Finding Reliable Solutions in Bilevel Optimization Problems Under Uncertainties

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ABSTRACT

Bilevel optimization problems are referred to as having a nested inner optimization problem as a constraint to a outer optimization problem in the domain of mathematical programming. It is also known as Stackelberg problems in game theory. In the recent past, bilevel optimization problems have received a growing attention because of its relevance in practice applications. However, the hierarchical structure makes these problems difficult to handle and they are commonly optimized with a deterministic setup. With presence of constrains, bilevel optimization problems are considered for finding reliable solutions which are subjected to a possess a minimum reliability requirement under decision variable uncertainties. Definition of reliable bilevel solution, the effect of lower and upper level uncertainties on reliable bilevel solution, development of efficient reliable bilevel evolutionary algorithm, and supporting simulation results on test and engineering design problems amply demonstrate their further use in other practical bilevel problems.

1. INTRODUCTION

Bilevel optimization problems are characterised by a hierarchical two level structure, in which there is a nested inner optimization problem as a constraint to a outer optimization problem. The outer optimization problem is often referred as the upper level task, and the inner optimization problem is often referred as the lower level task. Bilevel optimization problems are typically challenging to handle because the feasibility of a solution (upper level) can only be determined after another optimization process (lower level). However, due to bilevel optimization problem's relevance in practice [6, 15], it has received a growing attention from researchers across various disciplines in the recent past.

In the domain of evolutionary computation, researchers have shown strong interests in solving bilevel optimization problems because evolutionary algorithms' flexibility and population approach offer a viable mean to overcome various difficulties from bilevel optimization problems' nested structure, many researching effort could be found in [5, 1].

In practical applications, constraints are most likely presented. Under this scenario, a deterministic optimal solution is most likely to lie on one or more constraint boundaries and an uncertainty in variables may cause the solution to become infeasible in most occasions. In such cases, a solution that lies well inside the feasible search region and satisfies all constraints with a pre-specified reliability against constraint violation is declared as the 'reliable' solution. Evolutionary algorithms have been adequately used to address uncertainties in variables for single-level optimization problems [3, 7, 10].

In bilevel optimization problems, uncertainties in variables can occur in both lower and upper levels or in one of the levels only. Since each level has an unequal importance to the overall outcome of a bilevel optimization task, the effect of uncertainties in upper and lower level variables on the final reliable solutions of the problem is also expected to be different. Thus, searching for reliable bilevel solutions is likely to be more complicated than the uncertainty-based studies for single-level optimization problems [3, 4]. In this pilot systematic study, we make an attempt to understand these effects and demonstrate their importance through a number of simple numerical test problems and then apply the developed method to an navy ship design problem.

In the remainder of this paper, we provide a brief introduction to the nature of bilevel optimization problems in Section 2. In Section 2.1, we discuss constrained bilevel problems under uncertainties and discuss modifications to EBO algorithms for solving such problems efficiently. Thereafter, in Section 3, we present results for single and multi-variable constrained bilevel problems. An application a navy ship design problem is presented next in Subsection 3.3. Finally, conclusions of this extensive study are made in Section 4.

2. DEFINITIONS

A bilevel optimization problem has two levels of optimization tasks that involve two sets of variables $\mathbf{x} \in R^n$ (upper level) and $\mathbf{y} \in R^m$ (lower level), described below:

$$\begin{aligned} \text{Minimize} \quad & F(\mathbf{x}, \mathbf{y}), \\ \text{subject to} \quad & \mathbf{y} = \text{argmin}\left\{ f(\mathbf{x}, \mathbf{y}) | g_j(\mathbf{x}, \mathbf{y}) \le 0,\ j = 1, \ldots, J_L \right\}, \\ & G_j(\mathbf{x}, \mathbf{y}) \le 0,\ j = 1, 2, \ldots, J_U. \end{aligned} \tag{1}$$

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The pair $(\mathbf{x}, \mathbf{y})$ represents a particular bilevel solution. The solution $(\mathbf{x}, \mathbf{y})$ is feasible (i) if $\mathbf{y}$ is an optimal solution of the lower level optimization problem of minimizing $f(\mathbf{x}, \mathbf{y})$ subject to satisfaction of J_L constraints $g_j(\mathbf{x}, \mathbf{y}) \leq 0$, and (ii) if it also satisfies J_U upper level constraints $G_j(\mathbf{x}, \mathbf{y}) \leq 0$. The optimality of upper level solution $(\mathbf{x}, \mathbf{y})$ is associated with an upper level objective function $F(\mathbf{x}, \mathbf{y})$. Due to the need for solving the lower level optimization problem for every upper level solution, the above bilevel optimization problem is also known as a *nested* optimization problem and is, in general, a computationally intensive task.

2.1 Proposed Definition of Reliable Bilevel Solutions

In this section, we consider constrained optimization problems for which deterministic optimal solution(s) lie on one or more constraint boundary. We extend the definition of single-level reliable solutions [4] to define reliable solutions for bilevel optimization problems.

Definition 1. **Reliable Bilevel Solution:** A solution $(\mathbf{x}, \mathbf{y})^{r1}$ is called a reliable bilevel solution, if it is the optimal solution to the following bilevel minimization problem defined with respect to a pair of desired reliability measures R and r for upper and lower levels, respectively:

$$\begin{array}{ll} \text{Minimize} & F(\mathbf{x}, \mathbf{y}), \\ \text{subject to} & \mathbf{y}^* = \text{argmin}\left\{ f(\mathbf{x}, \mathbf{y}) | (p(\bigwedge_{j=1}^{J_L} g_j(\mathbf{x}, \mathbf{y}) \geq 0)) \geq r \right\}, \\ & (P(\bigwedge_{j=1}^{J_U} G_j(\mathbf{x}, \mathbf{y}) \geq 0)) \geq R, \end{array} \tag{2}$$

where $\mathbf{x}$ and $\mathbf{y}$ denote mean of lower level variable $\mathbf{x}$ and upper level variable $\mathbf{y}$, respectively. The terms $p()$ and $P()$ signify the joint probability of the solution $\mathbf{x}$ and $\mathbf{y}$ being feasible from all J constraints (both lower level constraint g and upper level constraint G) under the uncertainty assumption. The quantities r and R are the desired reliability (within $[0, 1]$), respectively, for lower level and upper level constraints. Instead of the original constraint $g_j(\mathbf{x}, \mathbf{y}) \geq 0$, the probabilistic constraint is introduced. Hence for a desired pair of reliability measure (r and R), it is then expected to find a feasible bilevel solution that will ensure that the joint probabilities of satisfying all lower and upper level constraints are r and R, respectively. To determine a bilevel solution's reliability, ideally the reliability of the solution must be evaluated by examining whether the solution is adequately safe against all constraints. The simultaneous consideration of all constraints to obtain the joint probability for each level is mathematically and computationally challenging [9]. In this study, we simply break the above joint probability estimation into different chance constraints for upper and lower level, respectively, as follows:

$$P(G_j(\mathbf{x}, \mathbf{y}) \geq 0) \quad \geq \quad R_j, \quad j = 1, 2, ..., J_U, \tag{3}$$

$$p(g_k(\mathbf{x}, \mathbf{y}) \geq 0) \quad \geq \quad r_k, \quad k = 1, 2, ..., J_L, \tag{4}$$

where R_j and r_k are the desired probabilities of constraint satisfaction of the jth and k-th constraint in upper and lower level problem, respectively. The feasibility check in our implementation requires every constraint to be over the desired reliability measures. Readers should refer to [4] for a more elaborated discussion on the joint probability estimation.

From the perspective of optimization-based reliability measures, the underlying idea is to locate a *most probable point* (MPP) on the constraint boundary with a minimum distance from the current solution [8]. There are many ways by which the MPP point can be calculated or approximated. To illustrate, let's consider one of the approaches, namely *Performance measure approach* (PMA) [2]. We first convert the $\mathbf{x}$ coordinate system into a standard normal coordinate system $\mathbf{U}$, through the Rosenblastt transformation [11] and rewrite a constraint function $g_j(\mathbf{x})$ as $g_j^n(\mathbf{U})$. The standard normal random variables are characterized by a zero mean and unit variance. In this space, we approximate the hyper-surface ($g_j(\mathbf{x})$ = 0 or equivalently $g_j^n(\mathbf{U})$ = 0) by a first-order approximation at the MPP. In other words, the MPP corresponds to a reliability index β_j, which makes a first-order approximation of $\mathbf{P}_j = \Phi(-\beta_j)$, where $\Phi()$ is the standard normal density function. Then, the following optimization problem is solved in PMA to find the MPP:

$$\begin{array}{ll} \text{Minimize} & g_j^n(\mathbf{U}), \\ \text{subject to} & \|U\| = \beta_j^R, \end{array} \tag{5}$$

where β_j^R is the desired reliability index computed from the required reliability R_j as $\beta_j^R = \Phi^{-1}(R_j)$. The above formulation finds a $\mathbf{U}^*$ point which lies on a circle of radius β_j^R and minimizes $g_j^n(\mathbf{U})$. The original probability constraint is replaced by the following constraint:

$$g_j^n(\mathbf{U}^*) \geq 0. \tag{6}$$

To illustrate the method graphically, let us consider a hypothetical problem shown in Figure 1. The figure shows a probabilistic constraint g_j in the $\mathbf{U}$-space. The corresponding constraint boundary $g_j^n(\mu_1, \mu_2) = 0$ and the correspondent feasible region are shown. The circle represents a solution that corresponds to a reliability index of β_j^R. Thus, the PMA approach finds a point $\mathbf{U}^*$ on the circle for which the function $g_j^n(\mathbf{U})$ takes the minimum value. Then, if the corresponding constraint function value is non-negative (i.e. $g_j^n(\mathbf{U}^*) \geq 0$), the probabilistic constraint $P(g_j(\mathbf{x}) \geq 0) \geq R_j$ is considered to have been satisfied. In this study, we implement a faster

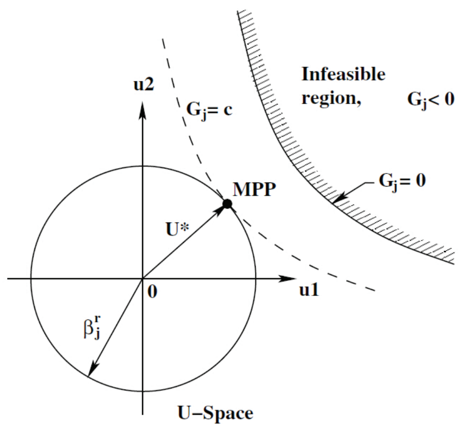

Figure 1: PMA approach is illustrated.

version of PMA , together with BLEAQ (a bilevel evolutionary algorithm based on quadratic approximation of lower level optimal solution as a function of upper level variables), to approximate MPP for reliable bilevel solutions acquisition. More information regarding *FastPMA* approach and BLEAQ can be found respectively in [4] and [13].

We now illustrate the effect of uncertainties in variables of each level one at a time and simultaneously in the following subsections.

2.2 Uncertainties in Lower Level alone

First, we consider that the lower level variables $\mathbf{y}$ are uncertain with a normal distribution having mean at the current solution and known standard deviation σ_y-vector. We construct a three-variable problem (with one upper level variable and two lower level variables) for this purpose.

$$\begin{array}{ll}\text{Minimize} & F(\mathbf{x},\mathbf{y}) = \left(\frac{y_2-50}{30}\right)^2 + \left(\frac{x-2.5}{0.2}\right)^2 \\ \text{subject to} & \mathbf{y} = \text{argmax}\left\{\begin{array}{l} f(\mathbf{x},\mathbf{y}) = y_2; \\ g_j(\mathbf{x},\mathbf{y}) \geq 0,\ j = 1, 2, \ldots, J_L \end{array}\right\}, \\ & 2 \leq x \leq 3, -4 \leq y_1 \leq 10, -100 \leq y_2 \leq 200. \end{array} \quad (7)$$

The respective lower level constraints are given below:

$$\begin{array}{l} g_1(\mathbf{x},\mathbf{y}) = x(y_1-2)^2 - y_2, \\ g_2(\mathbf{x},\mathbf{y}) = y_2 - 12.5x(y_1-5), \\ g_3(\mathbf{x},\mathbf{y}) = 5(y_1+4-x)(y_1+8-x) - y_2 \end{array} \quad (8)$$

Figure 2 shows the search space with lower level constraint surfaces. For any given upper level variable value ($\mathbf{x}$), the solution on line AA is the optimal solution, as this corresponds to the maximum value of y_2 for all feasible solutions. Since this solution lies on the constraint boundary of g_1 and g_2, it is sensitive to uncertainties in y_1 and y_2 variables. The narrow band of the feasible space close to this deterministic optimum makes it risky for constraint violation. However, the local optimal solution for the same $\mathbf{x}$ lies on the intersection of constraints g_1 and g_3 (near line BB). Since the feasible search region near this optimum is more wide, this local optimum is less sensitive to uncertainties in y_1 and y_2 variables. Hence, this local optimum solution is a reliable solution. Line BB marks the MPP points for all $\mathbf{x}$. Figure 3 shows the lower level feasible space with the reliable solution marked for a particular value of upper level variable x. For different values of $\mathbf{x}$, a similar situation occurs and if reliable, instead of global optimum, is desired, the lower level problem should find the reliable solution (line BB) for every $\mathbf{x}$.

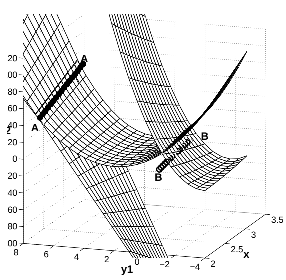

Figure 2: **Optimal and reliable solutions of lower level function of Case A are shown.**

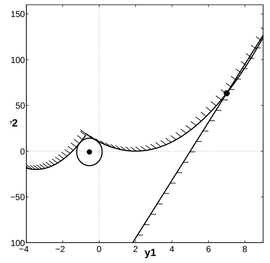

Figure 3: **Lower level feasible region with optimal and reliable solutions marked at fixed upper level variable $x^* = 2.5$.**

Figure 4 shows the upper level objective function $F(x, y_2)$ and marks both global optimum (AA) and reliable (BB) solutions of the lower level problem. Figure 5 shows the contour plot of upper level objective function on the x-y_2 plane and the relationship between these two variables for global optimal and reliable solutions. It is clear from the figures that although AA solution would make upper level function minimum, if reliable solutions at the lower level is desired, the solutions on line BB would correspond to the overall reliable solution of the bilevel problem. The upper level optimization problem considers only solutions on line BB feasible and then should find the optimal solution from the set BB. Since, solutions on set BB makes the lower level problem reliable, the final solution is the reliable bilevel solution of the problem.

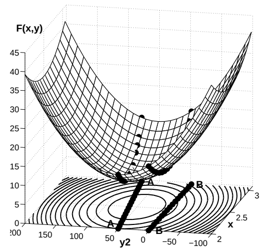

Figure 4: **Upper level function of Case A with mapped lower level optimal and reliable solutions.**

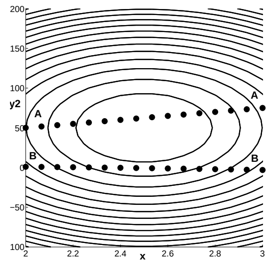

Figure 5: **Upper level function contour of Case A with optimal and reliable solutions from lower level.**

2.3 Uncertainties in Upper Level alone

Next, we consider a problem having uncertain upper variables only. The structure of both level problems from above subsection are interchanged here. There are two upper level variables and one lower level variable.

$$\begin{array}{ll}\text{Maximize} & F(\mathbf{x},\mathbf{y}) = x_2, \\ \text{subject to} & y = \text{argmin}\left\{f(\mathbf{x},\mathbf{y}) = \left(\frac{x_1-50}{28}\right)^2 + \left(\frac{y-2.5}{0.2}\right)^2\right\}, \\ & G_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1, 2, ..., J, \\ & 2 \leq y \leq 4,\ -80 \leq x_1 \leq 200,\ -100 \leq x_2 \leq 200. \end{array} \quad (9)$$

The upper level constraints are given below:

$$\begin{array}{l} G_1(\mathbf{x},\mathbf{y}) = y(\frac{x_1}{20}-2)^2 - x_2, \\ G_2(\mathbf{x},\mathbf{y}) = x_2 - 12.5y(\frac{x_1}{20}-5), \\ G_3(\mathbf{x},\mathbf{y}) = 5(\frac{x_1}{20}+4-y)(\frac{x_1}{20}+8-y) - x_2. \end{array} \quad (10)$$

Figure 6 shows the lower level objective function for an upper level variable x_1. Note that the above problem is simplified so that that lower level function is dependent only on one of the upper level variables. The y-x_1 search space and respective lower level optimal solutions are shown in Figure 7. It is clear that smaller optimal x_1 solution is less sensitive to $\mathbf{x}$ uncertainty.

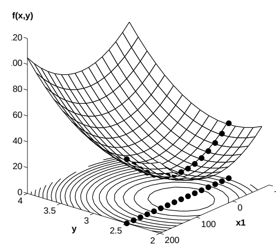

Figure 6: **Optimal solutions of lower level function $f(x, y)$ of the Case B are shown.**

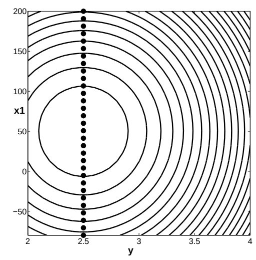

Figure 7: **Contour plot of lower level objective function shown with optimal lower level solutions marked in dots.**

When these lower level optimal solutions are considered to be feasible at the upper level, Figure 8 shows the corresponding variable combinations, marked with solid circles. Since variables x_1 and x_2 are uncertain, of the two optimal solutions for $F = x_2$, the smaller x_1 solution (right of the figure) is the reliable solution to the overall bilevel optimization problem. Figure 9 shows upper level variable space for an optimal lower level variable y value.

2.4 Uncertainties in Both Upper and Lower Levels

Finally, we consider a four-variable test problem in which

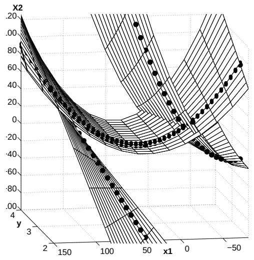

Figure 8: **Upper level function feasible search space - closed region bounded by three constraint surface.**

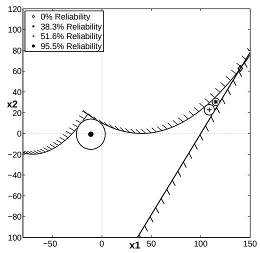

Figure 9: **Upper level feasible region w.r.t optimal lower level solution - four solutions with different reliability shown.**

all variables are uncertain:

$$\begin{array}{ll} \text{Maximize} & F(\mathbf{x},\mathbf{y}) = x_2 \\ \text{subject to} & \mathbf{y} = \text{argmax}\left\{ \begin{array}{l} f(\mathbf{x},\mathbf{y}) = y_2, \\ g_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1,2,\ldots,J_L \end{array} \right\}, \\ & G_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1,2,...,J, \\ & -4 \leq x_1 \leq 10, -100 \leq x_2 \leq 200, \\ & -4 \leq y_1 \leq 10, -100 \leq y_2 \leq 200. \end{array} \tag{11}$$

The upper and lower level constraint functions are given below.

$$\begin{array}{l} G_1(\mathbf{x},\mathbf{y}) = (\frac{y_1}{14} + \frac{16}{7})(x_1 - 2)^2 - x_2, \\ G_2(\mathbf{x},\mathbf{y}) = x_2 - 12.5(\frac{y_1}{14} + \frac{16}{7})(x_1 - 5), \\ G_3(\mathbf{x},\mathbf{y}) = 5(x_1 + 4 - (\frac{y_1}{14} + \frac{16}{7}))(x_1 + 8 - (\frac{y_1}{14} + \frac{16}{7})) - x_2, \\ \\ g_1(\mathbf{x},\mathbf{y}) = (\frac{x_1}{14} + \frac{16}{7})(y_1 - 2)^2 - y_2, \\ g_2(\mathbf{x},\mathbf{y}) = y_2 - 12.5(\frac{x_1}{14} + \frac{16}{7})(y_1 - 5), \\ g_3(\mathbf{x},\mathbf{y}) = 5(y_1 + 4 - (\frac{x_1}{14} + \frac{16}{7}))(y_1 + 8 - (\frac{x_1}{14} + \frac{16}{7})) - y_2. \end{array} \tag{12}$$

Note that lower level problem involves only one of the upper level variable (x_1) and upper level problem involves only one lower level variable y_1 for simplicity. Figure 10 shows the relationship of two lower lavel variables with x_1. For a given value of x_1, the respective reliable lower level solutions are shown to lie on line BB. When these solutions are sent to upper level and variable space x_1-y is shown in Figure 11. The reliable variable value x_2 is then found by solving the reliable upper level optimization problem. The solution on BB

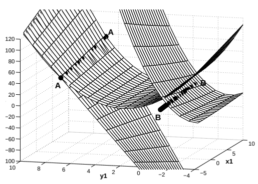

Figure 10: **Optimal and reliable solutions of lower level problem of Case C are shown.**

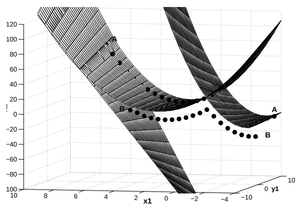

Figure 11: **Upper level function of Case C with mapped lower level optimal and reliable solutions. Reliable bilevel solution is marked as "X"**

close to the local but reliable part of the upper level objective function is the reliable bilevel solution.

3. RESULTS ON TEST PROBLEMS

We now present simulation results on the above illustrative and extended multi-variable test problems as well as a navy ship design problem using BLEAQ [14] in the following subsections. Brute force strategies, such as grid search is utilized to provide comparisons in case of illustrative test problems. As the number of variables at both levels increase, the grid search approach becomes exhaustive and computationally expensive to execute. We design higher-dimensional problems in a way so that all bilevel optimal variables take the same single-variable value. This way, we avoid executing the grid search method and simply compare the BLEAQ-obtained results with grid search solutions. Results are presented in terms of upper level elite function value(ULEFV), lower level elite function value(LLEFV), upper level variable solution(ULVS) and lower level variable solution(LLVS).

Uncertainties, that may occur in lower level variables alone, upper level variables alone, or both, are considered to have a normal distribution with the mean (μ) as its deterministic value and the variance (σ^2). The $FastPMA$ technique, described in Section 2.1, is used to identify the MPP in each case in order for BLEAQ to determine a solution's feasibility. A user-defined reliability index R and r (as the case may be) needs to be supplied. Here, all results use a default setting of 95% reliability unless stated otherwise.

The first subsection below contains results obtained on illustrative bilevel test problems, followed by the results obtained on 12-variable bilevel test problems in the second subsection. Results are also obtained for six-variable test problems, but are not included here for brevity. Finally, results obtained for a navy ship design problem are presented in the third subsection.

3.1 Low-dimensional Test Problems

- **Case 1: Lower level constrained only:** Test problems with constraints in lower level alone are studied in this case. The upper and lower level objective functions are provided in Equation 7 and corresponding constraints are provided in Equation 8. Both upper and lower level problems use a population size of 50 and both are restricted to a maximum generation of 800. The lower level variables are modeled with 0.35 and 7.5 variance, respectively, for y_1 and y_2. The desired lower level reliability index is 95%. For the grid search procedure, we use a step size of 1% for each variable dimension, thereby making a total of 100^3 or one million solution evaluations. Results obtained with deterministic, reliability-based, and grid search are provided in Table 1.

Table 1: Illustrative bilevel solutions for Case 1.

3-Variable		ULEFV	LLEFV	ULVS	LLVS	
Det.	Grid	0.1736	62.500	2.5	7.00	62.50
	Best	0.1691	62.1051	2.4842	7.00	62.1051
	Median	0.1689	62.1622	2.4865	7.00	62.1622
	Worst	0.1121	61.95	2.4986	6.8999	61.95
Reliable	Grid	2.8335	-0.70	2.5	-0.50	-0.70
	Best	2.8573	-0.6914	2.4907	-0.5710	-0.6914
	Median	2.8589	-0.7116	2.4922	-0.5697	-0.7116
	Worst	2.8597	-0.6821	2.4850	-0.5753	-0.6821

It is clear from the table that our proposed BLEAQ method with PMA approach of handling uncertainty is able to match the reliable solutions obtained using the grid search method with a much smaller number of function evaluations (7160 versus 1M).

- **Case 2: Lower level unconstrained only:** Test problems with constraints in upper level alone are studied here. The upper and lower level objective functions are provided in Equation 9 and corresponding constraints are provided in Equation 16. The population size for upper and lower level are 100 and 20, respectively. And both levels are restricted to a maximum generation of 800. The upper level variables are modeled with 0.35 and 7.5 variance, respectively, for x_1 and x_2. The desired upper level reliability index is 95%. For grid search method, the step size is 1% for each variable dimension. Results obtained with deterministic, reliability-based, and grid search are provided in Table 2. The proposed BLEAQ and PMA

Table 2: Illustrative bilevel solutions for Case 2.

3-Variable		ULEFV	LLEFV	ULVS		LLVS
Det.	Grid	62.50	10.3316	7.00	62.50	2.50
	Best	62.50	10.3316	7.00	62.50	2.50
	Median	62.50	10.3316	7.00	62.50	2.50
	Worst	62.50	10.3316	7.00	62.50	2.50
Reliable	Grid	-0.7288	4.7878	-0.5633	-0.7288	2.50
	Best	-0.7288	4.7878	-0.5633	-0.7288	2.50
	Median	-0.7288	4.7878	-0.5633	-0.7288	2.50
	Worst	-0.7288	4.7878	-0.5633	-0.7288	2.50

combination is able to match the results of the grid search method, but with a much smaller number of overall function evaluations (4536 versus 1M).

- **Case 3: Active constraints presented in both levels:** Test problems with constraints in both levels are studied in this case. The upper and lower level objective functions are provided in Equation 11 and corresponding constraints are provided in Equation 12. Both upper and lower level problems use a population size of 80 and both are restricted to a maximum generation of 1,000. Both upper and lower level variables are modeled with 0.35 and 7.5 variance, respectively, for x_1 and x_2 and y_1 and y_2. The desired upper and lower level reliability index are both 95%. For teh gird search problem, the step size is 1% for each variable dimension. Results obtained with deterministic, reliability-based and grid search are provided in Table 3.

Table 3: Illustrative bilevel solutions for Case 3.

4-Variable		ULEFV	LLEFV	ULVS		LLVS	
Det.	Grid	70.00	70.00	7.00	70.00	7.00	70.00
	Best	69.64	64.64	7.00	69.64	7.00	64.64
	Median	69.61	69.68	6.9988	69.61	7.0011	69.68
	Worst	69.50	69.63	6.9952	69.50	7.0001	69.63
Reliable	Grid	0.32	0.32	-0.80	0.30	-0.80	0.30
	Best	0.3022	0.3022	-0.7917	0.3022	-0.7917	0.3022
	Median	0.2745	0.2862	-0.7859	0.2745	-0.7903	0.2862
	Worst	0.2354	0.3009	-0.7859	0.2354	-0.7914	0.3009

As can be seen from the table, BLEAQ with PMA approach is able to match the reliable solution obtained by the exhaustive search method (12,486 vs 100M).

3.2 Higher-dimensions Test Problem Results

This subsection studies bilevel test problems with multiple variables in both levels, in particular, results are obtained on problems with six and nine variables. However, due to space restrictions, results are presented only for nine-variable problems and similar results are obtained for the six-variable problem. Cases 1, 2 and 3 for different uncertainty models are considered.

- **Case 1: Lower level constrained only:** Multi-variable test problems with constraints in lower level alone are studied in this case. The upper and lower level optimization problem formulations are provided below:

$$\begin{aligned} \text{Minimize} \quad & F(\mathbf{x},\mathbf{y}) = \textstyle\sum_{i\in E}^{m} \left(\frac{y_i-50}{30}\right)^2 + \sum_{k}^{n} \left(\frac{x_k-2.5}{0.2}\right)^2, \\ \text{subject to} \quad & \mathbf{y} = \text{argmax} \left\{ \begin{array}{l} f(\mathbf{x},\mathbf{y}) = \sum_{i\in E}^{m} y_i, \\ g_{cp}(\mathbf{x},\mathbf{y}) \geq 0,\ c = 1,2,\ldots,J_L \end{array} \right\}, \\ & 2 \leq x_k \leq 3, -4 \leq y_j \in O \leq 10, -100 \leq y_i \leq 200. \end{aligned} \tag{13}$$

where m and n denote the number of variables in upper and lower level respectively. Let O and E denote the set of positive odd and even integers, which are less than the dimensionality of each level's variable respectively. Lower level constraints are then modified as follows:

$$\begin{aligned} & g_{1k}(\mathbf{x},\mathbf{y}) = x_k(y_j-2)^2 - y_i, \\ & g_{2k}(\mathbf{x},\mathbf{y}) = y_i - 12.5x_k(y_j-5), \\ & g_{3k}(\mathbf{x},\mathbf{y}) = 5(y_j+4-x_k)(y_j+8-x_k) - y_i, \\ & \forall k \in \{1,2,...,n\}. \end{aligned} \tag{14}$$

Optimization tasks are performed with $m = 6$ and $n = 3$ comparing with and without reliability consideration in bilevel solution. A population of size of 150 is used for both lower and upper level. A maximum of 2,000 generations are allowed. The desired lower level reliability index is set at 2 which provides 95.5% reliability. And the modeled uncertainties are normally distributed with 10% variance over each variable's defining domain. Results are presented in Table 4.

Recall that the illustrative three-variable results for Case 1 was ULVS={2.50}, LLVS={7.00, 62.50} and ULVS={2.4907}, LLVS={-0.6914}, respectively, for deterministic and reliable solutions. Since the multi-variable functions are created by adding the respective identical pairs of three-objective functions for each variable pair, it is expected that the optimal value for each variable will be identical to the illustrative three-variable case. The optimal function values are multiples of the three-variable case.

BLEAQ solutions indicate similar variable values as the illustrative three-objective case. Importantly, the ability of BLEAQ method to solve a 9-variable problem is demonstrated here.

- **Case 2: Upper level constrained only:** Multi-variable test problems with constraints in upper level alone are studied in this case. The upper and lower level optimization problem formulations are provided below:

$$\begin{aligned} \text{Maximize} \quad & F(\mathbf{x},\mathbf{y}) = \textstyle\sum_{i\in E}^{n} x_i \\ \text{subject to} \quad & y = \text{argmin} \left\{ f(\mathbf{x},\mathbf{y}) = \textstyle\sum_{j\in O}^{m} \left(\frac{x_j-50}{28}\right)^2 \right. \\ & \qquad \left. + \textstyle\sum_{k}^{m} \left(\frac{y_k-2.5}{0.2}\right)^2 \right\}, \\ & G_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1,2,...,J, \\ & 2 \leq y_k \leq 4, -80 \leq x_j \leq 200, -100 \leq x_i \leq 200. \end{aligned} \tag{15}$$

where m and n denote the number of variables in upper and lower level respectively. Let O and E denote the set of positive odd and even integers, which are less than the dimensionality of each level's variable respectively. Upper level constraints are then modified as following:

$$\begin{aligned} & G_{1k}(\mathbf{x},\mathbf{y}) = y_k(\tfrac{x_j}{20}-2)^2 - x_i, \\ & G_{2k}(\mathbf{x},\mathbf{y}) = x_i - 12.5y_k(\tfrac{x_j}{20}-5), \\ & G_{3k}(\mathbf{x},\mathbf{y}) = 5(\tfrac{x_j}{20}+4-y_k)(\tfrac{x_j}{20}+8-y_k) - x_i, \\ & \forall k \in \{1,2,...,m\}. \end{aligned} \tag{16}$$

Table 4: 9-variable reliable bilevel solutions for Case 1.

9-Variable		ULEFV	LLEFV	ULVS			LLVS					
Det.	Best	0.5306	187.5047	2.4994	2.5003	2.5005	7.00	62.4845	7.00	62.5076	7.00	62.5127
	Median	0.5275	187.6351	2.5021	2.5114	2.4999	7.00	62.5526	6.9936	62.5855	7.00	62.4971
	Worst	0.5142	187.1778	2.5122	2.4984	2.5015	6.98	62.1819	7.00	62.4590	7.00	62.5369
Reliable	Best	8.5005	-2.0742	2.4963	2.4986	2.4930	-0.5712	-0.7014	-0.5699	-0.7121	-0.5705	-0.6994
	Median	8.5767	-2.1348	2.4933	2.4924	2.4922	-0.5695	-0.7113	-0.5702	-0.6998	-0.5697	-0.7116
	Worst	8.5791	-2.0463	2.4859	2.4896	2.4850	-0.5752	-0.6822	-0.5685	-0.7002	-0.5753	-0.6821

Table 5: 9-variable reliable bilevel solutions for Case 2.

9-Variable		ULEFV	LLEFV	ULVS						LLVS		
Det.	Best	187.50	30.9949	7.00	62.50	7.00	62.50	7.00	62.50	2.50	2.50	2.50
	Median	187.4985	30.9947	7.00	62.4997	7.00	62.4994	7.00	62.4994	2.50	2.50	2.50
	Worst	187.4041	30.9773	6.9979	62.4462	7.00	62.4999	6.9983	62.4580	2.50	2.50	2.50
Reliable	Best	-2.1864	14.3633	-0.5633	-0.7288	-0.5633	-0.7288	-0.5633	-0.7288	2.50	2.50	2.50
	Median	-2.1864	14.3633	-0.5633	-0.7288	-0.5633	-0.7288	-0.5633	-0.7288	2.50	2.50	2.50
	Worst	-2.2039	14.3651	-0.5639	-0.7427	-0.5632	-0.7299	-0.5634	-0.7313	2.50	2.50	2.50

Optimization tasks are performed with $m = 3$, $n = 6$ comparing with and without reliability consideration in bilevel solution. A population of size of 150 is used for both lower and upper level. A maximum of 2,000 generations are allowed. The desired upper level reliability index is set at 2 which provides 95.5% reliability. The modeled uncertainties are normally distributed with 10% variance over each variable's defining domain. Results are presented in Table 5.

Recall that the illustrative three-variable results for Case 2 was ULVS={7.00, 62.50}, LLVS={2.50} and ULVS={-0.5633, -0.7288}, LLVS={2.50}, respectively, for deterministic and reliable solutions. Since the multi-variable functions are created by adding the respective identical pairs of three-objective functions for each variable pair, it is expected that the optimal value for each variable will be identical to the illustrative three-variable case. And the optimal function values are multiples of the three-variable case.

- **Case 3: Active constraints presented in both levels:** Multi-variable test problems with constraints in both levels are studied in this case. The upper and lower level objective functions are provided in Equation 17:

$$\begin{array}{ll} \text{Maximize} & F(\mathbf{x},\mathbf{y}) = \sum_{i\in E}^{m} x_i \\ \text{subject to} & \mathbf{y} = \text{argmax}\left\{ \begin{array}{l} f(\mathbf{x},\mathbf{y}) = \sum_{i\in E}^{n} y_2, \\ g_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1,2,\ldots,J_L \end{array} \right\}, \\ & G_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1,2,...,J, \\ & -4 \leq x_{i\in O} \leq 10, -100 \leq x_{i\in E} \leq 200, \\ & -4 \leq y_{i\in O} \leq 10, -100 \leq y_{i\in E} \leq 200. \end{array} \tag{17}$$

where m and n denote the number of variables in upper and lower level respectively. Let O and E denote the set of positive odd and even integers, which are less than the dimensionality of each level's variable respectively. The constraint definitions for each level are provided in Appendix A.

Optimization tasks are performed comparing with and without reliability consideration in bilevel solution, with the number of total variables equal up to 16, due to the limited space, results are presented with m = n = 4. Maximum allowable generations for each level are set at 2000 and the desired reliability index for both levels are fixed at 2, which provide 95.5% reliability. The modeled uncertainties are normally distributed with 10% variance over each variable's defining domain. Results are provided in Table 6.

Recall that the illustrative four-variable results for Case 3 was ULVS = {7.00, 69.64}, LLVS = {7.00, 69.68} and ULVS = {-0.7917, 0.3022}, LLVS = {-0.7917, 0.3022}, respectively, for deterministic and reliable solutions. Since the multi-variable functions are created by adding the respective identical pairs of three-objective functions for each variable pair, it is expected that the optimal value for each variable will be identical to the illustrative four-variable case. And the optimal function values are multiples of the three-variable case.

3.3 Results on a Navy Ship Design Problem

For the purpose of applying our proposed reliability-based uncertainty handling technique in bilevel problems, we consider Sen-Bulker [12] ship model design problem. The original Sen-Bulker ship model is a tri-objective optimization problem consists of six design variables and nine constraints. The three objectives are minimization of light ship mass (LS), minimization of transportation cost (TC), and maximization of annual cargo capacity (AC). The full problem formulation is provided in Appendix B. Although a trade-off between transportation cost and ship mass is intuitive, it is expected that the cargo capacity and transportation cost will be correlated. Hence, we first eliminate one objective of maximizing AC and impose hierarchy into the remaining two objectives to redefine the original problem as a bilevel problem.

Now, we consider one objective of minimization of TC as the lower level optimization task and the other objective of minimization of LS as the upper level optimization task. The six design variables are separated accordingly and all nine constraints are assigned to the lower level optimization task. The new problem definition is provided as following:

$$\begin{array}{ll} \text{Min.} & F(\mathbf{x},\mathbf{y}) = \text{LightShipMass}(\mathbf{x},\mathbf{y}) \\ \text{s.t.} & \mathbf{y} = \text{argmin}\left\{ \begin{array}{l} f(\mathbf{x},\mathbf{y}) = \text{TransportationCost}(\mathbf{x},\mathbf{y}), \\ g_j(\mathbf{x},\mathbf{y}) \geq 0, \quad j = 1,2,\ldots,9. \end{array} \right\} \end{array} \tag{18}$$

The upper level variable vector **y** = {ShipLength(L), Beam Width(B)} specifies the physical dimension of a ship; and the lower variable vector **x** = {Draft(T), Depth(D), Coefficient(C_B), Velocity(V) } specifies the operating mode of a ship design. It is reasonable to propose the original Sen-Bulker ship model in this manner, in which we first obtain the best performance of a particular ship configuration measured by transportation cost (lower level objective), then optimize the shape of the ship by minimizing the ship mass (upper level objective). The new definition of the Sen-Bulker model with hierarchy in objectives transforms the problem into a bilevel optimization task, but still inherits the original concepts of the problem.

To better understand the Sen-Bulker navy ship model with

Table 6: 8-variable reliable bilevel solutions for Case 3.

8-Variable		ULEFV	LLEFV	ULVS				LLVS			
Det.	Best	139.2416	139.2830	6.9994	69.6270	6.9990	69.6146	7.0000	69.6418	7.0000	69.7411
	Median	139.1720	139.2790	6.9980	69.5844	6.9982	69.5875	7.0000	69.6394	7.0000	69.6396
	Worst	139.0419	139.2701	6.9961	69.5356	6.9951	69.5063	7.0000	69.6360	7.0000	69.6341
Reliable	Best	0.6045	0.6045	-0.7917	0.3022	-0.7917	0.3022	-0.7917	0.3022	-0.7917	0.3022
	Median	0.6041	0.6040	-0.7917	0.3022	-0.7915	0.3020	-0.7917	0.3022	-0.7918	0.3023
	Worst	0.5989	0.6045	-0.7915	0.2999	-0.7918	0.2990	-0.7917	0.3022	-0.7917	0.3023

hierarchy introduced, a upper level search space are obtained by forming a mesh grid of upper level variables, and for each combination of upper level variables, its corresponding lower level solution are optimized, as provided in Figure 12. It's noticeable that not every combination of upper level variables could result in a feasible lower level solution. For the purpose of completeness, the lower level solutions with least constraint violations are preserved in case of no feasible solutions exist.

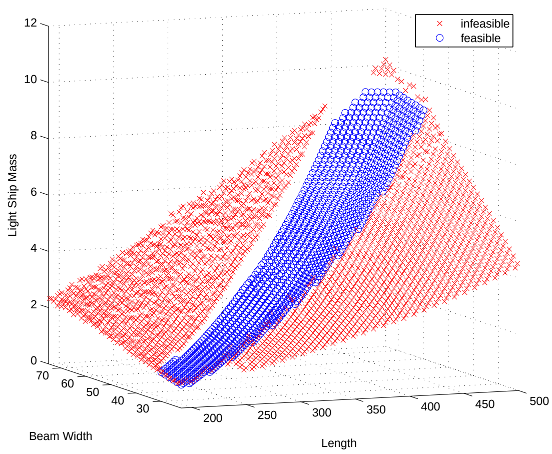

Figure 12: Upper level search space with respect to optimal lower level solutions correspondingly. In case of infeasible, the lower level solutions with the least constraint violations are preserved.

BLEAQ is used to solve the bilevel Sen-Bulker navy ship design problem (18) described above. Due to the computational complexity involved of obtaining optimization-based MPP ($FastPMA$) for each of nine constraints, Monte-Carlo sampling-based MPP acquisition is used instead. Depending on the desired reliability, the number of samples evaluated for each candidate solution needs to be updated. For the following simulation results, we have chosen to set the number of samples at 100. The population sizes are both 48 for upper and lower level. The maximum allowed generations for each level are set equally at 1,200. The obtained results are shown in Table 7.

To further explore the problem, the original Sen-Bulker navy ship model with elimination of the maximization of AC is solved using NSGA-II with and without the consideration of reliability. The results are shown in Figure 13

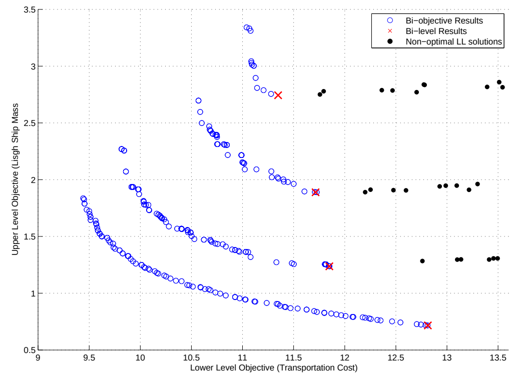

Figure 13: Bi-level vs. Bi-objective Results on Sen-Bulker Navy Ship design model

4. CONCLUSIONS

In this pilot study, we have introduced the concept of reliability in bilevel optimization problems arising from uncertainties in both lower and upper level decision variables. In the presence of constraints, uncertainties in decision variables cause optimal or near-optimal solutions to violate constraints some of the times. By restricting an upper bound on allowable failures in lower as well as upper level problems, reliable bilevel solutions can be achieved. The reliability based bilevel optimization methods have been proposed using the standard single-level stochastic optimization methods and results on test and a ship design problem have been presented.

The topic of uncertainty handling in bilevel problems is highly practical and timely with the overall growth in research in bilevel evolutionary algorithms and in uncertainty handling methods. This paper remains as the first systematic study in this direction and should spur interests for further research and application in the coming years.

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Table 7: Navy Ship Design Bilevel Results

Desired Reliability	xu		xl				LLFV	ULFV
Deterministic	195.1830	24.1525	10.2728	13.6754	0.6300	14.0000	12.8130	0.7163
40% Reliable	214.8231	33.1589	12.2449	18.8168	0.6541	14.0011	11.1811	1.1792
60% Reliable	241.4192	36.6471	13.8405	22.1056	0.7257	14.0183	11.9853	1.7223
70% Reliable	278.5342	40.2525	15.9080	24.8135	0.7367	14.0071	11.158	2.3803

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6. APPENDIX

6.1 Constraint violation evaluation of Multi-variable Case 3 Reliability model

A solution having positive value for all constraints is considered feasible. The defining domain for each variable is provided in Equation 17.

$$\begin{aligned}
G_1(\mathbf{x},\mathbf{y}) &= (\tfrac{y_1}{14}+\tfrac{16}{7})(x_1-2)^2 - x_2,\\
G_2(\mathbf{x},\mathbf{y}) &= x_2 - 12.5(\tfrac{y_1}{14}+\tfrac{16}{7})(x_1-5),\\
G_3(\mathbf{x},\mathbf{y}) &= 5(x_1+4-(\tfrac{y_1}{14}+\tfrac{16}{7}))(x_1+8-(\tfrac{y_1}{14}+\tfrac{16}{7})) - x_2,\\
G_4(\mathbf{x},\mathbf{y}) &= (\tfrac{y_3}{14}+\tfrac{16}{7})(x_3-2)^2 - x_4,\\
G_5(\mathbf{x},\mathbf{y}) &= x_4 - 12.5(\tfrac{y_3}{14}+\tfrac{16}{7})(x_3-5),\\
G_6(\mathbf{x},\mathbf{y}) &= 5(x_3+4-(\tfrac{y_3}{14}+\tfrac{16}{7}))(x_3+8-(\tfrac{y_3}{14}+\tfrac{16}{7})) - x_4,
\end{aligned} \quad (19)$$

$$\begin{aligned}
g_1(\mathbf{x},\mathbf{y}) &= (\tfrac{x_1}{14}+\tfrac{16}{7})(y_1-2)^2 - y_2,\\
g_2(\mathbf{x},\mathbf{y}) &= y_2 - 12.5(\tfrac{x_1}{14}+\tfrac{16}{7})(y_1-5),\\
g_3(\mathbf{x},\mathbf{y}) &= 5(y_1+4-(\tfrac{x_1}{14}+\tfrac{16}{7}))(y_1+8-(\tfrac{x_1}{14}+\tfrac{16}{7})) - y_2.\\
g_4(\mathbf{x},\mathbf{y}) &= (\tfrac{x_3}{14}+\tfrac{16}{7})(y_3-2)^2 - y_4,\\
g_5(\mathbf{x},\mathbf{y}) &= y_4 - 12.5(\tfrac{x_3}{14}+\tfrac{16}{7})(y_3-5),\\
g_6(\mathbf{x},\mathbf{y}) &= 5(y_3+4-(\tfrac{x_3}{14}+\tfrac{16}{7}))(y_3+8-(\tfrac{x_3}{14}+\tfrac{16}{7})) - y_4.
\end{aligned} \quad (20)$$

6.2 Evaluation of a design using original Sen-Bulker Model

The function forms of $a(C_B)$ and $b(C_B)$ are given elsewhere [12].A solution having positive value for all constraints is considered feasible. The allowable variable values are: 190 $\leq$ L $\leq$ 500 m, 10 $\leq$ T $\leq$ 27 m, 12 $\leq$ D $\leq$ 51 m, 0.63 $\leq C_B \leq$ 0.75, 22 $\leq$ B $\leq$ 75 m, and 14 $\leq$ V $\leq$ 18 Knots.

$[f, Constr] = Sen{-}Bulker\ ship\ model(x)\{$
$(L, T, D, C_B, B, V) = x;$
$Displ = 1.025 * L * B * T * C_B;$
$P = Displ^{2/3} * V^3/(b(C_B) * V/(9.8065 * L)^{0.5} + a(C_B));$
$SteelMass = 0.0034 * L^{1.7} * B^{0.7} * D^{0.4} * C_B^{0.5};$
$OutfitMass = L^{0.8} * B^{0.6} * D^{0.3} * C_B^{0.1};$
$MachineMass = 0.17 * P^{0.9};$
$DeadWeight = Displ - ShipMass;$
$DailyConsmp = 0.2 + 0.19 * P * 0.024;$
$SeaDays = 5000/(24 * V);$
$FuelCarried = DailyConsmp * (SeaDays + 5);$
$Crew = 2 * DeadWeight^{0.5};$
$CargoDw = DeadWeight - FuelCarried - Crew;$
$PortDays = 2 * (CargoDw/8000 + 0.5);$
$ShipCost = 1.3{*}(2000{*}SteelMass^{0.85}{+}3500{*}OutfitMass{+}$ $2400 * P^{0.8});$
$RunningCost = 40000 * DeadWeight^{0.3};$
$FuelCost = 1.05 * DailyConsmp * SeaDays * 100;$
$PortCost = 6.3 * DeadWeight^{0.8};$
$VoyageCost = FuelCost + PortCost;$
$RTPA = 350/(SeaDays + PortDays);$
$AnnualCost = 0.2{*}ShipCost{+}RunningCost{+}VoyageCost{*}$ $RTPA;$
$AnnualCargo = CargoDw * RTPA;$
$LightShipMass = SteelMass{+}OutfitMass{+}MachineMass;$
$TranspCost = AnnualCost/AnnualCargo;$
$f = [TranspCost, LightChipCass/10^4, -AnnualCargo/10^6];$
$/ * ConstraintEvaluation * /$
$GM = 0.53 * T + (0.085 * C_B - 0.002) * B^2/(T * C_B) + 0.52 *$ $D + 1;$
$Constr(1) = L/B - 6; Constr(2) = 15 - L/D;$
$Constr(3) = 19 - L/T;$
$Constr(4) = 0.45 * DeadWeight^{0.31} - T;$
$Constr(5) = 0.7 * D + 0.7 - T; Constr(6) = DeadWeight -$ $3000;$
$Constr(7) = 500000 - DeadWeight;$
$Constr(8) = 0.32 - V/(9.8065 * L)^{0.5};$
$Constr(9) = GM - 0.07 * B; \}$