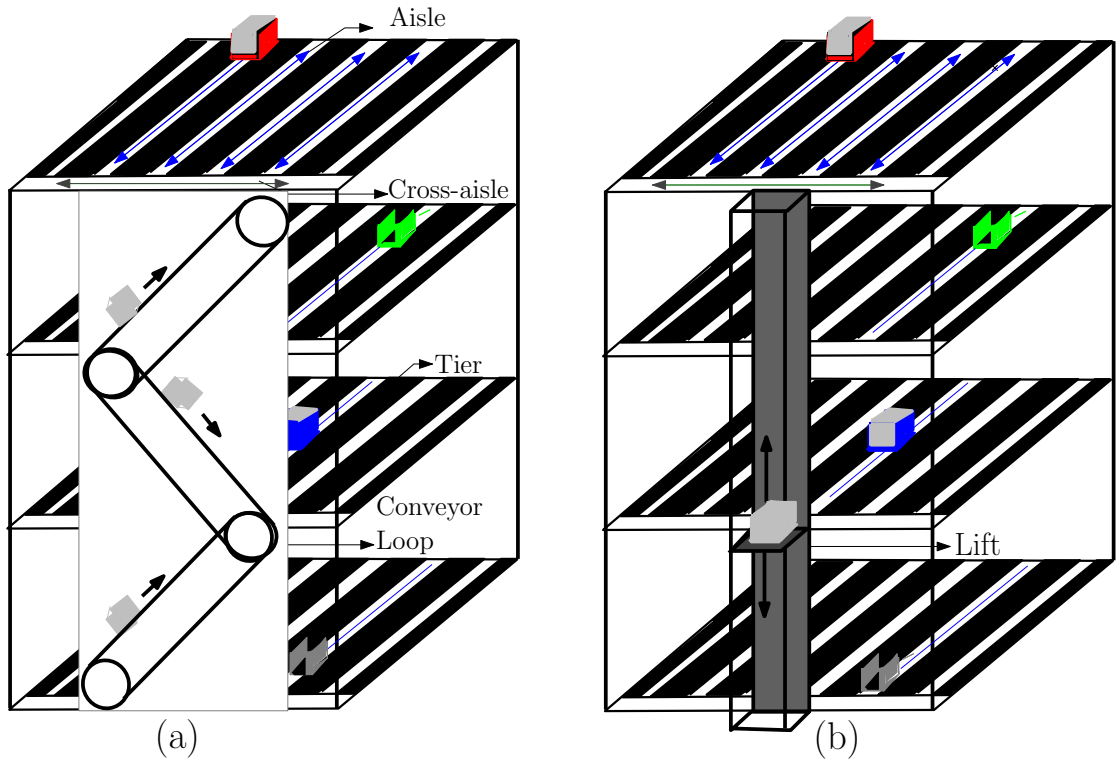

Figure 2: AVS/RS with tier-captive vehicles and (a) conveyor mechanism and (b) lift mechanism

motion to move a pallet up, then the loop rotates in a counter-clockwise motion to move a pallet down. At any point in time, the conveyor is either idle, or moving a pallet up or moving a pallet down.

To retrieve a pallet in a lift-based system, the vehicle in tier i retrieves the pallet from the storage location and deposits it at the LU point of tier i. The lift travels from its dwell point and picks up the pallet from the LU point of tier i. After loading the pallet, the lift travels to the LU point of tier 1 and unloads the pallet. Similarly, to store a pallet, the lift travels from its dwell point to pick up the pallet from the LU point of tier 1. The lift then travels to the LU point of the storage tier and unloads the pallet.

2.2 Modeling Approach

The transaction cycle time and throughput of AVS/RS depend on several factors including vehicle utilization, conveyor (lift) utilization, and tier configuration parameters. Therefore, a queuing network model is needed to model the system dynamics and estimate performance measures. An integrated queuing network model is proposed here for an AVS/R system with T tiers. It is composed of: 1) a conveyor (lift) subsystem supporting vertical movement and 2) T single-tier subsystems supporting horizontal movement (Figure 3a). Note that the departures of storage transactions from the conveyor (lift) subsystem form the arrivals of storage transactions to the tier subsystems. Similarly, the departures of retrieval transactions from the tier subsystems form the arrivals of retrieval transactions to the conveyor/ lift subsystems. Hence, we adopt a decomposition-based modeling approach that recognizes these relationships between the subsystems. The steps of the analysis approach are as follows.

1. First, queuing models for individual tiers are analyzed in isolation. This analysis provides, among other measures, parameters that characterize the departure process (in terms of the moments of inter-departure times) from each tier (see Sections 3 and 4 for details).

2. Then, the queuing model for the vertical transfer mechanism (lift or conveyor subsystem) is analyzed in isolation. This analysis provides parameters that characterize the departure process (in terms of the moments of inter-departure times) from all conveyor loops (see Section 5 for details).

3. Subsequently, the departures and arrivals to different subsystems are linked together through a linking algorithm (see Section 6 for details).

4. After linking all subsystems, the performance measures for individual tiers (average queue length measures, resource utilization, and throughput times) and vertical transfer mechanism (average queue length measures, resource utilization, and throughput times) are estimated (see Section 7 for details).

We discuss next the model assumptions for the tier and the vertical transfer subsystems before providing details of each step in our modeling approach.

2.3 Modeling Assumptions

The main assumptions for the analysis of single-tier subsystems are as follows. Within a tier, the vehicle dwells at the LU point after processing a transaction. This implies that a vehicle that completes a retrieval transaction dwells at the LU point. After a vehicle completes a storage transaction, it travels to the LU point to serve the next transaction. The system operates under single-command cycle only, that is, vehicles either process a storage transaction or a retrieval transaction in one cycle. All vehicles are pooled within a tier, that is, any free vehicle can process any type of transaction. Without loss of generality, the number of aisles in the tier is assumed to be even. The storage and retrieval transaction arrival rates for a system with T tiers are Poisson with rates $\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_T}$ and $\lambda_{r_1}, \lambda_{r_2}, \ldots, \lambda_{r_T}$ respectively. Without loss of generality, it is assumed that λ_{s_i} equals to λ_{r_i} for each tier i. For simplicity of exposition, all tier subsystems are assumed to have V dedicated vehicles. The LU points in all tiers have sufficient buffer space to load/unload the pallets.

The main assumptions for the analysis of the vertical transfer mechanism (conveyor/ lift subsystem) are as follows. Each conveyor loop/ lift transfers at most one pallet at any time. The pallets for storage and retrieval are transferred by each conveyor loop/ lift in an

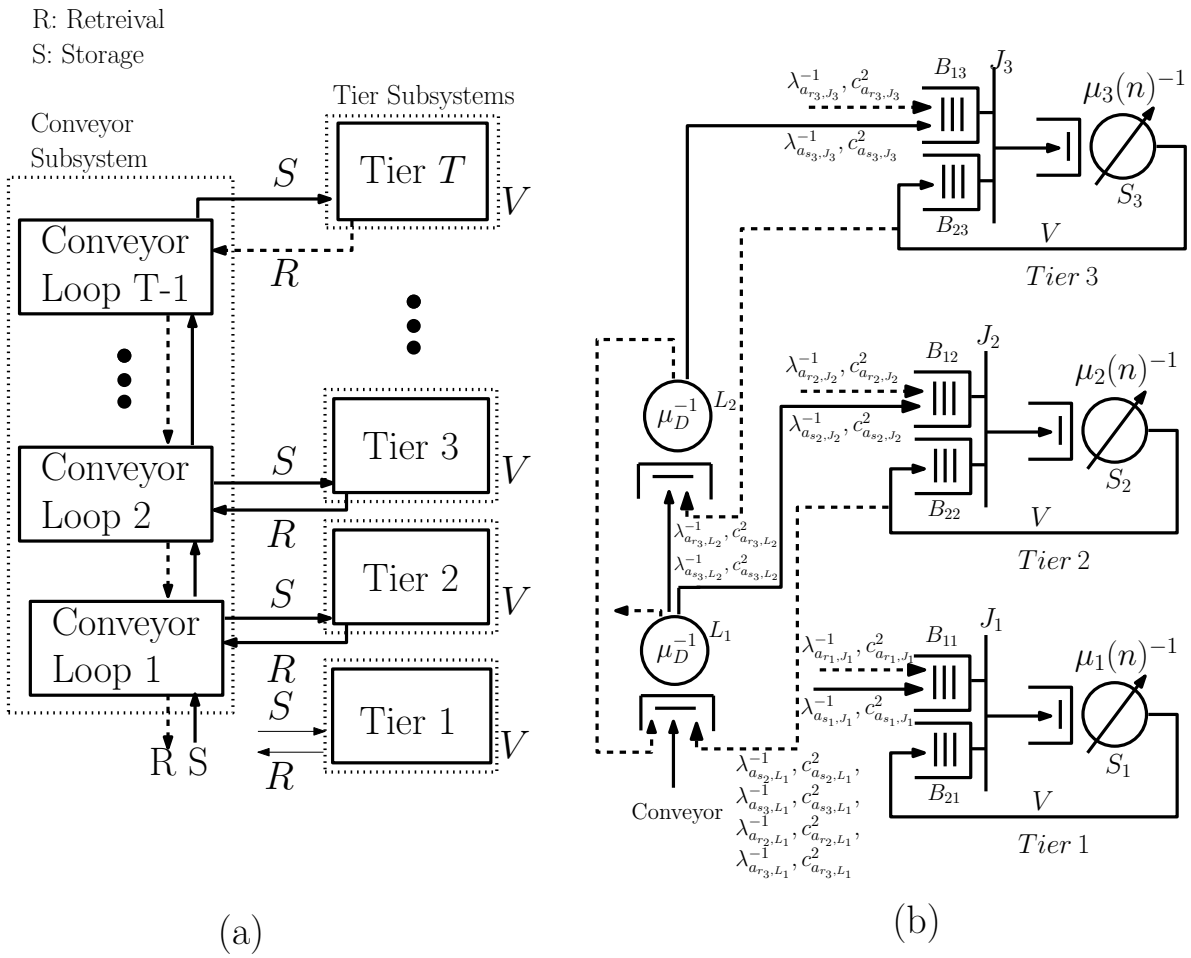

Figure 3: Analysis approach of the multi-tier system with conveyors: (a) block representation and (b) queuing network

FCFS fashion.

Note that the assumptions can be relaxed, and the proposed approach can still be used albeit with additional model complexity. Some instances of systems with different assumptions and their analysis have been reported in Roy et al. [2012], Roy et al. [2014], and Roy et al. [2015]. The queuing network model for horizontal movement in a tier is discussed in the next section.

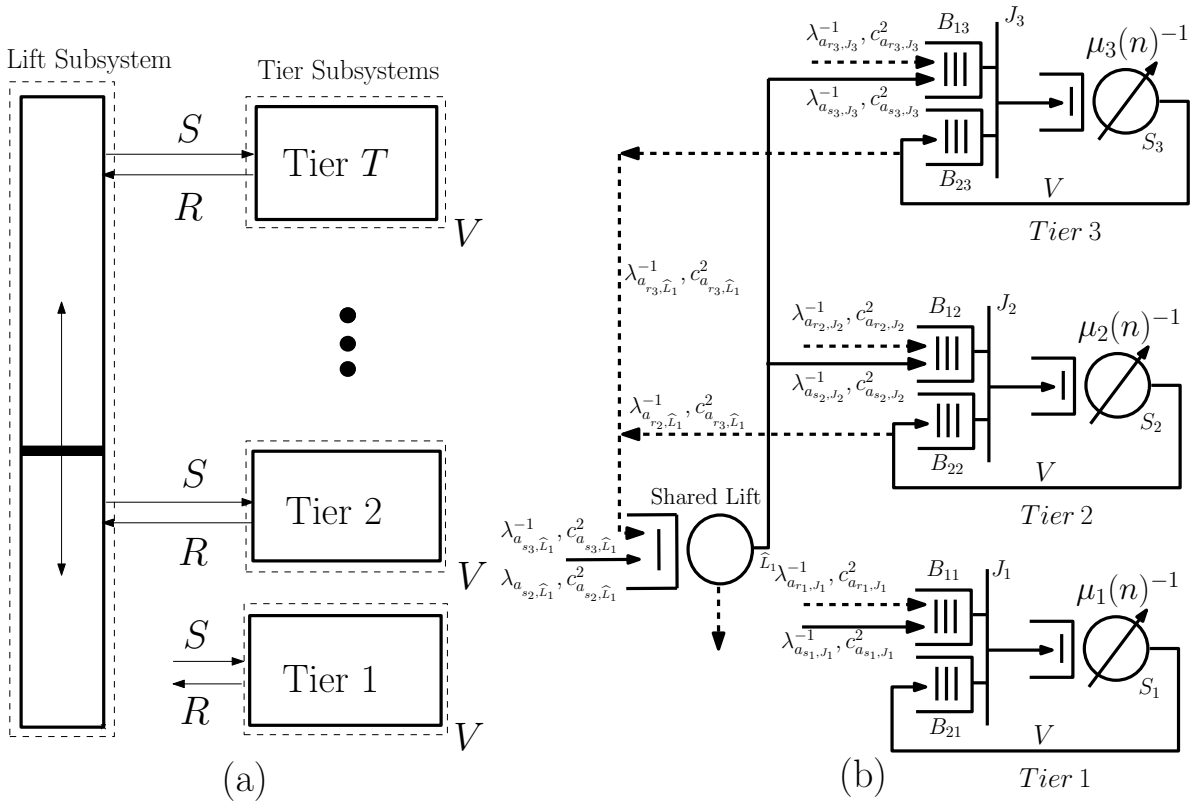

Figure 4: Analysis approach of the multi-tier system with lift: (a) block representation and (b) queuing network

3 Queuing Network for Horizontal Movement in a Tier

The process of either storing (retrieving) a pallet at (from) a location involves the horizontal movement of a vehicle within the aisles and cross-aisles of a tier in addition to vertical travel using lifts or conveyors. Hence a key subsystem of the integrated queuing network model for AVS/RS is the model of a single tier. This single tier model must capture the movement dynamics within a single tier as well as the departure process from the single tier as they form inputs to the subsystem modeling the vertical transfer. The SOQN model for the single tier is described in Figure 3. In the SOQN model of a tier i, there are V vehicles

processing transactions. These vehicles belong to two classes, storage class and retrieval class, denoted by s_i and r_i respectively.

A key input required in this analysis is the arrival process to the SOQN. Note that the mean and the squared coefficient of variation (SCV) of the inter-arrival times for retrieval transactions (denoted by $\lambda^{-1}_{a_{r_i,J_i}}$ and $c^2_{a_{r_i,J_i}}$, where r_i is the retrieval class index and J_i is the synchronization station index in tier i) are known inputs. Since the pallets to be retrieved are directly sent to the LU point of the retrieval tier in the tier subsystem, the inter-arrival times for class i retrieval transaction to the tier i are assumed to be exponential with mean $\lambda^{-1}_{a_{r_i,J_i}}$, $c^2_{a_{r_i,J_i}} = 1$, where $i = \{1, \ldots, T\}$. For the storage transactions to tier i, where $i > 1$, let $\lambda^{-1}_{a_{s_i,J_i}}$ and $c^2_{a_{s_i,J_i}}$ denote the mean and the SCV of the inter-arrival times. (Note that tier 1 is located at the ground level and does not need vertical transfer. Therefore, the inter-arrival times of storage transactions are exponential at tier 1, $c^2_{a_{s_1,J_1}} = 1$.)

To simplify the analysis for a single tier i, the external arrival streams corresponding to the storage and retrieval transaction classes are aggregated into a single transaction class (A_i). (Note that we later use a disaggregation technique to estimate the performance measure for each transaction class.) Aggregation implies that the mean of the inter-arrival time for the aggregated class ($\lambda^{-1}_{a_{A_i,J_i}}$) is given by Equation 1.

$$\lambda^{-1}_{a_{A_i,J_i}} = (\lambda_{a_{s_i,J_i}} + \lambda_{a_{r_i,J_i}})^{-1} \tag{1}$$

Note that the inter-arrival time distribution of the retrieval class is exponential whereas the inter-arrival time distribution of the storage class is not exponential. Therefore, the aggregated SCV of the transaction inter-arrival times ($c^2_{a_{A_i,J_i}}$) to the buffer B_1 of the synchronization station J in tier i is determined using Equation 2. Note that ($c^2_{a_{r_i,J_i}}$)=1. The

SCV of arrivals of class A_i, $c^2_{a_{A_i,J_i}}$, is given by Equation 2 and a convex combination of the inter-arrival time SCV for the storage and retrieval transactions to a tier i (Whitt [1983]).

$$c^2_{a_{A_i,J_i}} = \frac{\lambda_{a_{s_i,J_i}}}{\lambda_{a_{s_i,J_i}} + \lambda_{a_{r_i,J_i}}}(c^2_{a_{s_i,J_i}}) + \frac{\lambda_{a_{r_i,J_i}}}{\lambda_{a_{s_i,J_i}} + \lambda_{a_{r_i,J_i}}}(c^2_{a_{r_i,J_i}}) \tag{2}$$

The notations used in the queuing analysis of the horizontal movement within a tier are described in Table 1. We use the model from Roy et al. [2014] to estimate the throughput of storage and retrieval transactions from a single tier with V vehicles. Subsequently, in the integrated queuing network model of the whole system, the subnetwork corresponding to tier i (which consists of N aisles, cross-aisle (left and right), and an LU point station) is replaced with an equivalent single load-dependent station S_i (see Figure 3b). The service rate of the load-dependent station is assumed to be exponentially distributed with mean $\mu_i(n)^{-1}$, where $\mu_i(n)$ is the throughput of a closed queuing network with n vehicles, for $n = \{0, \ldots, V\}$.

4 Departure Process Analysis from a Single Tier

The objective of the departure process analysis is to determine the moments (in particular the mean and SCV) of the inter-departure times from the tier i for each class of transactions (s_i and r_i). The parameters describing the departure process from each tier and the performance measures are estimated using a three-step approach: 1) fit a two-phase Coxian distribution to the interarrival times, 2) define the embedded Markov chain and form the transition matrix, P_D and 3) analyze the inter-departure times from the load-dependent station using an embedded Markov chain analysis. The departure process from the load-dependent station S_i is studied as a Markov renewal process to determine the mean ($\lambda^{-1}_{d_{A_i,S_i}}$) and SCV ($c^2_{d_{A_i,S_i}}$) of the inter-departure times from tier i. The details of the approach are

Table 1: Notations used in the analysis of horizontal movement within a tier

Notation	Description
T	Number of tiers
V	Number of vehicles/tier
S_i	Load-dependent station of tier i
J_i	Synchronization station of tier i
B_{1i}, B_{2i}	Virtual buffers in tier i for waiting transactions and vehicles respectively
$\mu_i(n)^{-1}$	Mean service time of S_i with n vehicles
A_i	Aggregated transaction class in tier i
$\lambda_s^{-1}, \lambda_r^{-1}$	Mean inter-arrival times for all storage and retrieval transaction classes
$\lambda_{s_i}^{-1}, \lambda_{r_i}^{-1}$	Mean inter-arrival times for storage and retrieval transaction classes with destination tier i
$\lambda_{a_{s_i,J_i}}^{-1}, c_{a_{s_i,J_i}}^2$	Mean and SCV of the inter-arrival time for storage transaction class to J_i
$\lambda_{a_{r_i,J_i}}^{-1}, c_{a_{r_i,J_i}}^2$	Mean and SCV of the inter-arrival time for retrieval transaction class to J_i
$\lambda_{a_{A_i,J_i}}^{-1}, c_{a_{A_i,J_i}}^2$	Mean and SCV of the inter-arrival time for the aggregated transaction class to J_i
$\lambda_{d_{A_i,S_i}}^{-1}, c_{d_{A_i,S_i}}^2$	Mean and SCV of the inter-departure time for the aggregated transaction class from S_i
S_D	State space for the embedded Markov chain
P_D	Transition probability matrix for the embedded Markov chain
Π_D	Steady state probability distribution for the embedded Markov chain

discussed in the following paragraphs.

4.1 Step 1: Fit a 2-phase Coxian Distribution

Each tier is analyzed assuming that the mean and SCV of inter-arrival times for storage and retrieval transactions are known. Using this information, a 2-phase Coxian distribution is fit to model the inter-arrival times to the tier. Let λ_{1_i} and λ_{2_i} denote the two phases of the Coxian distribution and p_i denote the probability with which the transaction proceeds to the second arrival phase after completing the first phase of arrival. Note that we assume a balanced 2-phase Coxian distribution to determine λ_{1_i}, λ_{2_i}, and p_i that satisfy the mean and SCV of the inter-arrival times, $\lambda_{a_{A_i,J_i}}^{-1}$ and $c_{a_{A_i,J_i}}^2$ (see Bolch et al. [2006]).

4.2 Step 2: Develop the Transition Probability Matrix (P_D)

The departure process from the load-dependent station (S_i), corresponding to tier i (see Figure 3b), is studied as a Markov renewal process and the mean and SCV of the transaction inter-departure times from a tier i ($\lambda^{-1}_{d_{A_i,S_i}}$ and $c^2_{d_{A_i,S_i}}$) are obtained by analyzing the Markov chain embedded at departure instants from S_i. First, the transition probability matrix (P_D) is developed and the steady state stationary probability vector (Π_D) is obtained. Using Π_D, the mean and SCV of the inter-departure times from S_i are obtained.

The state of the embedded Markov chain (X_k) has two tuples (i_1, i_2). The component i_1 corresponds to the difference between the number of transactions waiting in buffer B_{1i} and the number of idle vehicles waiting in buffer B_{2i} whereas the component i_2 corresponds to the phase of the 2-phase Coxian distribution of the pending arrival. Since the buffer size for transactions at buffer B_{1i} is K, at the departure instant, component i_1 takes a value from the set $\{-V, \ldots, -1, 0, 1, \ldots, K-1\}$ and component i_2 takes a value from the set $\{1, 2\}$. Therefore, the cardinality of the statespace, S_D, is $2(K+V)$.

Since the arrivals to the buffer B_{1i} are composed of exponential phases of a Cox-2 distribution and the load-dependent service times are exponentially distributed, the transition matrix P_D has a special structure. The non-zero portion of P_D has four main regions and the entries in P_D are denoted by $P(X_i, X_j)$ where X_i, X_j are the states observed at two consecutive departure time instants. The components of X_i and X_j are denoted by $(i_1,\ i_2)$ and $(j_1,\ j_2)$ respectively. For a semi-open queuing network with $V = 2$ and $K = 3$, the states and the regions are described in Table 2. Next we provide an example to illustrate how each $P(X_1, X_2)$ is determined. The detailed expressions to estimate $P(X_i, X_j)$ are obtained by considering subregions within these four main regions and are listed in Appendix A.

Table 2: Different regions in the P_D matrix

$X_i(i_1,i_2)$,$X_j(j_1,j_2)$	$-2,1$	$-2,2$	$-1,1$	$-1,2$	$0,1$	$0,2$	$1,1$	$1,2$	$2,1$	$2,2$
$-2,2$	2	2	2	2	2	2	2	2	2	4
$-2,1$	2	2	2	2	2	2	2	2	2	4
$-1,1$	2	2	2	2	2	2	2	2	2	4
$-1,2$	-	3	3	3	3	3	3	3	3	4
$0,1$	-	-	1	1	1	1	1	1	1	4
$0,2$	-	-	-	1	1	1	1	1	1	4
$1,1$	-	-	-	-	1	1	1	1	1	4
$1,2$	-	-	-	-	-	1	1	1	1	4
$2,1$	-	-	-	-	-	-	1	1	1	4
$2,2$	-	-	-	-	-	-	-	1	1	4

Consider the case when $X_i = (0,2)$ and $X_j = (1,1)$ (see region 1 in Table 2). Since $j_1 - i_1 = 1$, two arrivals occur prior to a departure. Further, $i_1 = 0$ implies that in state X_i, other vehicles ($V = 2$) are busy processing transactions. Since the arrival is in phase 2 of the arrival process ($i_2 = 2$), the probability that the arrival occurs prior to the service completion is $\frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(2)}$. The probability that the second arrival also occurs prior to the service completion is given by $\left[\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(2)}\right]\left[p_i\left(\frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(2)}\right)+(1-p_i)\right]$. Finally, the probability that the service is complete prior to a third arrival is given by $\frac{\mu_i(2)}{\lambda_{1_i}+\mu_i(2)}$. Therefore, $P(X_i,X_j)$ for $X_i = (0,2)$ and $X_j = (1,1)$ is given by Equation 3. Using similar logic, we derive all the other expressions (see Appendix A).

$$\begin{aligned} P(X_i,X_j) &= \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(2)}\left[\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(2)}\right]\left[p_i\left(\frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(2)}\right)+(1-p_i)\right]\frac{\mu_i(2)}{\lambda_{1_i}+\mu_i(2)} \\ &= \frac{\mu_i(2)\lambda_{1_i}\lambda_{2_i}\left[\lambda_{2_i}+\mu_i(2)(1-p_i)\right]}{(\lambda_{1_i}+\mu_i(2))^2(\lambda_{2_i}+\mu_i(2))^2} \end{aligned} \tag{3}$$

Let $\Pi_D = \{\Pi_D(X_k) : X_k \in S_D\}$, where $\Pi_D(X_k)$ is the steady state probability that the load-dependent station is in state X_k at a departure instant. Using P_D, the stationary

probability vector Π_D of the underlying Markov chain is obtained by solving the system of linear Equations 4 and 5.

$$\Pi_D P_D = \Pi_D \quad (4)$$

$$\sum_{k \in S_D} \Pi_D(X_k) = 1 \quad (5)$$

After deriving the steady state probability distribution, Π_D, the first two moments of the inter-departure times ($E[D_i]$ and $E[D_i^2]$) are estimated using an approach presented in the next section.

4.3 Step 3: Estimate Parameters of the Inter-departure Time Distribution

After estimating the steady state probability vector Π_D, the first and second moment of the inter-departure time, D_i, from the load-dependent station S_i are determined. Note that at the departure instant, the transaction leaves the system in one of the $2(K+V)$ states in S_D. Note that the time to the subsequent departure from the load-dependent queue would depend on the state, X_i, at the instant of a departure. Correspondingly, we partition S_D into five sets, G_1, G_2, G_3, G_4, and G_5. The description of these sets is given below.

1. $G_1 = \{(-V, 1)\}$: In this state, there are no vehicles processing transactions in the load-dependent station S_i. Therefore, the next departure occurs when a transaction arrives and completes its service.

2. $G_2 = \{(-V, 2)\}$: In this state, there are no vehicles in the load-dependent station S_i. However, a transaction has completed phase 1 of its arrival process. Therefore, the next departure occurs when phase 2 of the arrival process completes followed by

completion of the service of this transaction.

3. $G_3 = \{(-V+1,1),(-V+2,1),\ldots,(-1,1)\}$: In these states, there are one or more vehicles at the load-dependent station S_i and the arriving transaction is in phase 1. Therefore, the next departure occurs when the transaction at station S_i completes its service.

4. $G_4 = \{(-V+1,2),(-V+2,2),\ldots,(-1,2)\}$: In these states, there are one or more vehicles at the load-dependent station S_i and the arriving transaction is in phase 2. Therefore, the next departure occurs when the transaction at station S_i completes its service.

5. $G_5 = \{(0,1),(0,2),\ldots,(K-1,1),(K-1,2)\}$: In these states, all vehicles are present at the load-dependent station S_i and the arriving customer is either in phase 1 or phase 2. Therefore, the next departure occurs when the transaction at station S_i completes its service.

We next describe the procedure used to determine the parameters of the inter-departure time using states in G_1 as an example.

Departure Analysis for States in G_1: If a departure leaves the system in state $s = (-V,1)$, the following events need to occur for the subsequent departure. First, a transaction should arrive and then its service needs to be completed. Let the notations A and $S^{'}$ denote the events corresponding to an arrival and service completion respectively. Note that the inter-arrival time follows a Cox-2 distribution with rates λ_{1_i} and λ_{2_i} corresponding to phase 1 and 2 respectively. The service completion time, however, could vary depending on the number of vehicles present at station S_i. The service time at S_i follows a load-dependent exponential service time with mean $\mu_i(n)^{-1}$ when there are n vehicles in tier i. We denote

$\mathbb{S}_v^1$ as a sequence with v arrivals followed by a service completion, i.e., $\mathbb{S}_v^1 = (A, \ldots, A, S')$ where $v = 1, \ldots, V$. One of the following sequence of events ($\mathbb{S}_v^1$) needs to occur before the next departure. The first part of the service is completed at rate $\mu_i(1)$, the second part of the service is completed at rate $\mu_i(2)$. Likewise, the $(v-1)^{th}$ part of the service time is completed at rate $\mu_i(v-1)$, and the residual service time is completed at a rate $\mu_i(v)$. Note that the estimation of the first and second moment of the inter-departure time corresponding to each sequence of event, e, $E[D_i|\mathbb{S}_e^1], E[D_i^2|\mathbb{S}_e^1]$, involves determining the distribution of the residual service time after the last arrival. Determining these residual service times requires conditioning on the exact times of each of the previous arrivals, which can get very cumbersome. Hence, we develop an approximation for the first and the second moments of the inter-departure times.

Note that $\mu_i(n)$ is the throughput of the closed queuing network with n resources. As the number of resources increases, the throughput increases monotonically, that is, $\mu_i(n) > \mu_i(n-1) > \ldots > \mu_i(1)$. If there are n vehicles present at the load-dependent queue before the departure instant, then using a $\mu_i(n)$ service rate would give a lower bound estimate on the expected inter-departure times whereas using a service rate corresponding to the number of vehicles present at S_i at the inception of the service would give an upper bound estimate of the first and second moment of the inter-departure times. Since performance measurement under high vehicle utilization is more practical, we use lower bound estimates for the two moments ($E[D_i|\mathbb{S}_e^1]_l, E[D_i^2|\mathbb{S}_e^1]_l$) as our approximation. At high vehicle utilization, all vehicles will be present more often at the load-dependent station.

To compute the probability associated with each sequence $\mathbb{S}_e^1$, we need to estimate the probability q_n of an arrival prior to service completion at S_i with n customers operating at rate $\mu_i(n)$. Let the random variables, Y and Z_n, denote the Cox-2 inter-arrival times at

station J_i and the exponentially distributed service times at the load-dependent station S_i with n busy vehicles. Further, let the random variables Y_1 and Y_2 denote the first and the second exponential phase of the 2-phase Coxian random variable, Y.

Formally, the probability distribution of Cox-2 inter-arrival times, $f_Y(t)$ is shown in Equation 6, where C_1 and C_2 are expressed as $\left(\frac{\lambda_{1_i}(1-p_i)-\lambda_{2_i}}{\lambda_{1_i}-\lambda_{2_i}}\right)$ and $\left(1-\frac{\lambda_{1_i}(1-p_i)-\lambda_{2_i}}{\lambda_{1_i}-\lambda_{2_i}}\right)$ respectively.

$$f_Y(t) = C_1\lambda_{1_i}e^{-\lambda_{1_i}t} + C_2\lambda_{2_i}e^{-\lambda_{2_i}t},\ t \geq 0 \tag{6}$$

The probability distribution function for Z_n is expressed as follows.

$$f_{Z_n}(t) = \mu_i(n)e^{-\mu_i(n)t},\ t \geq 0 \tag{7}$$

Then the probability q_n, which is $P[Y \leq Z_n]$ is given by Equation 8.

$$P[Y \leq Z_n] = \left(C_1\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(n)} + C_2\frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(n)}\right) \tag{8}$$

With this set of information, the probability corresponding to each sequence of events ($\mathbb{S}_e^1$ for state $s = (-V, 1)$), the conditional lower bounds for the two moments of the expected inter-departure times ($E[D_i|\mathbb{S}_e^1]_l$, $E[D_i^2|\mathbb{S}_e^1]_l$) are determined (Table 8 in Appendix). The estimation of the conditional lower bound for the expected inter-departure time is described for $\mathbb{S}_2^1 = \{A, A, S^{'}\}$. The expected time for an arrival is $E[Y]$. The expected time for an arrival in the first and second phase of an arrival are denoted by $E[Y_1]$ and $E[Y_2]$ respectively. After two arrivals, the expected time to complete a service is $E[Z_2] = \frac{1}{\mu_i(2)}$. Therefore, the lower bound is given by $E[D_i|\mathbb{S}_2^1]_l$, which is $E[Y] + E[Z_2]$. Note that the estimate of lower

bound follow the order: $E[D_i|\mathbb{S}_2^1]_l \leq E[D_i|\mathbb{S}_2^1]$. The probability ($p_{\mathbb{S}_2^1}$) that this sequence occurs is the probability of exactly two arrivals taking place before the service completion, which is $q_1(1-q_2)$. Similarly, the conditional lower bound for the second moment of $\mathbb{S}_2^1$ is given by the expression $\left[p_i\left((Var[Y_1]+Var[Y_2]+Var[Z_2])+(E[Y_1]+E[Y_2]+E[Z_2])^2\right)\right]+\left[(1-p_i)\left((Var[Y_1]+Var[Z_2])+(E[Y_1]+E[Z_2])^2\right)\right]$.

Likewise, the conditional expected lower bounds for the first and the second moment are determined for all sequences ($\mathbb{S}_e^1$) in s. Then the expressions for the lower bound for the first and the second moment of the inter-departure times of the sequences along with their occurrence probabilities are used to determine the expressions for the lower bound for the first and the second moment of the inter-departure times corresponding to a state $s \in G_1$. Equations 9 and 10 provide the relationship for the lower bound of the first and second moments of the inter-departure time.

$$\sum_{\mathbb{S}_e^1 \in s} p_{\mathbb{S}_e^1} E[D_i|\mathbb{S}_e^1]_l = E[D_i|s \in G_1]_l \leq E[D_i|s \in G_1] \tag{9}$$

$$\sum_{\mathbb{S}_e^1 \in s} p_{\mathbb{S}_e^1} E[D_i^2|\mathbb{S}_e^1]_l = E[D_i^2|s \in G_1]_l \leq E[D_i^2|s \in G_1] \tag{10}$$

A similar analysis is done for all states in $G_2, \ldots, G_5$. The analysis details and summary of the expressions are included in Appendix B. Using the steady state probability distribution, Π_D, the unconditional estimates of the lower bound for the first and second moment of the inter-departure times are given by Equations 11 and 12. These lower bounds are used as approximations for the first and second moments of the inter-departure times.

$$\sum_{i=1}^{5} \sum_{s \in G_i} \Pi_D(s) E[D_i|s \in G_i]_l = E[D_i]_l \leq E[D_i] \tag{11}$$

$$\sum_{i=1}^{5} \sum_{s \in G_i} \Pi_D(s) E[D_i^2|s \in G_i]_l = E[D_i^2]_l \leq E[D_i^2] \tag{12}$$

Now, the SCV of inter-departure times of transactions from S_i can be estimated using Equations 13 and 14. Equation 13 provides the expression to estimate the SCV of the inter-departure times for all transactions from station S_i in tier i ($c^2_{d_{A_i,S_i}}$) whereas Equation 14 provides the expression to estimate the SCV of the inter-departure times for the retrieval transactions from station S_i in tier i, where q_o is the proportion of transactions that belongs to retrieval class r_i (Whitt [1983]).

Note that the gap between the lower bound estimate for the expected inter-departure time and the actual value widens when the number of arrivals (before a service completion) in the sequence, $\mathbb{S}^1_e$, increases. We use the maximum service rate in the lower bound, which weakens the bound estimate with an increase in the number of arrivals. However, the probability of such an event occurrence also decreases, especially under heavy traffic conditions (high vehicle utilization). Hence, the overall bound estimate may not be affected to a large extent. Using a similar analysis, we can also develop an upper bound estimate for the first two moments of the inter-departure times. However, the upper bound would be a weak approximation because the transaction at the load-dependent station would be serviced at the lowest possible rate, $\mu_i(1)$.

$$c^2_{d_{A_i,S_i}} = \frac{E[D_i^2]_l - E[D_i]_l^2}{E[D_i]_l^2} \tag{13}$$

$$c^2_{d_{r_i,S_i}} = q_o c^2_{d_{A_i,S_i}} + 1 - q_o \tag{14}$$

The queuing analysis of the vertical transfer mechanism (conveyor/ lift subsystem) is described in the subsequent section.

5 Queuing Models for Vertical Movement between Tiers

We describe the queuing network models for the conveyor and the lift subsystems in this section. The objective of analyzing the conveyor system is to determine the mean and the SCV of the inter-departure times for the transactions from the conveyor loops and to estimate the performance measures. The notations used in the analysis of the conveyor subsystem are described in Table 3. The details of the queuing model and the analysis approach are discussed in the following paragraphs. In the conveyor system, the pallet is

Table 3: Notations used in the analysis of vertical transfer with conveyors and lifts

Notation	Description
L_k	Conveyor loop $k = 1, \ldots, T-1$
$\lambda^{-1}_{a_{s_i,L_k}}, c^2_{a_{s_i,L_k}}$	Mean and SCV of the inter-arrival time for storage transaction class s_i to L_k
$\lambda^{-1}_{a_{r_i,L_k}}, c^2_{a_{r_i,L_k}}$	Mean and SCV of the inter-arrival time for retrieval transaction class r_i to L_k
$\lambda^{-1}_{d_{s_i,L_k}}, c^2_{d_{s_i,L_k}}$	Mean and SCV of the inter-departure time for storage transaction class s_i from L_k
$\lambda^{-1}_{d_{r_i,L_k}}, c^2_{d_{r_i,L_k}}$	Mean and SCV of the inter-departure time for retrieval transaction class r_i from L_k
$\mu_D^{-1}, c^2_{\hat{s}_{r_i,L_k}}$	Mean and SCV of the service time for retrieval (or storage) transaction class r_i (or s_i) at L_k
C_{L_k}	Set of all transaction classes that visit conveyor loop L_k
ρ_{r_i,L_k}	Utilization of conveyor loop L_k due to retrieval class r_i
ρ_{L_k}	Utilization of conveyor loop L_k

transferred vertically using one or more conveyor loops. Each loop (L_k) transfers a pallet between consecutive tiers k and $k+1$ where $k = 1, \ldots, T-1$. Therefore, to transfer pallets in a multi-tier system with T tiers, a maximum of $T-1$ conveyor loops are required. Loop L_1 transfers a load between the first and the second tier whereas loop L_{T-1} transfers a load between $T-1^{th}$ and T^{th} tier. For each loop k, the pallets, to be stored, queue at the LU point on tier k and the pallets, to be retrieved, queue at the LU point on tier $k+1$.

Next, the queuing analysis is discussed. Each conveyor loop segment is modeled as an open $GI/G/1$ queue with deterministic service time, μ_D^{-1}, implying that a network of $T-1$ open $GI/G/1$ queues are used to model the conveyor system. The conveyor stations are

indexed as $L_1, L_2, \ldots, L_{T-1}$. There are T transaction classes corresponding to the storage transaction and T transaction classes corresponding to the retrieval transaction. The index i for storage and retrieval classes: $1, 2, \ldots, T$ correspond to tiers $1, 2, \ldots, T$. Note that class 1 storage and retrieval transactions do not use the conveyor. A storage class i transaction is routed through the conveyor stations in the following order: $L_1, L_2, \ldots, L_{i-1}$ whereas a retrieval class i transaction is routed through the conveyor stations in the following order: $L_{i-1}, L_{i-2}, \ldots, L_1$. Figure 5 shows the queuing network for the conveyor system with four tiers.

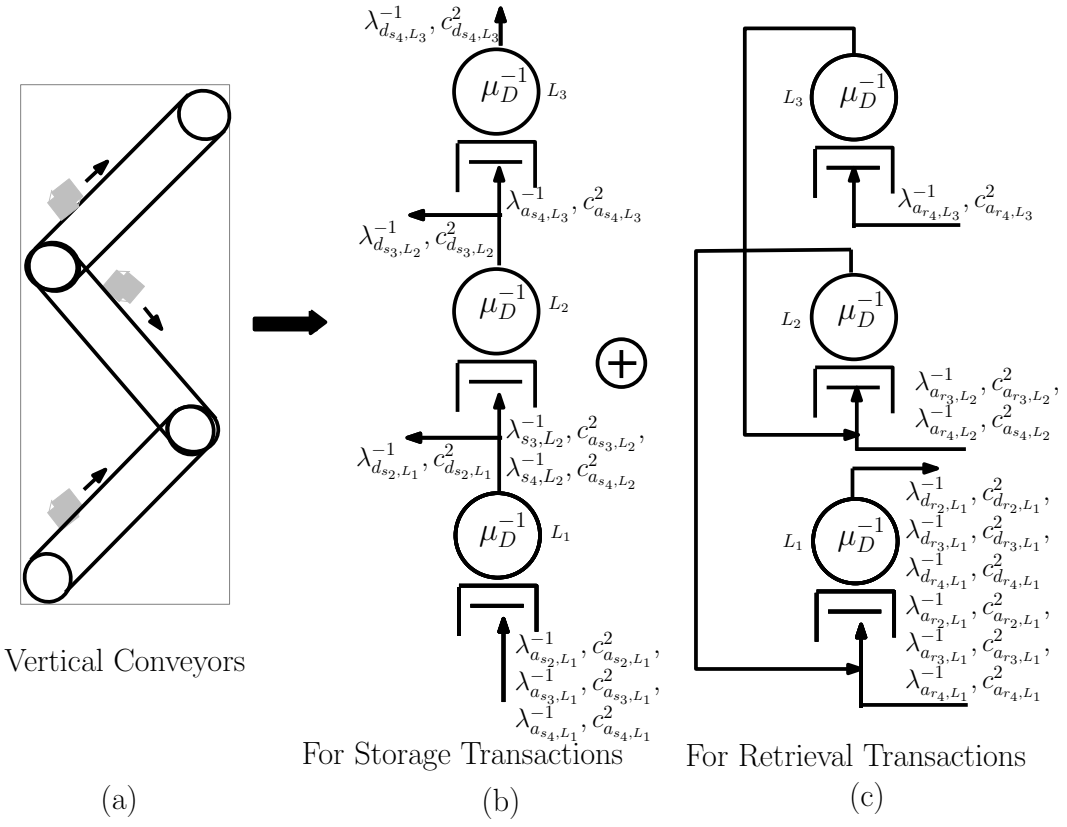

Figure 5: Vertical conveyor queuing network for a four-tier system: (a) vertical conveyors, (b) flow of storage transactions, and (c) flow of retrieval transactions

Since pallets to be stored are first conveyed to the destination tiers using the conveyor subsystem, the storage transaction requests arrive directly to the conveyor subsystem from an external source. The distribution of the inter-arrival times for storage transaction class (s_i) to the conveyor loop L_1 is exponential with mean, $\lambda^{-1}_{a_{s_i,L_1}}$, and SCV, $c^2_{a_{s_i,L_1}} = 1$, where $i = \{2, \ldots, T\}$. However, the tier subsystem is involved in the first processing step of the retrieval transactions. The vehicle in the tier retrieves the pallet from the storage address and then deposits at the LU point of the tier. Using a conveyor subsystem, the pallet is transferred from the LU point of the retrieval tier to the LU point of tier 1. Therefore, the distribution of the inter-arrival times of the retrieval transactions to the conveyor loops is not exponential. The mean of the inter-arrival time for the retrieval transaction class r_i to the conveyor loop L_{i-1} from tier i is $\lambda^{-1}_{a_{r_i,L_{i-1}}}$. Further, for the retrieval transaction class r_i, the SCV of the inter-arrival times to the conveyor loops ($c^2_{a_{r_i,L_{i-1}}}$), is unknown. The inputs to the analysis are the mean and SCV of the inter-arrival times of the storage and retrieval transactions to the conveyor loops. Note that the SCV of the inter-arrival times for retrieval transactions are not known and will be subsequently determined by linking the departure processes from the tier and the conveyor subsystems. However, for the analysis of the conveyor system in isolation, these are assumed to be known inputs with mean, $\lambda^{-1}_{a_{r_i,L_{i-1}}}$ and SCV, $c^2_{a_{r_i,L_{i-1}}} = 1$. Within the network, the routing of the transactions and the service times at each node of the tier are also known. With this information, the departure process from each conveyor loop and the performance measures are estimated using a parametric-decomposition approach.

The conveyor model, which is a multi-class open queuing network with tandem stations, is a non product-form queuing network that is solved using a parametric-decomposition approach (Whitt [1983],Whitt [1994]). To solve a queuing model using the decomposition approach, the inputs are the mean and the SCV of the transaction inter-arrival time to all

stations, and the mean and SCV of the service times for all stations in the network. The outputs are the performance measures for each station, such as utilization, expected cycle time and the expected number of transactions waiting in the queue.

In the conveyor subsystem, the mean inter-arrival and the inter-departure times for all transaction classes at conveyor station, L_k are given by Equations 15 and 16 respectively. Though the mean inter-arrival time, mean service time, and the SCV of the service time at all stations are known, the SCV of inter-arrival times at stations are not all known. For instance, in Figure 5, the SCV of the inter-arrival times for storage classes 2, 3, and 4 at conveyor loop L_1 queue are known, but the SCV of the inter-arrival times for storage classes 3 and 4 to conveyor loop L_2 are unknown. Similarly, while the SCV of the inter-arrival times for retrieval classes 2, 3, and 4 at conveyor loop L_1, L_2, and L_3 are known, the SCV of the inter-arrival times for retrieval classes 3 and 4 at loop L_1 queue are unknown. The approach to determine the unknown SCVs of the inter-arrival times is described for a retrieval class r_i at conveyor loop k. Let C_{L_k} denote the set of all transaction classes that visit station L_k (for instance, $C_{L_1} = \{r_2, \ldots, r_T, s_2, \ldots, s_T\}$). The expression for estimating $c^2_{d_{r_i,L_k}}$ is provided by Whitt [1994] (Equation 17), where ϕ_{r_i,L_k} is defined as $\lambda_{a_{r_i,L_k}} / \sum_{j \in C_{L_k}} \lambda_{a_{j,L_k}}$. Also note that the inter-departure time SCV and the inter-arrival time SCV of a transaction class are linked across consecutive conveyor stations by the following relationship. The inter-departure time SCV of a transaction class at a conveyor station is equal to the inter-arrival time SCV for the same class at its next station in the routing (Equations 18 and 19). Using this approach, the SCV of the inter-departure time for all classes from the conveyor loops can be determined.

$$\lambda^{-1}_{a_{j,L_k}} = \lambda^{-1}_{j} \quad \forall j \in C_{L_k} \tag{15}$$

$$\lambda^{-1}_{d_{j,L_k}} = \lambda^{-1}_{a_{j,L_k}} \quad \forall j \in C_{L_k} \tag{16}$$

$$
\begin{aligned}
c^2_{d_{r_i,L_k}} &= \rho^2_{r_i,L_k} c^2_{\hat{s}_{r_i,L_k}} + (1 - 2\rho_{r_i,L_k}\rho_{L_k} + \rho^2_{r_i,L_k}) c^2_{a_{r_i,L_k}} + \\
&\quad \phi_{r_i,L_k} \sum_{j \neq r_i, \forall j \in C_{L_k}} \frac{\rho^2_{j,L_k}}{\phi_{j,L_k}} (c^2_{\hat{s}_{j,L_k}} + c^2_{a_{j,L_k}}) \qquad (17) \\
c^2_{d_{r_i,L_k}} &= c^2_{a_{r_i,L_{k-1}}} \qquad (18) \\
c^2_{d_{s_i,L_k}} &= c^2_{a_{s_i,L_{k+1}}} \qquad (19) \\
&\quad \text{where } i \in \{1, \ldots, T\} \text{ and } k \in \{1, \ldots, T-1\}
\end{aligned}
$$

Since the travel time in the conveyor loop is assumed to be deterministic, $c^2_{\hat{s}_{r_i,L_k}} = 0$ and $c^2_{\hat{s}_{s_i,L_k}} = 0 \ \forall i \in \{1, \ldots, T\}$ and $\forall k \in \{1, \ldots, T-1\}$, the values of two variables, for each transaction class $j : j \in C_{L_k}$ at a station L_k, $c^2_{d_{j,L_k}}$ and $c^2_{a_{j,L_k}}$, are unknown. The number of transaction classes routed to conveyor station L_k is $2(T-k)$. Therefore, the total number of initial variables is $2T(T-1)$. Amongst them, the inter-arrival time SCVs of the storage classes at station 1, $(T-1)$ quantities, and the inter-arrival time SCVs of retrieval transactions from the tiers to the conveyor stations, $(T-1)$ quantities, are initialized to 1 (assuming an exponential distribution). Therefore, the remaining number of unknown quantities is $2(T-1)^2$.

To estimate the SCV of inter-arrival times of retrieval and storage classes at all stations, a system of linear equations is formed using the following two steps: 1) The expression for the inter-departure time SCV for all classes at each conveyor station is known from Equation 17. This gives a set of $T(T-1)$ linear equations, 2) Further, note that the inter-departure time SCV of a transaction class from a conveyor station forms the inter-arrival time SCV of the same class to the consecutive station. This gives an additional set of $(T-2)(T-1)$ linear equations (Equations 18 and 19).

Now, we have a system of $2(T-1)^2$ linear equations and $2(T-1)^2$ unknown variables, which is solved to obtain the inter-arrival time SCVs for all classes to the conveyor stations.

Next, each station can be solved in isolation and the performance measures such as conveyor loop utilization, average queue length, and storage and retrieval vertical transfer cycle times can be evaluated using standard approximation for $GI/G/1$ queues (Refer Whitt [1983]). The expressions for the performance measures are provided in Section 7.2. The vertical movements with lifts are modeled in a similar fashion where the lift resource is modeled using a $GI/G/1$ queue. The details of the lift analysis are included in Appendix C.

6 Linking Models for Horizontal and Vertical Movements

In the previous sections, the queuing analysis of individual tiers and the vertical transfer subsystem (lifts or conveyors) have been studied in isolation. However, in reality, these queuing systems are inter-related. For instance, for storage transactions, the departure process from the vertical transfer subsystem forms the arrival process to the tier subsystems. Similarly, for retrieval transactions, the departure process from the tier subsystems forms the arrival process to the vertical transfer subsystem (see Figure 3b). The departure processes for the tier and vertical transfer subsystems are linked by a set of equations that are solved using an iterative algorithm. Figure 7 illustrates the approach described in detail in Sections 6.1 and 6.2.

6.1 Linking Equations for Vertical Transfer with Conveyors

First, the queuing model of the conveyor system is solved assuming the SCV of the inter-arrival time for the retrieval transaction class r_i, $(c^2_{a_{r_i},L_{i-1}})^{curr}$ $\forall i = (2,\ldots,T)$, to the conveyor loop L_{i-1} to be equal to 1. Since the inter-arrival times for storage transactions have an exponential distribution, the inter-arrival time SCV for all classes of storage transactions to the conveyor loop L_1 is indeed equal to 1. With this initialization, the conveyor queuing

network is solved using the method described in Section 5. After solving the queuing network, the inter-departure time SCVs for all classes of storage transactions are determined. With this information, the inter-arrival time SCV for the aggregated class ($c^2_{a_{A_i,J_i}}$) to the buffer B_{1i} of synchronization station J in tier $i : i = (2, \ldots, T)$ is calculated using Equation 2 described in Step 1 of Section 4. This step is followed by aggregating the subnetwork of each tier into a load-dependent station S_i and estimating $\mu_i(n)$. Note that this step is executed only once because the value of $\mu_i(n)$ is independent of the inter-arrival time distribution of the transactions. Then, the inter-departure time SCV for aggregate transaction classes from all tiers is analyzed using the approach described in Step 2 of Section 4 and the SCV of the inter-departure times for the retrieval transaction class r_i, $c^2_{d_{r_i,S_i}}$, from S_i is determined. Since this inter-departure time SCV forms the inter-arrival time SCV for transaction class r_i to the conveyor loop L_{i-1}, the error component (δ_i), which is defined as the absolute difference between $c^2_{d_{r_i,S_i}}$ and $(c^2_{a_{r_i,L_{i-1}}})^{curr}$, is computed for $i = (2, \ldots, T)$. If the maximum absolute difference (δ_{max}) is less than ϵ then the algorithm is terminated else $(c^2_{a_{r_i,L_{i-1}}})^{curr}$ is updated using the step-size rule and all steps are repeated. The flowchart shown in Appendix E summarizes the steps of this algorithm. The next section presents the model and the expressions to determine the performance measures for the tier, conveyor, and integrated multi-tier system.

6.2 Linking Equations for Vertical Transfer with Lift

Similar to the conveyor model, the departure process from the lift and the tier subsystems is analyzed and linked together using the algorithm described in Section 6.1. The linking algorithm for multiple tiers with a lift is similar to the one developed with a conveyor except that there is a single server representing the lift resource ($\widehat{L}_1$) instead of a series of single-server stations representing the conveyor segments. First, the queuing model of the

lift system, $\widehat{L}_1$ is solved with the SCV of the inter-arrival time for the retrieval transaction class r_i, $(c^2_{a_{r_i,\widehat{L}_1}})^{curr}$ $\forall i = (2,\ldots,T)$, to the lift resource $\widehat{L}_1$, is assumed to be 1. Since the inter-arrival times for storage transactions have an exponential distribution, the inter-arrival time SCV for all classes of storage transactions to the lift resource $\widehat{L}_1$ is indeed 1. With this initialization, the lift queuing network is solved using the method described in Appendix C and the SCV of the inter-departure times for the individual tiers are obtained. The remaining linking steps where the tier networks are evaluated and new estimates for the SCV of the inter-arrival times for the transactions to the lifts are identical to that discussed in Section 6.1.

7 Estimating Performance Measures

The following subsections explain the model and list the expressions to estimate the performance measures for the subsystems and the multi-tier system. Section 7.1 discusses the equations to estimate the measures corresponding to a tier whereas Section 7.2 discusses the equations to estimate the measures corresponding to a vertical transfer unit (both conveyors and lifts).

7.1 Performance Measures for Horizontal Movement within a Tier

The performance estimate for each tier corresponding to the model illustrated in Figure 3b is obtained by solving a continuous time Markov chain. The state space for the CTMC is described by a two-tuple vector (i_1, i_2), which is used earlier in the analysis of the embedded Markov chain except that the value for the tuples i_1 is no longer restricted to $K-1$. The tuples i_1 and i_2 take the values from the set $\{-V, -V+1, \ldots, 0, \ldots, \infty\}$ and $\{1, 2\}$ respectively. The expected inter-arrival times corresponding to the first and the second phase

of the Cox-2 arrival process are $\lambda_{1_i}^{-1}$ and $\lambda_{2_i}^{-1}$ respectively. The expected load-dependent service time is denoted by $\mu_i(n)^{-1}$. With this information, the flow balance equations are solved and the steady state probability distribution for the CTMC, π_t is obtained. Using π_t, the vehicle utilization (U_{V_i}) and the expected number of transactions waiting to be processed at buffer B_1 ($Q_{B_{1i}}$) for tier $i : i \in \{1, \ldots, T\}$ can be estimated. The expressions for the performance measures of a tier are provided now.

Vehicle Utilization: To estimate vehicle utilization, the expected number of idle vehicles ($E[I_{V_i}]$) needs to be determined. The expressions to determine $E[I_{V_i}]$ and vehicle utilization (U_{V_i}) are given by Equations 20 and 21 respectively. Note that when $i_1 < 0$, there are $|i_1|$ number of idle vehicles at buffer B_{2i}. Therefore, the expected number of idle vehicles is estimated by taking an expectation on the number of idle vehicles corresponding to states $i_1 < 0$.

$$E[I_{V_i}] = \sum_{i_1, i_2 : i_1 < 0} \pi_t(i_1, i_2)|i_1| \tag{20}$$

$$U_{V_i} = 1 - \frac{E[I_{V_i}]}{V} \tag{21}$$

Average Number of Transactions Waiting for Service: The expression for the average number of transactions waiting for service ($Q_{B_{1i}}$) is given by Equation 22.

$$Q_{B_{1i}} = \sum_{i_1, i_2 : i_1 > 0} \pi_t(i_1, i_2) i_1 \tag{22}$$

Expected Transaction Cycle Times in a Tier: To estimate these measures, the expected number of busy vehicles in the tier subsystem is determined by the expression $V - E[I_{V_i}]$. Since we assume $\lambda_{s_i} = \lambda_{r_i}$ for each tier, the expected number of busy vehicles processing storages and retrievals is equal to $\frac{V - E[I_{V_i}]}{2}$.

The expected retrieval cycle time in a tier, $E[CT_{tr_i}]$, is composed of two components: waiting time for an available vehicle and processing time in a tier. Both the components are estimated by applying Little's law in the buffer B_{1i} and in the tier network. Since $Q_{B_{1i}}$ is the expected number of transactions waiting in buffer B_{1i}, $\frac{Q_{B_{1i}}}{\lambda_{r_i}+\lambda_{s_i}}$ is the expected waiting time for an available vehicle. Similarly, $\frac{V-E[I_{V_i}]}{2}$ is the expected number of vehicles processing retrieval transactions within a tier. Therefore, $\frac{V-E[I_{V_i}]}{2\lambda_{r_i}}$ is the average time to process a retrieval transaction within a tier. While the waiting time component can be estimated in a similar fashion for the storage transactions, the expected processing time for a storage transaction cannot be directly estimated.

Note that the processing of a storage transaction is complete when the pallet is unloaded at the storage location within an aisle. Therefore, only a fraction of storage class vehicles within an aisle are processing storage transactions while the rest are on their return travel to the LU dwell point. To estimate the expected time spent by a storage class vehicle within an aisle until unloading the pallet is complete, the following approach is adopted. The total expected time spent within an aisle is the difference between the expected processing time within a tier and the sum of the expected times spent by the storage class vehicle at the cross-aisles and the LU point. Hence, the expected time spent by a storage class vehicle at an aisle is determined using the expression $\frac{V-E[I_{V_i}]}{2\lambda_{s_i}} - (2\mu_{CA_L}^{-1} + \mu_{LU}^{-1})$. Further, this expression is multiplied by a term α, which is the ratio of time spent in the aisle until a storage transaction is complete and the total expected time spent within an aisle to obtain $E[CT_{a_i}]$, which is the expected time spent by the vehicle in the aisle until the storage transaction is complete. With this information, the expected cycle time for processing storage and retrieval transactions in a tier i ($E[CT_{ts_i}]$ and $E[CT_{tr_i}]$) can be obtained by

the expressions provided in Equations 23 and 24.

$$E[CT_{tr_i}] = \frac{Q_{B_{1i}}}{\lambda_{r_i} + \lambda_{s_i}} + \frac{V - E[I_{V_i}]}{2\lambda_{r_i}} \quad (23)$$

$$E[CT_{ts_i}] = \frac{Q_{B_{1i}}}{\lambda_{r_i} + \lambda_{s_i}} + \mu_{CA_L}^{-1} + \mu_{LU}^{-1} + E[CT_{a,i}] \quad (24)$$

where $E[CT_{a_i}] = \alpha\big(\frac{V-E[I_{V_i}]}{2\lambda_{s_i}} - (2\mu_{CA_L}^{-1} + \mu_{LU}^{-1})\big)$ is the expected aisle time spent by a storage transaction and $\alpha = \frac{\frac{W}{2v_h} + \frac{x_w}{v_h} + U_{vt}}{\frac{W}{v_h} + \frac{2x_w}{v_h} + U_{vt}}$.

7.2 Performance Measures for the Vertical Transfer Unit

The performance estimates for the conveyor subsystem such as conveyor utilization (U_C), expected number of transactions waiting for conveyor (Q_C), and expected conveyor cycle time for processing storage and retrieval transactions ($E[CT_{rc}]$ and $E[CT_{sc}]$). These measures are calculated using the SCV of the inter-arrival times for the transaction classes obtained after the convergence of the linking algorithm.

Conveyor Utilization: The utilization of conveyor loop L_1 (ρ_{L_1}) is of prime interest to design engineers because all transactions that require conveyors use loop L_1. Hence, it is the most utilized conveyor loop among all loops and used as a measure of the conveyor system utilization (Equation 25).

$$U_C = \sum_{j \in C_{L_1}} \rho_{j,L_1} \quad (25)$$

Expected Cycle Times for the Conveyor System: Equation 26 provides the expression to estimate the expected cycle time ($E[R_{L_k}]$) for all classes of transactions at conveyor loop L_k where $E[W_{L_k}]^{GI/G/1}$ denotes the expected waiting time at loop L_k. In this equa-

tion, $E[W_{L_k}]^{GI/G/1}$ denotes the expected waiting time in a $GI/G/1$ queue (Whitt [1983]). Equations 27 and 28 provide the expressions to determine the expected conveyor cycle time for class i retrieval and class i storage transactions ($E[CT_{cr_i}]$ and $E[CT_{cs_i}]$) respectively using the values for $E[R_{L_k}]$. The expected cycle time component to retrieve and store a pallet using the conveyor subsystem are denoted by $E[CT_{cr}]$ and $E[CT_{cs}]$ respectively (Equations 29 and 30).

$$E[R_{L_k}] = E[W_{L_k}]^{GI/G/1} + \mu_D^{-1} \ \forall k \in \{1, \ldots, T-1\} \tag{26}$$

$$E[CT_{cr_i}] = \sum_{k=1}^{i-1} E[R_{L_k}] \ \forall i \in \{2, \ldots, T\} \tag{27}$$

$$E[CT_{cs_i}] = \sum_{k=1}^{i-1} E[R_{L_k}] \ \forall i \in \{2, \ldots, T\} \tag{28}$$

$$E[CT_{cr}] = \frac{\sum_{i=2}^{T} E[CT_{cr_i}]}{T-1} \tag{29}$$

$$E[CT_{cs}] = \frac{\sum_{i=2}^{T} E[CT_{cs_i}]}{T-1} \tag{30}$$

Average Number of Transactions Waiting for Vertical Transfer: The average number of transactions waiting at conveyor loop L_k (Q_{L_k}), is estimated using Little's law. The expression to estimate the total number of transactions (Q_C) waiting in the conveyor subsystem is shown in Equation 31.

$$Q_C = \sum_{k=1}^{T-1} Q_{L_k} \tag{31}$$

From the lift queuing model, the following performance measures can be obtained: the expected storage and retrieval lift cycle time $E[CT_{ls_i}]$ and $E[CT_{lr_i}]$ for transaction class i (Equations 32 and 33), the lift utilization (U_L), and the average number of transactions

waiting for the lift (Q_L).

$$E[CT_{ls_i}] = E[W_L]^{GI/G/1} + E[S_{s_i}] \quad (32)$$

$$E[CT_{lr_i}] = E[W_L]^{GI/G/1} + E[S_{r_i}] \quad (33)$$

7.3 Performance Measures for the Overall System

For the integrated system, the expected transaction cycle times ($E[CT_{s_c}]$ and $E[CT_{r_c}]$), average vehicle utilization (U_V), and the expected number of transactions waiting for service ($E[Q_W]$) in all tiers are estimated.

Expected Transaction Cycle Times: The total expected cycle time for storage and retrieval transactions, which is the weighted sum of the cycle time across all tiers, are given by Equations 34 and 35 respectively.

$$E[CT_{s_c}] = \frac{1}{T}(E[CT_{ts_1}]) + \frac{1}{T}\sum_{i=2}^{T}(E[CT_{ts_i}] + E[CT_{cs_i}]) \quad (34)$$

$$E[CT_{r_c}] = \frac{1}{T}(E[CT_{tr_1}]) + \frac{1}{T}\sum_{i=2}^{T}(E[CT_{tr_i}] + E[CT_{cr_i}]) \quad (35)$$

Average Vehicle Utilization: The average vehicle utilization across all tiers is given by Equation 36.

$$U_V = \frac{\sum_{i=1}^{T} U_{V_i}}{T} \quad (36)$$

Average Number of Transactions Waiting for Service: The average number of trans-

actions waiting across all tiers is given by Equations 37.

$$Q_{B_1} = \sum_{i=1}^{T} Q_{B_{1i}} \tag{37}$$

To determine the expected transaction cycle times ($E[CT_{s_l}]$ and $E[CT_{r_l}]$), $E[CT_{ls_i}]$ and $E[CT_{lr_i}]$ are substituted in place of $E[CT_{cs_i}]$ and $E[CT_{cr_i}]$ in Equations 34 and 35 respectively. The next section presents the numerical results and insights.

8 Numerical Experiments

This section describes the design of experiments conducted to validate the model results and develop insights with respect to the design parameters. For the multi-tier system, the expected queue length at the vertical transfers, the expected transaction throughput times, and the vehicle and vertical transfer resource utilization are of interest for system sizing. To validate the analytical model, we obtain input data by partnering with Savoye Logistics (*www.savoye.com*), a leading manufacturer of autonomous vehicle-based systems. For experimentation, we consider a tier with two levels of $\frac{D}{W}$ ratio: 1 and 2. A tier with 30 aisles and 81 columns (4860 storage locations per tier) has a $\frac{D}{W}$ ratio of 1 whereas a tier with 44 aisles and 60 columns (5280 storage locations per tier) has a $\frac{D}{W}$ ratio of 2. The number of tiers is also varied at two levels: 5 and 7. The transaction rate is varied at 10 equally spaced intervals from 270 pallets/hr to 400 pallets/hrs. To maintain the utilization of both vehicles as well as the vertical transfer between 60% to 90%, we consider 5 vehicles per tier for the conveyor-based system. However, for the lift-based system, we consider 2 vehicles per tier and 3 vehicles per tier for the 7 tier and the 5 tier system, respectively. In sum, 40 cases each ($2 \times 2 \times 10$) were analyzed for both conveyor and lift systems.

Based on practical application data, the vehicle horizontal velocity (v_h), lift velocity (v_l), and conveyor velocity (v_c) are initialized to 8.2 ft/sec, 4.9 ft/sec, and 1.5 ft/sec respectively. We assume that lifts have an additional load/unload time of 2 seconds. The simulation model is build using AutoMod^{TM} v12.2.1. (see Roy et al. [2015] for details). For each scenario, 15 replications are run with a warm-up period of at least 6,500 transactions and a run time of at least 65,000 transactions. The analytical model takes less than 30 seconds of computational time on a standard PC.

Performance of the Analytical Model: For AVS/RS with conveyor mechanism, the average absolute error percentage $\left|\frac{y_a - y_s}{y_s}\right|$ in the total expected conveyor transaction cycle times, conveyor utilization and the expected number of transactions waiting for the conveyor are 8%, 0.1%, and 22% respectively whereas for AVS/RS with lift mechanism, the average absolute error percentage in the total expected transaction cycle times, lift utilization and expected number of transactions waiting for the lift are 6%, 0.1%, and 12% respectively, where y_a and y_s denote the performance measure estimates obtained from the analytical and simulation models respectively. The linking algorithm converges in less than 25 iterations for a seven-tier system. Figure 8a in Appendix F shows the distribution of the absolute errors for the conveyor-based system such as vehicle utilization, expected conveyor retrieval and storage cycle time, expected number of transactions waiting for the conveyor, and conveyor utilization. Similarly, Figure 8b in Appendix F shows the distribution of the absolute errors for the lift-based system such as vehicle utilization, expected lift retrieval and storage cycle time, expected number of transactions waiting for lift, and lift utilization. It can be seen that the overall errors for all measures are within 15% except for the expected number of transactions that wait for conveyor, Q_C. The expected number of transactions waiting for the conveyor is low (0.3-0.5 per tier), hence the errors appear high (See Table 6). Tables 4 and 5 provide a summary of the averages as well as the range (min-max) for the

performance measures corresponding to the conveyor system and lift system respectively.

Table 4: Model performance (Output for conveyor-based system)

Statistic	U_V	$E[CT_{cr}]$	$E[CT_{cs}]$	Q_C	U_C
Average	0.90%	7.66%	8.17%	21.95%	0.11%
Range	-0.24%-1.99%	4.73%-12.26%	4.49%-15.81%	-6.98%-60.8%	0.01%-0.30%

Table 5: Model performance (Output for lift-based system)

Statistic	U_V	$E[CT_{lr}]$	$E[CT_{ls}]$	Q_L	U_L
Average	1.34%	6.84%	4.91%	11.16%	0.13%
Range	-0.35%-3.01%	1.01%-14.49%	14.4%-11.79%	4.45%-21.63%	0.0%-0.30%

Performance Measures for Conveyor and Lift-based Systems: Tables 6 and 7 provide the numerical results from the analytical models of the conveyor and lift-based systems respectively. For the conveyor-based system, the results for the performance measures: vehicle utilization, conveyor utilization, expected transaction cycle times, expected conveyor cycle times, and the average number of transactions waiting for vehicles and conveyor are shown whereas for the lift-based system, the results for the performance measures: vehicle utilization, lift utilization, expected transaction cycle times, expected lift cycle times, and the average number of transactions waiting for vehicles and lift are shown. The configurations for both systems are seven tiers, and 5280 storage locations/tier. Note that the lift system becomes a bottleneck resource with 2 vehicles/tier. However, the conveyor system permits an increase in the number of vehicles from 2 to 5 vehicles/tier, which allows an increase in the throughput capacity of the system by 150%. These experiments suggest that the conveyor mechanism can substantially improve the throughput capacity of AVS/RS. Also note that by using multiple conveyor loops, the expected cycle time for

vertical transfer is less than that of the lift system.

Table 6: Performance estimates for conveyor-based AVS/RS with 5 vehicles/tier

λ_s, λ_r (pall./hr)	Type	Q_{B_1}	U_V (%)	$E[CT_{r_c}]$(sec)	$E[CT_{s_c}]$ (sec)	$E[CT_{cr}]$(sec)	$E[CT_{cs}]$(sec)	Q_C	U_C (%)
648	y_a	4.3	66%	180	131	32	32	2.3	77%
	y_s	3.0	66%	173	123	29	28	1.7	77%
662	y_a	4.9	68%	184	135	33	33	2.5	79%
	y_s	3.3	67%	176	125	30	29	1.9	79%
677	y_a	5.7	70%	190	140	35	35	2.8	81%
	y_s	4.0	69%	180	130	31	30	2.1	81%
691	y_a	6.7	71%	196	146	36	36	3.1	82%
	y_s	4.7	71%	185	134	32	32	2.4	82%
706	y_a	7.8	73%	203	153	38	38	3.4	84%
	y_s	5.6	72%	192	140	34	33	2.7	84%
720	y_a	9.1	75%	211	161	40	40	3.9	86%
	y_s	6.1	73%	196	144	36	35	3.0	86%
734	y_a	10.7	76%	220	170	43	43	4.4	87%
	y_s	7.0	75%	202	150	38	37	3.5	87%
749	y_a	12.6	78%	231	181	46	46	5.1	89%
	y_s	8.2	77%	211	158	42	40	4.2	89%
763	y_a	14.8	80%	245	195	51	51	6.0	91%
	y_s	9.4	78%	221	167	46	44	5.0	91%
778	y_a	16.6	82%	263	212	57	57	7.3	93%
	y_s	10.8	80%	234	179	53	50	6.3	93%

Table 7: Performance estimates for lift-based AVS/RS with 2 vehicles/tier

λ_s, λ_r (pall./hr)	Type	Q_{B_1}	U_V (%)	$E[CT_{r_c}]$(sec)	$E[CT_{s_c}]$(sec)	$E[CT_{lr}]$(sec)	$E[CT_{ls}]$(sec)	Q_L	U_L (%)
270	y_a	5.7	63%	236	191	50	49	2.3	83%
	y_s	3.5	62%	204	159	47	46	2.2	83%
274	y_a	6.0	64%	242	197	53	52	2.6	84%
	y_s	3.8	62%	208	165	50	50	2.4	84%
277	y_a	6.4	65%	249	204	57	55	2.8	86%
	y_s	4.0	63%	214	169	53	52	2.6	86%
281	y_a	6.8	65%	257	212	61	60	3.2	87%
	y_s	4.3	64%	221	177	57	56	2.9	87%
284	y_a	7.2	66%	266	221	66	65	3.5	88%
	y_s	4.4	65%	227	183	61	60	3.2	88%
288	y_a	7.7	67%	276	231	72	70	4.0	89%
	y_s	4.6	66%	233	189	66	66	3.6	89%
292	y_a	8.2	68%	287	242	79	78	4.5	90%
	y_s	4.8	66%	244	201	75	74	4.3	90%
295	y_a	8.7	69%	300	255	88	87	5.2	91%
	y_s	5.1	67%	255	211	84	83	5.0	91%
299	y_a	9.3	70%	316	271	100	99	6.1	92%
	y_s	5.3	68%	266	222	93	92	5.6	92%
302	y_a	9.9	71%	335	290	116	114	7.4	93%
	y_s	5.7	69%	287	243	110	109	6.9	93%

Comparison of Expected Transaction Cycle Times: Further, for the multi-tier system with 7 tiers, 28,560 storage locations, and 3 vehicles/tier, the λ_s, λ_r are varied from 270 to 306 pallets/hr. For these set of system configurations, it is observed that the conveyor system decreases the expected transaction cycle times by 17%-64% (Figure 6). Since the conveyor throughput capacity is greater than the lift throughput capacity, the lift waiting time is more than the conveyor waiting time for the same transaction arrival rates. Hence, we notice that as the arrival rates increase, the expected transaction time with the lift grows rapidly. However, note that the decision to select a conveyor vertical transfer over a lift vertical transfer is subject to many other factors such as cost and space considerations. For instance, the lift unit is compact and typically requires less space than the conveyor unit.

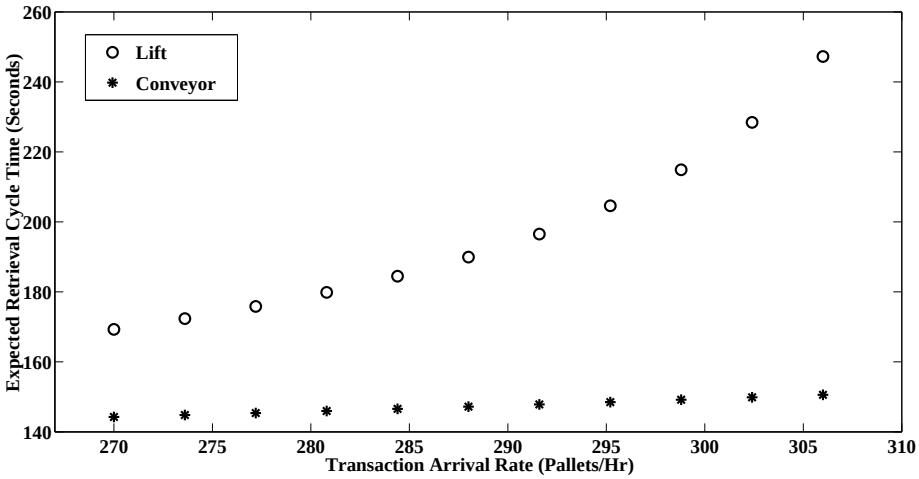

Figure 6: Comparing retrieval transaction cycle times with lift and conveyor system

Throughput Capacity: The throughput capacity of each tier i is $min(X(V_i), \lambda_{s_i} + \lambda_{r_i})$ where $X(V_i)$ is the throughput of the closed queuing network corresponding to a tier with V vehicles. However, for the multi-tier system, the throughput capacity is

$min(\sum_{i=1}^{T} X(V_i), \sum_{i=1}^{T} \lambda_{s_i} + \lambda_{r_i}, \mu_v)$ where μ_v is the throughput capacity of the vertical transfer unit. While the number of vehicles in the system can be increased to increase the throughput capacity, at some point, the throughput capacity of the vertical transfer mechanism will constrain the throughput capacity of the system. Due to multiple conveyor loops, which process transactions in parallel, the throughput capacity of the system is improved by multiple times when compared to the lift-based system.

9 Summary and Conclusions

During the last decade, a new generation of vehicle-based storage and retrieval systems (AVS/RS) that provides additional throughput capacity flexibility has emerged. We develop a modular decomposition-based queuing network framework to analyze such systems. Our approach captures several distinguishing features of AVS/RS such as sequential rectilinear vehicle movement in a tier, service protocols for accessing resources, transaction requests competing for shared vertical transfer resources from multiple tiers, and resource synchronization requirements. We illustrate the use of this approach using two types of vertical transfer mechanisms: lifts and conveyors. The solution approach is efficient and scalable, and can accommodate a wide variety of design parameter settings such as different tier depth-to-width ratio, number of tiers, and number of vertical transfer units.

A key building block of the approach is the detailed model of the horizontal movement dynamics within a tier. Each tier is modeled as an SOQN to capture the transaction waiting times for vehicles. To ensure computational tractability of a system with multiple tiers, each tier is modeled in an aggregate way as a single load-dependent queue, with the service rate for this queue being obtained from the analysis of the respective SOQNs.

The vertical transfer subsystem is modeled as a multi-class queuing network with

$GI/G/1$ queues corresponding to different vertical transfer segments. An analysis of the entire system requires effectively capturing the linkage between arrivals and departures in the tier subsystem and vertical transfer units. To do so, we develop approximations using embedded Markov chain analysis to estimate the first and second moments of inter-departure times from the load-dependent queue present in the semi-open queue. Then, using a detailed departure process analysis and a novel linking algorithm, the models are solved. Detailed simulations are carried out to show the efficacy of the analytical model. A comparison of the results with simulation shows that the errors are low. Our approximations for the departure process in SOQN and the methodology for linking multiple SOQNs also addresses a major limitation in the current state-of-the-art SOQN literature.

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Online Supplemental Material

A Embedded Markov Chain Analysis for SS-SOQN

In this appendix, the transition probabilities from X_i to X_j are described for the four regions corresponding to transition probability matrix, P_D (Refer Section 4.

Region 1

$$P(X_i, X_j) = \frac{\mu_i(V)}{\lambda_{1_i} + \mu_i(V)} \text{ if } i_1 - j_1 = 1, i_2 = 1, j_2 = 1, i_1 >= 0$$

$$P(X_i, X_j) = \frac{\mu_i(V)}{\lambda_{2_i} + \mu_i(V)} \text{ if } i_1 - j_1 = 1, i_2 = 2, j_2 = 2, i_1 >= 0$$

$$P(X_i, X_j) = \frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(V)} \frac{\mu_i(V)}{\lambda_{2_i} + \mu_i(V)} \text{ if } i_1 - j_1 = 1, i_2 = 1, j_2 = 2, i_1 >= 0$$

$$P(X_i, X_j) = \frac{\lambda_{2_i}}{\lambda_{2_i} + \mu_i(V)} \frac{\mu_i(V)}{\lambda_{1_i} + \mu_i(V)} \text{ if } i_1 - j_1 = 0, i_2 = 2, j_2 = 1, i_1 >= 0$$

$$P(X_i, X_j) = \theta_1^{|i_1 - j_1 + 1|} \frac{\mu_i(V)}{\lambda_{1_i} + \mu_i(V)} \text{ if } i_1 - j_1 <= 0, i_2 = 1, j_2 = 1, i_1 >= 0$$

$$P(X_i, X_j) = \theta_1^{|i_1 - j_1 + 1|} \frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(V)} \frac{\mu_i(V)}{\lambda_{1_i} + \mu_i(V)} \text{ if } i_1 - j_1 <= 0, i_2 = 1, j_2 = 2, i_1 >= 0$$

$$P(X_i, X_j) = \theta_1^{|i_1 - j_1|} \frac{\lambda_{2_i}}{\lambda_{2_i} + \mu_i(V)} \frac{\mu_i(V)}{\lambda_{1_i} + \mu_i(V)} \text{ if } i_1 - j_1 < 0, i_2 = 2, j_2 = 1, i_1 >= 0$$

$$P(X_i, X_j) = \theta_1^{|i_1 - j_1|} \frac{\lambda_{2_i}}{\lambda_{2_i} + \mu_i(V)} \frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(V)} \frac{\mu_i(V)}{\lambda_{2_i} + \mu_i(V)} \text{ if } i_1 - j_1 < 0, i_2 = 2, j_2 = 2, i_1 >= 0$$

where $\theta_1 = p_i \left(\frac{\lambda_{1_i}}{\lambda_{1_i} + \mu_i(V)} \frac{\lambda_{2_i}}{\lambda_{2_i} + \mu_i(V)} \right) + (1 - p_i) \left(\frac{\lambda_{1_i}}{\lambda_{1_i} + \mu_i(V)} \right)$

The expression for transition probability, $P(X_i, X_j)$, is now explained for $i_1 - j_1 < 0, i_2 = 2, j_2 = 2, i_1 >= 0$. Since i_1 is less than j_1 there are $j_1 - i_1 + 1$ arrival events by the next departure instant. This requires completion of phase 2 of the first arrival, followed by completion of $j_1 - i_1$ arrivals, followed by completion of phase 1 of the $(j_1 - i_1 + 2)^{th}$

arrival, and finally completion of the service process prior to the completion of phase 2 of the $(j_1 - i_1 + 2)^{th}$ arrival. This yields $P(X_i, X_j)$ is $\theta_1^{|i_1 - j_1|} \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(V)} \frac{\lambda_{1_i} p_i}{\lambda_{1_i}+\mu_i(V)} \frac{\mu_i(V)}{\lambda_{2_i}+\mu_i(V)}$.

Region 2

$$P(X_i, X_j) = \theta_2(|i_1 - j_1|) \frac{\mu_i(min(|i_1 - j_1| + 1, V))}{\lambda_{1_i} + \mu_i(min(|i_1 - j_1| + 1, V))}$$

$$\text{if } (i_1 = -V, i_2 \in \{1, 2\}, j_2 = 1 \text{ or } i_1 = -V + 1, i_2 = 1, j_2 = 1)$$

$$\text{and } j_1 \in \{-V, \ldots, K - 1\}$$

$$P(X_i, X_j) = \theta_2(|i_1 - j_1|) \frac{\mu_i(min(|i_1 - j_1| + 1, V))}{\lambda_{2_i} + \mu_i(min(|i_1 - j_1| + 1, V))} \frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(min(|i_1 - j_1| + 1, V))}$$

$$\text{if } (i_1 = -V, i_2 \in \{1, 2\}, j_2 = 2 \text{ or } i_1 = -V + 1, i_2 = 1, j_2 = 2)$$

$$\text{and } j_1 \in \{-V, \ldots, K - 2\}$$

where $\theta_2(n) = \prod_{i=1}^{n} p_i \left(\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(i)} \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(i)} \right) + (1 - p_i) \left(\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(i)} \right)$, and

$$\mu_i(i) = \begin{cases} \mu_i(V) & i > V \\ \mu_i(i) & \text{otherwise} \end{cases}$$

The expression for transition probability, $P(X_i, X_j)$, is now explained for $i_1 = -V, i_2 \in \{1, 2\}, j_1 \in \{-V, \ldots, K - 2\}, j_2 = 2$. After the first transaction arrives, the number of additional transactions that arrive before the next departure is $j_1 - i_1$. The probability of an arrival prior to service completion is given by the expression $p_i \left(\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(i)} \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(i)} \right) + (1 - p_i) \left(\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(i)} \right)$ where i is the number of vehicles processing transactions at S_i. Since all arrival events are independent of each other, the joint probability of $|i_1 - j_1|$ arrivals is given by the term $\theta_2(|i_1 - j_1|)$. Since j_2 is 2, the $(j_1 - i_1 + 1)^{th}$ arrival must complete phase 1 of the arrival process and enter phase 2 before the next departure. This probability is given by the

expression $\frac{\lambda_{1_i} p_i}{\lambda_{1_i}+\mu_i(min(|i_1-j_1|+1,V))}$ where $min(|i_1-j_1|+1, V)$ is the number of vehicles present at the load-dependent station. However, the departure occurs before the phase 2 arrival process is complete and this happens with probability $\frac{\mu_i(min(|i_1-j_1|+1,V))}{\lambda_{2_i}+\mu_i(min(|i_1-j_1|+1,V))}$. By taking a product of these event probabilities, the final expression for $P(X_i, X_j)$ is determined.

Region 3

$$P(X_i, X_j) = \frac{\mu_i(min(V+i_1, V))}{\lambda_{1_i} + \mu_i(min(V+i_1, V))}$$

$$\text{if } i_1 > -V+1, i_1 < 0, i_1 - j_1 = 1, i_2 = 1, j_2 = 1$$

$$P(X_i, X_j) = \frac{\mu_i(min(V+i_1, V))}{\lambda_{2_i} + \mu_i(min(V+i_1, V))}$$

$$\text{if } i_1 > -V+1, i_1 < 0, i_1 - j_1 = 1, i_2 = 2, j_2 = 2$$

$$P(X_i, X_j) = \frac{\mu_i(min(V+i_1, V))}{\lambda_{2_i} + \mu_i(min(V+i_1, V))} \frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(min(V+i_1, V))}$$

$$\text{if } i_1 > -V+1, i_1 < 0, i_1 - j_1 = 1, i_2 = 1, j_2 = 2$$

$$P(X_i, X_j) = \theta_3(|i_1 - j_1 + 1|, V+i_1) \frac{\mu_i(min(|i_1 - j_1|, V))}{\lambda_{1_i} + \mu_i(min(|i_1 - j_1|, V))}$$

$$\text{if } i_1 > -V+1, i_1 < 0, i_1 - j_1 < 1, i_2 = 1, j_2 = 1$$

$$P(X_i, X_j) = \theta_3(|i_1 - j_1 + 1|, V+i_1) \frac{\mu_i(min(j_1 + V + 1, V))}{\lambda_{2_i} + \mu_i(min(j_1 + V + 1, V))}$$

$$\frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(min(j_1 + V + 1, V))} \text{ if } i_1 > -V+1, i_1 < 0, i_1 - j_1 < 1, i_2 = 1, j_2 = 2$$

$$P(X_i, X_j) = \theta_3(|j_1 - i_1|, V+i_1+1) \frac{\mu_i(min(i_1 + V + 1, V))}{\lambda_{1_i} + \mu_i(min(i_1 + V + 1, V))}$$

$$\frac{\lambda_{2_i}}{\lambda_{2_i} + \mu_i(min(i_1 + V, V))} \text{ if } i_1 >= -V+1, i_1 < 0, i_1 - j_1 < 1, i_2 = 2, j_2 = 1$$

$$P(X_i, X_j) = \theta_3(|j_1 - i_1|, V+i_1+1) \frac{\mu_i(min(j_1 + V + 1, V))}{\lambda_{1_i} + \mu_i(min(j_1 + V + 1, V))} \frac{\lambda_{2_i}}{\lambda_{2_i} + \mu_i(min(i_1 + V, V))}$$

$$\frac{\lambda_{1_i} p_i}{\lambda_{1_i} + \mu_i(min(j_1 + V + 1, V))} \text{ if } i_1 >= -V+1, i_1 < 0, i_1 - j_1 < 1, i_2 = 2, j_2 = 2$$

where $\theta_3(n, v_1) = \prod_{j=1:n} p_i \left(\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(j)} \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(j)}\right) + (1-p_i)\left(\frac{\lambda_{1_i}}{\lambda_{1_i}+\mu_i(j)}\right)$,

v_1 is the number of vehicles present in the load-dependent station corresponding to state X_i, and

$$\mu_i(j) = \begin{cases} \mu_i(V) & j + v_1 + 1 > V \\ \mu_i(j + v_1) & \text{otherwise} \end{cases}$$

The expression for transition probability, $P(X_i, X_j)$, is now explained for $i_1 >= -V+1, i_1 < 0, i_1 - j_1 < 1, i_2 = 2, j_2 = 2$. In this case, there are one or more vehicles already in service at S_i. The number of additional arrivals that occur before the next departure is $j_1 - i_1 + 1$. At the previous departure instant, the impending arrival was in phase 2 of arrival process. Therefore, the probability that this arrival occurs before the service completion is $\frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(min(i_1+V,V))}$. Note that the service rate of the load-dependent station is $\mu_i(min(i_1 + V, V))$ because the number of vehicles present at S_i is $min(i_1 + V, V)$. The probability that $j_1 - i_1$ additional arrivals occur prior to the service completion is $\theta_3(|j_1 - i_1|, V + i_1 + 1)$ where $V + i_1 + 1$ is the number of vehicles present at the load-dependent station. Further, $(j_1 - i_1 + 2)^{th}$ arrival must complete phase 1 of the arrival process and join phase 2 before the departure. This probability is given by the expression $\frac{\lambda_{1_i} p_i}{\lambda_{1_i}+\mu_i(min(j_1+V+1,V))}$ where $min(j_1 + V + 1, V)$ is the number of vehicles present at the load-dependent station before the departure instant. However, the departure occurs before the phase 2 arrival process is complete. This probability is given by the expression $\frac{\mu_i(min(j_1+V+1,V))}{\lambda_{1_i}+\mu_i(min(j_1+V+1,V))}$. By taking a product of these event probabilities, the final expression for $P(X_i, X_j)$ is obtained as $\theta_3(|j_1 - i_1|, V + i_1 + 1) \frac{\mu_i(min(j_1+V+1,V))}{\lambda_{1_i}+\mu_i(min(j_1+V+1,V))} \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(min(i_1+V,V))} \frac{\lambda_{1_i} p_i}{\lambda_{1_i}+\mu_i(min(j_1+V+1,V))}$.

Region 4

$$P(X_i, X_j) = 1 - \sum_{j_1=-V}^{K-2} \sum_{j_2=1}^{2} P(X_i, X_j) - P(X_i, (K-1, 1))$$

where $X_i \in S_D$ and $X_j = (K-1, 2)$

The expression for transition probability, $P(X_i, X_j)$, is now explained for $X_i \in S_D$ and $X_j = (K-1, 2)$. There are K vehicles before the service completion and another arrival is in phase 2 of the arrival process. To estimate $P(X_i, X_j)$, we use the law of total probability, i.e, the sum of all transition probabilities from a X_i to $X_j : X_j \in S_D$ is 1.

B Inter-departure Time Analysis

The following paragraphs describe the analysis of departure process for states in G_2, G_3, G_4, and G_5.

Departure Analysis for States in G_2: If a departure leaves the system in state $(-V, 2)$, the following events need to occur for the subsequent departure. First, a transaction should arrive and then this transaction's service needs to be completed. Since the arrival is already in phase 2, let the notation A_2 reflect a transaction arrival from the second phase. Similar to the analysis of states in G_1, the service completion time could vary depending on the number of vehicles present at the load-dependent station S_i affecting $\mu_i(n)^{-1}$. Therefore, one of the following sequence of events could occur before the next departure instant.

- $\mathbb{S}_1^2 = (A_2, S^{'})$: Completion of the second phase of an arrival followed by a service completion by rate $\mu_i(1)$.
- $\mathbb{S}_2^2 = (A_2, A, S^{'})$: Completion of two arrivals (second phase of the first and both phases of the second) followed by a service completion. A part of the service is completed at rate $\mu_i(1)$ and the residual service time requirement is completed at a rate $\mu_i(2)$.
- $\mathbb{S}_V^2 = (A_2, A, \ldots, A, S^{'})$: Completion of V arrivals (second phase of the first and both phases of the remaining arrivals) followed by a service completion. The first part of the service is completed at rate $\mu_i(1)$, the second part of the service is completed at rate $\mu_i(2)$. Likewise, the $(V-1)^{th}$ part of the service time is completed at rate $\mu_i(V-1)$ and the residual service time is completed at a rate $\mu_i(V)$.

Let $p_{\mathbb{S}_e^2}$ denote the probabilities of these sequence of events. Then the relationship for the lower bound of the first and second moments of the inter-departure time is obtained as

follows.

$$\sum_{\mathbb{S}_e^2 \in s} p_{\mathbb{S}_e^2} E[D_i|\mathbb{S}_e^2]_l = E[D_i|s \in G_2]_l \leq E[D_i|s \in G_2] \tag{38}$$

$$\sum_{\mathbb{S}_e^2 \in s} p_{\mathbb{S}_e^2} E[D_i^2|\mathbb{S}_e^2]_l = E[D_i^2|s \in G_2]_l \leq E[D_i^2|s \in G_2] \tag{39}$$

$$\tag{40}$$

Departure Analysis for States in G_3: For states in G_3, the previous departure leaves one or more transactions at S_i. Therefore, a transaction immediately begins its service after the previous transaction departure. Hence, the time to the next departure equals the time to complete one service. Similar to the analysis of states in sets G_1 and G_2, the service completion time would vary depending on the number of vehicles present at the load-dependent station S_i affecting $\mu_i(n)^{-1}$ and during their service time multiple arrivals could occur leading to the following sequence of events. The sequence of events and further analysis results are described with respect to state $(-V+1, 1)$.

- $\mathbb{S}_1^3 = (S^{'})$: Service completes at rate $\mu_i(1)$.
- $\mathbb{S}_2^3 = (A, S^{'})$: One arrival followed by a service completion. A part of the service is completed at rate $\mu_i(1)$ and the residual service time requirement is completed at a rate $\mu_i(2)$.
- $\mathbb{S}_V^3 = (A, \ldots, A, S^{'})$: Likewise, $V-1$ arrivals followed by a service completion. The first part of the service is completed at rate $\mu_i(1)$, the second part of the service is completed at rate $\mu_i(2)$. Likewise, the $(V-1)^{th}$ part of the service time is completed at rate $\mu_i(V-1)$ and the residual service time is completed at a rate $\mu_i(V)$.

Let $p_{\mathbb{S}_e^3}$ denote the probabilities of these sequence of events. Then the relationship for the

lower bound of the first and second moments of the inter-departure time is obtained as follows.

$$\sum_{\mathbb{S}_e^3 \in s} p_{\mathbb{S}_e^3} E[D_i|\mathbb{S}_e^3]_l = E[D_i|s \in G_3]_l \leq E[D_i|s \in G_3] \tag{41}$$

$$\sum_{\mathbb{S}_e^3 \in s} p_{\mathbb{S}_e^3} E[D_i^2|\mathbb{S}_e^3]_l = E[D_i^2|s \in G_3]_l \leq E[D_i^2|s \in G_3] \tag{42}$$

$$\tag{43}$$

Departure Analysis for States in G_4: Similar to states in G_3, the previous departure leaves one or more transactions at S_i. However, the next arrival is already in phase 2. Therefore, although a transaction immediately begins service after the previous departure, a departure occurs when service is complete. Similar to the analysis of states in G_1, G_2, G_3, the service completion time would vary depending on the number of vehicles present at the load-dependent station S_i affecting $\mu_i(n)^{-1}$ and during this time multiple arrivals could occur leading to the following sequence of events.The sequence of events and further analysis is described with respect to state $(-V+1, 2)$.

- $\mathbb{S}_1^4 = (S^{'})$: Service completes at rate $\mu_i(1)$.
- $\mathbb{S}_2^4 = (A_2, S^{'})$: Completion of phase 2 of the first arrival followed by a service completion. A part of the service is completed at rate $\mu_i(1)$ and the residual service time requirement is completed at a rate $\mu_i(2)$. The probability that phase 2 of arrival completes prior to completion of service, denoted by $\hat{q}_n$, is given by $P[Y_2 \leq Z_n] = \frac{\lambda_{2_i}}{\lambda_{2_i}+\mu_i(n)}$.
- $\mathbb{S}_V^4 = (A_2, A, \ldots, A, S^{'})$: Completion of phase 2 of the first arrival followed by completion of both phases of $(V-2)$ arrivals before the completion of service. The first part

of the service is completed at rate $\mu_i(1)$, the second part of the service is completed at rate $\mu_i(2)$. Likewise, the $(V-1)^{th}$ part of the service time is completed at rate $\mu_i(V-1)$ and the residual service time is completed at a rate $\mu_i(V)$.

Let $p_{\mathbb{S}_e^4}$ denote the probabilities of these sequence of events. Then the relationship for the lower bound of the first and second moments of the inter-departure time is obtained as follows.

$$\sum_{\mathbb{S}_e^4 \in s} p_{\mathbb{S}_e^4} E[D_i|\mathbb{S}_e^4]_l = E[D_i|s \in G_4]_l \leq E[D_i|s \in G_4] \tag{44}$$

$$\sum_{\mathbb{S}_e^4 \in s} p_{\mathbb{S}_e^4} E[D_i^2|\mathbb{S}_e^4]_l = E[D_i^2|s \in G_4]_l \leq E[D_i^2|s \in G_4] \tag{45}$$

$$\tag{46}$$

Analysis for a departure from states in G_5: Since the previous departure leaves V transactions to be processed S_i, for states in G_5, a transaction that immediately begins service after the previous departure. This service is not interrupted by future arrivals. Hence, the time to the next departure only depends on the completion of service with rate $\mu_i(V)$. (see Equations 47 and 48).

$$E[D_i|s \in G_5]_l = \frac{1}{\mu_i(V)} \tag{47}$$

$$E[D_i^2|s \in G_5]_l = \frac{2}{\mu_i(V)^2} \tag{48}$$

The first and second moment expressions are provided in Table 8.

Table 8: First and second moment expressions of the inter-departure times

State, Set	**Event Sequence** ($\mathbb{S}_e$)	**Probability** ($p_{\mathbb{S}_e}$)	$E[D_i\|\mathbb{S}_e]_l$	$E[D_i^2\|\mathbb{S}_e]_l$
$(-V,1) \in G_1$	A, S'	$1-q_1$	$p_i\left(E[Y_1]+E[Y_2]+E[Z_1]\right)+(1-p_i)\left((E[Y_1]+E[Z_1])\right)$	$\left[p_i\left((Var[Y_1]+Var[Y_2]+Var[Z_1])+(E[Y_1]+E[Y_2]+E[Z_1])^2\right)\right]$ $+\left[(1-p_i)\left((Var[Y_1]+Var[Z_1])+(E[Y_1]+E[Z_1])^2\right)\right]$
	A, A, S'	$q_1(1-q_2)$	$p_i\left((E[Y_1]+E[Y_2]+E[Z_2])\right)+(1-p_i)\left((E[Y_1]+E[Z_2])\right)$	$\left[p_i\left((Var[Y_1]+Var[Y_2]+Var[Z_2])+(E[Y_1]+E[Y_2]+E[Z_2])^2\right)\right]$ $+\left[(1-p_i)\left((Var[Y_1]+Var[Z_2])+(E[Y_1]+E[Z_2])^2\right)\right]$
	.	.	.	.
	.	.	.	.
	$A, \ldots, A, S'$	$q_1q_2\ldots q_{V-1}$	$p_i\left((E[Y_1]+E[Y_2]+E[Z_V])\right)+(1-p_i)\left((E[Y_1]+E[Z_V])\right)$	$\left[p_i\left((Var[Y_1]+Var[Y_2]+Var[Z_V])+(E[Y_1]+E[Y_2]+E[Z_V])^2\right)\right]$ $+\left[(1-p_i)\left((Var[Y_1]+Var[Z_V])+(E[Y_1]+E[Z_V])^2\right)\right]$
$(-V,2) \in G_2$	A_2, S'	$1-q_1$	$(E[Y_2]+E[Z_1])$	$(Var[Y_2]+Var[Z_1])+(E[Y_2]+E[Z_1])^2$
	A_2, A, S'	$q_1(1-q_2)$	$(E[Y_2]+E[Z_2])$	$(Var[Y_2]+Var[Z_2])+(E[Y_2]+E[Z_2])^2$
	.	.	.	.
	.	.	.	.
	$A_2, A, \ldots, A, S'$	$q_1q_2\ldots q_{V-1}$	$(E[Y_2]+E[Z_V])$	$(Var[Y_2]+Var[Z_V])+(E[Y_2]+E[Z_V])^2$
$(-V+1,1) \in G_3$	S'	$1-q_1$	$E[Z_1]$	$E[Z_1]^2$
	A, S'	$q_1(1-q_2)$	$E[Z_2]$	$E[Z_2]^2$
	.	.	.	.
	.	.	.	.
	$A, A, \ldots, A, S'$	$q_1q_2\ldots q_{V-1}$	$E[Z_V]$	$E[Z_V]^2$
$(-V+1,2) \in G_4$	S'	$1-\hat{q}_1$	$E[Z_1]$	$E[Z_1]^2$
	A_2, S'	$\hat{q}_1(1-q_2)$	$E[Z_2]$	$E[Z_2]^2$
	.	.	.	.
	.	.	.	.
	$A_2, A, \ldots, A, S'$	$\hat{q}_1q_2\ldots q_{V-1}$	$E[Z_V]$	$E[Z_V]^2$
$\forall s \in G_5$	S'	1	$E[Z_V]$	$E[Z_V]^2$

C Analysis of Vertical Movements with Lifts

In AVS/RS with lift mechanism, lifts are used to transfer pallets in the vertical direction instead of conveyors. Similar to the conveyor-based system, the lift-based system with T tiers can be divided into a single lift subsystem and T individual tier subsystems (see Figure 4a). The lift dwells at the point of service completion, i.e., after processing a storage transaction it dwells near the LU point of the destination tier, whereas after processing a retrieval transaction it dwells at the LU point of the tier 1.

Figure 4b shows the queuing network model for a system with three tiers that are linked by a lift unit, denoted by $\widehat{L}_1$. All tiers except the first tier is linked to the lift queue. The queuing models for the tier subsystems are identical to the conveyor-based system. But, the queuing model for the lift subsystem differs from the queuing model for the conveyor subsystem. Note that unlike multiple conveyor queues in conveyor subsystem, the lift subsystem has only one queue i.e., all transactions that require storage or retrievals with destination tier greater than 1 uses the shared lift subsystem. Further, the difference lies in estimating the service time at the $GI/G/1$ queue.

For vertical travel, lift is used for transactions with storage or retrieval destination tiers in $2, \ldots, T$. Since the service times of the lift vary depending on the type of transaction, dwell point of the lift and the destination tier location, each transaction type with a different expected lift service time is modeled as a different class. Therefore, there are T transaction classes corresponding to the storage transaction and T transaction classes corresponding to the retrieval transaction. The index i in storage and retrieval classes (s_i and r_i): $i = 1, 2, \ldots, T$ correspond to tiers $1, 2, \ldots, T$. Let C denote the set of all storage and retrieval classes. The inter-arrival times for the storage transactions is exponential with parameter $\lambda^{-1}_{a_{s_i,\widehat{L}_1}}$, where s_i is the class of storage transaction. Similar to the conveyor subsystem,

the inter-arrival times for the retrieval transactions have a general distribution with mean $\lambda^{-1}_{a_{r_i,\widehat{L}_1}}$. Note that the lift is not used by class 1 transactions (s_1 and r_1). Therefore, the lift is modeled as an open $GI/G/1$ queue with $2(T-1)$ transaction classes, general inter-arrival times for retrieval class, exponential inter-arrival times for storage class, and general service times with means $E[S_{r_i}]$ and $E[S_{s_i}]$ corresponding to class index r_i and s_i of retrieval and storage transactions respectively.

The lift service times, which correspond to the vertical travel time components, vary depending on the originating and the destination tier number of the lift. Therefore, the inputs to determine the lift service times for each class of transaction are the originating tier index, the destination tier number of the lift, and the probability mass function of the lift's originating tier index. The distance between any two tiers (i and j) is expressed by the absolute value of the difference between the tier numbers ($|i-j|$) and multiplying the difference by the height of a tier. Depending on the dwell point location of the lift, the lift could originate from any of the tiers. Let $p(i)$ represent the probabilities that the lift originates from tier $i : i = 1, 2, \ldots, T$. Therefore, $p(1)$ and $p(i > 1) = \sum_{i=2}^{T} p(i)$ denote the probabilities that the lift originates from tier 1 and the remaining tiers respectively. Therefore, the probability that the lift originates from a particular tier $i : i > 1$ is $p(i|i > 1)p(i > 1) = \frac{1}{T-1}p(i > 1)$. Since, the lift adopts a point of service completion dwell point policy, it dwells at tier 1 after completing a retrieval transaction and dwells at the destination tier $(2, \ldots, T)$ after completing a storage transaction. Further, if $\lambda_{s_i} = \lambda_{r_i}\ \forall i \in \{2, \ldots, T\}$, then it is expected that the probability of dwelling at any tier i is equally likely. Therefore, in this model, $p(1) = p(i > 1) = 0.5$. Equations 49 and 50 provide the expressions to calculate the lift service times for retrieval and storage transactions, respectively where $i = 2, \ldots, T$.

$$E[S_{r_i}] = \frac{p(i>1)}{T-1}\left\{\sum_{j=2}^{T}\frac{u_h}{v_l}\left(|i-j|+i-1\right)\right\} + p(1)\left\{\frac{2(i-1)u_h}{v_l}\right\} + L_{lt} + U_{lt} \quad (49)$$

$$E[S_{s_i}] = \frac{p(i>1)}{T-1}\left\{\sum_{j=2}^{T}\frac{u_h}{v_l}\left(i+j-2\right)\right\} + p(1)\left\{\frac{(i-1)u_h}{v_l}\right\} + L_{lt} + U_{lt} \quad (50)$$

The second moment of the service times for all transaction classes is based on Bayes' theorem, which relies on the property that the second moment of a mixture of distributions is the mixture of the second moments (Equations 51 and 52).

$$E[S_{r_i}^2] = \frac{p(i>1)}{T-1}\sum_{j=2}^{T}\left(\frac{u_h}{v_l}\left(|i-j|+i-1\right)+L_{lt}+U_{lt}\right)^2 + p(1)\left(\frac{2(i-1)u_h}{v_l}+L_{lt}+U_{lt}\right)^2 \quad (51)$$

$$E[S_{s_i}^2] = \frac{p(i>1)}{T-1}\sum_{j=2}^{T}\left(\frac{u_h}{v_l}\left(i+j-2\right)+L_{lt}+U_{lt}\right)^2 + p(1)\left(\frac{(i-1)u_h}{v_l}+L_{lt}+U_{lt}\right)^2 \quad (52)$$

In Equations 49-52, the notations u_h, v_l, L_{lt}, and U_{lt} denote the height of each tier, vertical velocity of the lift, and load/unload times of the lift respectively. The queuing model is solved using $GI/G/1$ queue with multiple customer classes.

D Cycle Time Expressions in AVS/RS

The cycle time expressions to complete storage and retrieval transactions in tier i for a conveyor-based AVS/R system are given in Equations 53 and 54. The cycle time of a transaction is composed of waiting time for resource, blocking delays, and horizontal and vertical travel time components. Let W_V denote the waiting time to access a free vehicle

and W_{c_i} denote the waiting time to access a free conveyor loop i. Let W_{clu} and W_{crk} denote the blocking delays at the cross-aisle when the vehicle is traveling from LU point to the racks and from racks to the LU point respectively. We denote W_{a_s} and W_{a_r} as the blocking delays at the aisles for storage and retrieval transactions respectively. Note that the storage transaction is assumed to be complete when the pallet is unloaded at the storage location. Hence, the blocking delay experienced by the vehicle in the aisle during its return travel is not included in the storage cycle time expression. However, the total blocking delay in the aisle is included in the cycle time expression for retrieval transactions. For vertical travel using conveyor segment i, the terms t, W_{c_i}, and t_c denote the storage/retrieval tier, the waiting time for conveyor loop i, and the travel time in each loop. The horizontal travel times include traveling from vehicle dwell point (x_{lu}, y_{lu}, which are x and y coordinates of LU point) to the storage/retrieval location (x_s, y_s or x_r, y_r) with a velocity v_h. Let L_{vt} and U_{vt} denote the time to load and unload the pallet by a vehicle.

$$\begin{aligned} CT_{s_c}(i) &= \sum_{j=1}^{i-1}(W_{c_j}+t_c) + W_V + L_{vt} + W_{clu} + \left|\frac{x_{lu}-x_s}{v_h}\right| + W_{a_s} \\ &+ \left|\frac{y_{lu}-y_s}{v_h}\right| + U_{vt} \qquad (53) \\ CT_{r_c}(i) &= W_V + W_{clu} + \left|\frac{x_{lu}-x_r}{v_h}\right| + \left|\frac{y_{lu}-y_r}{v_h}\right| + L_{vt} + W_{a_r} \\ &+ \left|\frac{y_r-y_{lu}}{v_h}\right| + W_{crk} + \left|\frac{x_r-x_{lu}}{v_h}\right| + U_{vt} + \sum_{j=1}^{i-1}(W_{c_j}+t_c) \qquad (54) \end{aligned}$$

For AVS/RS with lift mechanism, Equations 55 and 56 provide cycle time expressions (CT_{s_l}, CT_{r_l}) for storage and retrieval transactions respectively. For vertical travel using lift mechanism, the travel times include traveling from lift dwell point (z_{vd}) to LU point (z_{lu}), and from LU point to the destination tier (z_s or z_r). The load (unload) times for the

vehicles and lift are L_{vt} (U_{vt}) and L_{lt} (U_{lt}) respectively. The vehicle and the lift waiting times are denoted by W_V and W_L respectively. Note that unlike cycle time expressions (Equations 53 and 54) for the conveyor system, there are additional vertical travel time components such as lift travel time from its dwell point to the LU point.

$$\begin{aligned} CT_{s_l} &= W_L + \left|\frac{z_{vd} - z_{lu}}{v_l}\right| + L_{lt} + \left|\frac{z_{lu} - z_s}{v_l}\right| + U_{lt} + W_V + L_{vt} + W_{clu} + \left|\frac{x_{lu} - x_s}{v_h}\right| \\ &+ W_{a_s} + \left|\frac{y_{lu} - y_s}{v_h}\right| + U_{vt} \end{aligned} \tag{55}$$

$$\begin{aligned} CT_{r_l} &= W_V + W_{clu} + \left|\frac{x_{lu} - x_r}{v_h}\right| + \left|\frac{y_{lu} - y_r}{v_h}\right| + L_{vt} + W_{a_r} + \left|\frac{y_r - y_{lu}}{v_h}\right| + W_{crk} \\ &+ \left|\frac{x_r - x_{lu}}{v_h}\right| + U_{vt} + W_L + \left|\frac{z_{vd} - z_r}{v_l}\right| + L_{lt} + \left|\frac{z_r - z_{lu}}{v_l}\right| + U_{lt} \end{aligned} \tag{56}$$

E Flowchart of the Linking Algorithm

The flowchart used for linking the models corresponding to the horizontal and vertical movements is provided in Figure 7.

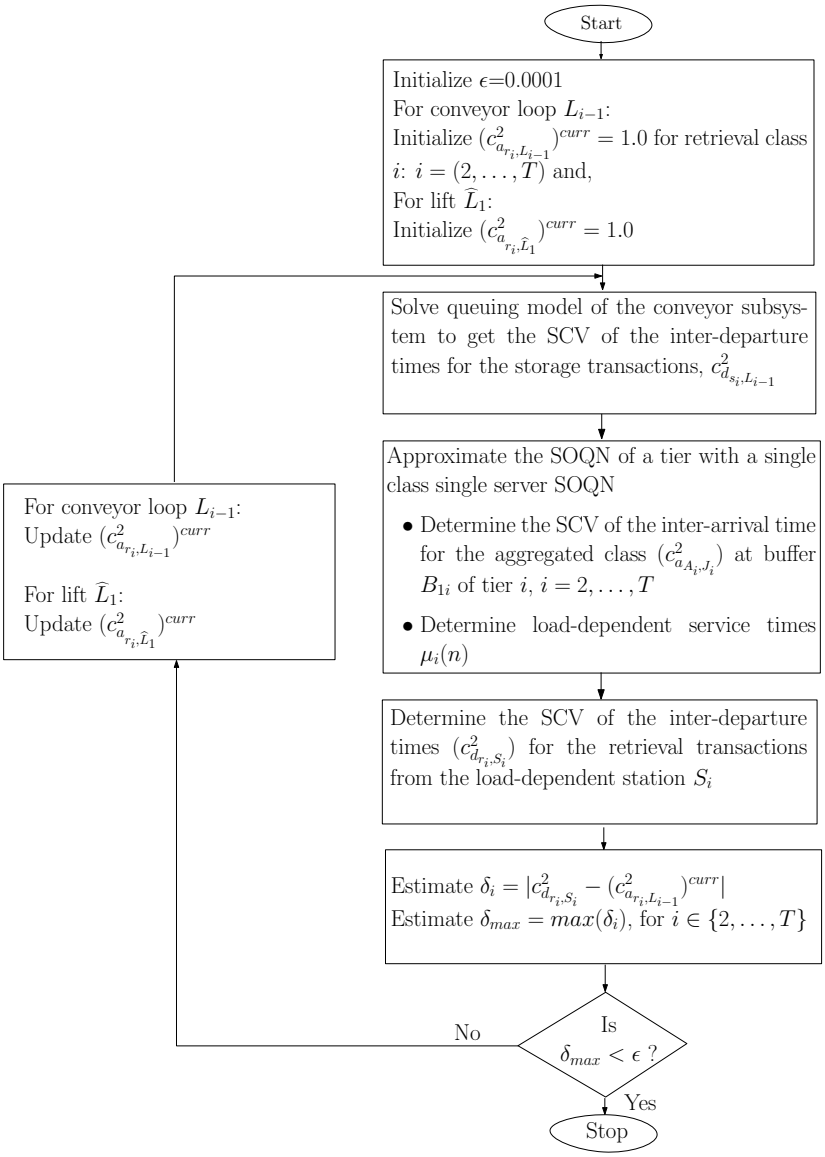
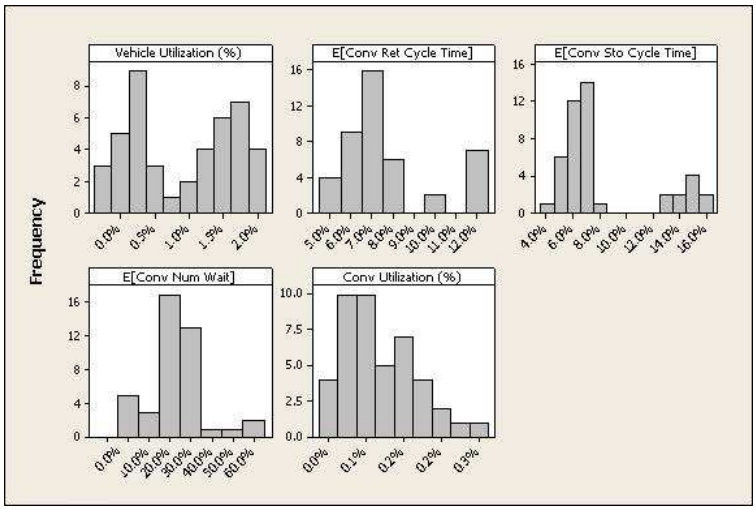
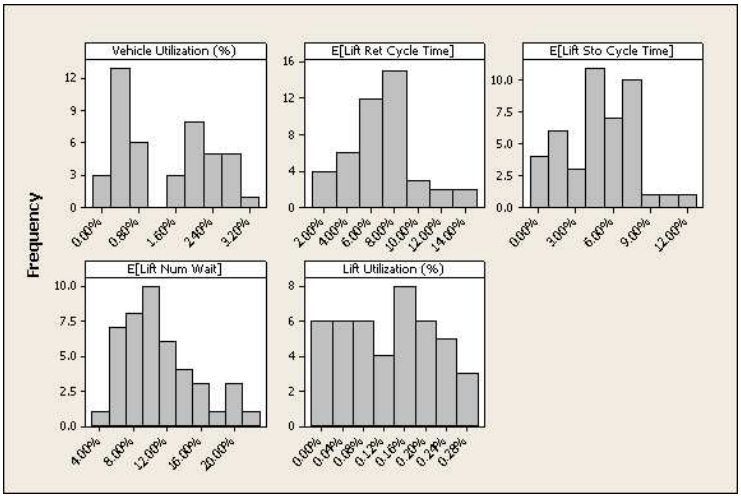


Figure 7: Flowchart to link the systems modeling horizontal and vertical movements

F Model Error Distribution



(a)



(b)

Figure 8: Summary of errors for (a) the conveyor-based system (the histograms - top (left to right) and bottom (left to right) correspond to the five measures: U_V, $E[CT_{cr}]$, $E[CT_{cs}]$, Q_C, and U_C) and (b) lift-based system (the histograms - top (left to right) and bottom (left to right) correspond to the five measures: U_V, $E[CT_{lr}]$, $E[CT_{ls}]$, Q_L, and U_L)