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Source sink flows with capacity installation in batches

Sunil Chopra[a,*] Itzhak Gilboa[a] S. Trilochan Sastry[b]

[a] J.L. Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Leverone Hall, Evanston IL 60208-2001, USA

[b] Indian Institute of Management, Ahmedabad, India

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Abstract

We consider the problem of sending flow from a source to a destination where there are flow costs on each arc and fixed costs toward the purchase of capacity. Capacity can be purchased in batches of C units on each arc. We show the problem to be NP-hard in general. If d is the quantity to be shipped from the source to the destination, we give an algorithm that solves the problem in time polynomial in the size of the graph but exponential in $\lfloor d/C \rfloor$. Thus, for bounded values of $\lfloor d/C \rfloor$ the problem can be solved in polynomial time. This is useful since a simple heuristic gives a very good approximation of the optimal solution for large values of $\lfloor d/C \rfloor$. We also show a similar result to hold for the case when there are no flow costs but capacity can be purchased either in batches of 1 unit or C units. The results characterizing optimal solutions with a minimum number of free arcs are used to obtain extended formulations in each of the two cases. The LP-relaxations of the extended formulations are shown to be stronger than the natural formulations considered by earlier authors, even with a family of strong valid inequalities added.

Keywords: min cost flow; Integer programming

1. Introduction

In this paper we consider the one-facility, one-commodity (OFOC) network design problem which can be stated as follows. Consider a directed graph $G = (V,A)$ with a source s and a destination t. Capacity on each arc can be purchased in integer multiples of C units with each batch of C units costing $w_a \geqslant 0$ on arc a. There is a flow cost of $p_a \geqslant 0$ per unit of flow on arc a. The total cost of the flow is the flow cost plus the cost of purchasing capacity. The objective is to design a minimum cost network to send d units of flow from s to t.

A more general form of the problem with several sources and sinks arises in the telecommunications and transportation industry. OFOC arises as a subproblem in these

* Corresponding author. E-mail: s-chopra@casbah.acns.nwu.edu.

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instances. OFOC has been studied by Magnanti and Mirchandani (1993) for the special case where the flow cost p_a is zero on each arc. They show that the problem reduces to the shortest path problem and can thus be solved in polynomial time. They also give an inequality description for which they show that all objective functions with $p_a = 0$ for all arcs a, have at least one optimal solution that is integral. A more general case with multiple commodities has been considered by Magnanti et al. and Bienstock et al. Another related problem has been considered by Leung et al., and Pochet and Wolsey, where they study the capacitated lot sizing problem. They provide families of facet defining inequalities for the associated polyhedron.

In this paper we show the problem OFOC to be NP-hard in general when flow costs are present. This is in contrast to the case where all flow costs are zero, which is polynomially solvable (see [6]). We provide an algorithm to solve OFOC in polynomial time for bounded values of $\lfloor d/C \rfloor$. This is valuable since a simple approximation heuristic is asymptotically optimal.

We also consider the two-facility one-commodity (TFOC) network design problem (see [6, 7]). The problem is similar to OFOC except that we assume that capacity can be purchased either in batches of size 1 at a cost of $w_a^1 \geqslant 0$ or size C at a cost of $w_a^2 \geqslant 0$. Magnanti and Mirchandani consider the problem for the case where all flow costs p_a are 0. However the status of the problem in terms of complexity was unresolved. In this paper we show TFOC to be NP-hard for the case where all flow costs are 0. For the case when flow costs are 0, we provide an algorithm to solve TFOC in polynomial time for bounded values of $\lfloor d/C \rfloor$.

We use the results characterizing optimal solutions to OFOC and TFOC to obtain extended formulations in each case. We show that the LP-relaxations of the extended formulations are stronger than the natural formulations considered by earlier authors, even with a family of strong valid inequalities added. We also characterize objective functions for which the LP-relaxations of the extended formulations give integer optima. Computational tests reported in Section 6 support our claim that the extended formulations are much stronger than the natural formulations and are very effective in solving OFOC and TFOC. In our computational tests, the extended formulations give integer optimal solutions for every problem instance attempted (189 each for OFOC and TFOC).

In Section 2, we show that OFOC is NP-hard in general. In Section 3, we give an algorithm that allows us to solve OFOC in polynomial time as long as $\lfloor d/C \rfloor$ is bounded. A simple approximation heuristic is seen to be asymptotically optimal. In Section 4, we show TFOC to be NP-hard even when all flow costs are 0. An algorithm similar to that given for OFOC allows us to solve TFOC in polynomial time for bounded $\lfloor d/C \rfloor$ if flow costs are 0. Section 5 contains the extended formulations and we show them to be stronger than the natural formulations even with additional facet defining inequalities included. In Section 6 we describe computational tests supporting this claim and showing the extended formulation to be very effective in solving OFOC and TFOC.

We assume basic familiarity with graphs and network flows (see, for instance [2]). An arc a, directed from u to v will be referred to as (u, v). A vector indexed by the

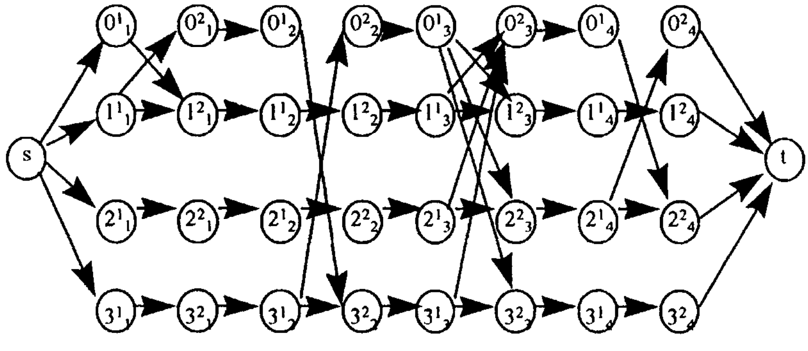

Fig. 1.

arc set will have variables referred to as x_{uv} or x_a depending upon the context. Given a node set $X \subseteq V$, define $\delta^+(X)$ to be the set of arcs directed from X to $V \setminus X$ and $\delta^-(X)$ to be the set of arcs directed from $V \setminus X$ to X. Given $\bar{A} \subseteq A$, and a vector y indexed by A, define $y(\bar{A}) = \sum_{a \in \bar{A}} y_a$.

2. OFOC is NP-hard

We prove that OFOC is NP-hard by transforming Minimum Cover (see [4]) into an instance of OFOC.

Proposition 2.1. *The problem OFOC is* NP-*hard.*

Proof. In an instance of Minimum Cover, we are given a collection $F = \{S_j, j = 1,\dots,m\}$ of subsets of a finite set $S = \{1,\dots,n\}$, and a positive integer $k \leqslant m$. The question is whether F contains a cover for S of size k or less, i.e., a subset $F' \subseteq F$ with $|F'| \leqslant k$ such that every element of S belongs to at least one member of F'.

Given the above instance of Minimum Cover, we construct the directed graph $G_F = (V_F, A_F)$, where

$$V_F = \{s,t\} \cup \{j_i^1, j_i^2, i = 1,\dots,n; j = 0,1,\dots,m\},$$

$$\begin{aligned} E_F = \; & \{(s, j_1^1), j = 0,1,\dots,m\} \cup \{(j_i^1, j_i^2), (j_i^2, j_{i+1}^1), j = 1,\dots,m, i = 1,\dots,n\} \\ & \cup (0_i^2, 0_{i+1}^1), i = 1,\dots,n\} \cup \{(0_i^1, j_i^2), (j_i^1, 0_i^2), \text{if } i \in S_j, \text{ for } i = 1,\dots,n\}. \end{aligned}$$

In the above description we have assumed that $j_{n+1}^1 = t$ for all j. The graph G_F contains a directed path $P_j, j = 1,\dots,m$ from s to t using the nodes j_i^1 and j_i^2 for each subset S_j in F. The arcs $(0_i^1, j_i^2)$ and $(j_i^1, 0_i^2)$ are present if and only if $i \in S_j$.

For the case where $n = 4$, $k = 2$, $m = 3$, $S_1 = \{1,3\}$, $S_2 = \{3,4\}$, $S_3 = \{2,3\}$, the graph G_F is as shown in Fig. 1.

On the graph G_F, consider the problem OFOC where $kC + \varepsilon$ units of flow is to be sent from s to t, where ε is close to 0. Arcs a along the paths P_j have $w_a = M$ and $p_a = 0$. The arcs of the form $(s, 0_1^1)$ and $(0_i^2, 0_{i+1}^1), i = 1, \ldots, n$ have $w_a = 0$ and $p_a = 2M$. Here M is a large positive integer ($M = (2n+1)k$ will suffice). All other arcs have $w_a = 0$ and $p_a = 0$.

Note that to send kC units of flow from s to t we must use k of the paths $P_j, j = 1, \ldots, m$ (multiple uses of a path are counted as multiple paths), with each path carrying C units, since any other path uses at least one arc with $p_a = 2M$ resulting in a cost at least as high if not higher. If flow is sent as described above, a total cost of $(2n+1)kM$ is incurred to send the kC units from s to t and we cannot send this portion any cheaper. This leaves ε units to be sent from s to t. Note that each arc in a path $P_j, j = 1, \ldots, n$, that has been used to send C units from s to t can now be used in the reverse direction to send the remaining ε units without incurring a cost w_a. If there is a solution to Minimum Cover, send C units along path P_j for each set S_j in the cover. Since the sets S_j corresponding to the paths P_j define a cover, for each i, $1 \le i \le n$, there exists a set $S_{r(i)}$ in the cover with $i \in S_{r(i)}$. The path $P_{r(i)}$ has been used to send C units of flow from s to t. Since $i \in S_{r(i)}$, there exists the arc $(0_i^1, r(i)_i^2)$ in E_F for $i \in \{1, \ldots, n\}$. Consider the path

$$P_\varepsilon = \{(s, 0_1^1), (0_1^1, r(1)_1^2), (r(1)_1^1, r(1)_1^2), (r(1)_1^1, 0_1^2), (0_1^2, 0_2^1), \ldots, (0_i^1, r(i)_i^2), (r(i)_i^1, r(i)_i^2), (r(i)_1^1, 0_i^2), (0_i^2, 0_{i+1}^1), \ldots, (0_n^2, 0_{n+1}^1)\}.$$

The path P_ε can be used to ship ε units from node s to node t, where the ε units flow on each arc $(r(i)_i^1, r(i)_i^2)$, $i \in \{1, \ldots, n\}$, in the reverse direction and on all other arcs in the path the flow is in the forward direction.

If there is no solution to Minimum Cover, no such path as P_ε exists to send ε units of flow where the ε units flow on arcs $(r(i)_i^1, r(i)_i^2)$ in the reverse direction (this can only occur if the path $P_r(i)$ has been used to send C units of flow as described earlier). Thus the cost incurred to send $k + \varepsilon$ units from s to t is at least $(2n+1)kM + M$, since at least one of the arcs in the paths P_j must be used in the forward direction to carry the ε units. On the other hand, if there exists a solution to Minimum Cover, the $k + \varepsilon$ units can be sent at a cost of $(2n+1)kM + 2(n+1)M\varepsilon < (2n+1)kM + M$, for ε sufficiently small. Thus OFOC on G_F has an optimal solution of value $(2n+1)kM + 2(n+1)M\varepsilon$ if and only if there exists a solution to Minimum Cover. The result thus follows. $\square$

Note that for the example in Fig. 1, there is no cover using two or fewer subsets. To send $2 + \varepsilon$ units from s to t in the graph G_F in Fig. 1, we have to use at least two of the paths P_j (multiple uses of a path being counted as multiple paths) to send 2 units and at least one of the arcs in the paths P_j in the forward direction to send the remaining ε units. The total cost incurred in this case is at least $19M$. On the other hand, if we set $S_3 = \{2, 4\}$, there is a cover using two subsets. In the corresponding graph G_F, it is possible to send $2 + \varepsilon$ units from s to t at a cost of $18M + 10M\varepsilon$.

3. OFOC for bounded $\lfloor d/C \rfloor$

In this section we show that if $d = kC + r, 1 \leqslant r \leqslant C - 1$, and k is bounded from above by some constant, then OFOC can be solved in polynomial time. We also give a polynomial heuristic that is shown to be asymptotically optimal. Thus, if the flow to be sent from s to t is small we can rely on the first algorithm to obtain the optimal solution, while if the flow to be sent is large we can rely on the polynomial heuristic to obtain a good approximation.

3.1. Structure of optimal solutions for OFOC

We identify certain structural properties of optimal solutions to OFOC. Consider an optimal solution vector (y^*, f^*) where y_a^* is the capacity installed on arc a and f_a^* is the flow through arc a. Note that since $w_a \geqslant 0$ and $p_a \geqslant 0$, given the flow vector f^* in any optimal solution, the optimal capacity installed can be assumed to be given by $y_a^* = \lceil f_a^*/C \rceil$. Given a solution (y^*, f^*), define an arc a with $f_a^* < Cy_a^*$ to be a *free arc*. We now characterize optimal solutions of OFOC with the minimum number of free arcs.

Proposition 3.1. *Let (y^*, f^*) be an optimal solution to OFOC with the minimum number of free arcs. All the free arcs defined by (y^*, f^*) lie on a path (ignoring direction) from s to t. Free arcs directed forward along this path have a flow from $\{lC+r\}_{l=0}^k$, and those directed backward have a flow from $\{lC-r\}_{l=1}^k$, in the optimal solution.*

Proof. We first show that the free arcs define exactly one path (ignoring direction) from s to t. Given the optimal solution (y^*, f^*), the flow f^* is an optimal solution to the min-cost flow problem on the graph G with arc capacities y_a^* and flow costs p_a. Since (y^*, f^*) has the minimum number of free arcs, y^* is an extreme optimal solution to the min-cost flow problem.

Define a path in G from s to t (ignoring arc direction) to be a *free path* if each arc a in the path has $0 < f_a^* < y_a^*$. Using standard network flow results (see [1]), since y^* is an extreme solution to the capacitated min-cost flow problem, there exists at most one free path from s to t. For any node set $X \subseteq V, s \in X, t \in V \setminus X$, note that

$$f^*(\delta^+(X)) - f^*(\delta^-(X)) = kC + r. \tag{1}$$

Thus, for every choice of X, at least one of the the sets $\delta^+(X)$ or $\delta^-(X)$ contains a free arc. This implies a path from s to t (ignoring direction) using only free arcs (using Menger's Theorem, see [2]). Thus, there exists exactly one free path from s to t.

Now we prove that all the free arcs must be on the free path P_F. To the contrary assume that there is a free arc (u, v) that does not belong to P_F. Assume that

$u \notin P_F$. Since flow is conserved at all nodes other than s and t, there must be another free arc incident to u. One can proceed from u along that arc. Continuing this procedure, one can extend the arc (u,v) using free arcs till it either forms a cycle (ignoring direction), or a path (ignoring direction) with both end points in P_F. In either case, by adjusting the flow on the cycle or the path and the flow on P_F, one can show that there exists another optimal solution with at least one less free arc than (y^*, f^*). This contradiction proves that all free arcs must be on the free path P_F.

Let A_F^{f} be the set of forward arcs in P_F and A_F^{b} be the set of reverse arcs in P_F when moving from s to t. The nodes in the free path P_F have a natural ordering as one proceeds from s to t. Assume that the nodes in P_F, between s and t, are ordered as $\{u_j\}_{j=1}^{p}$. Define $X_q = \{s\} \cup \{u_j\}_{j=1}^{q}$. Note that exactly one of $\delta^+(X_q)$ or $\delta^-(X_q)$ contains one free arc a_q. If $a_q \in A_F^{\mathrm{f}}$, then $a_q \in \delta^+(X_q)$ and if $x_q \in A_F^{\mathrm{b}}$ then $a_q \in \delta^-(X_q)$. From (1) we thus have

$$f_a^* = \begin{cases} lC + r, & l \in \{0,\dots,k\}, \quad a \in A_F^{\mathrm{f}}, \\ lC - r, & l \in \{1,\dots,k\}, \quad a \in A_F^{\mathrm{b}}. \end{cases}$$

The result thus follows. $\square$

From this point on we restrict attention to optimal solutions to OFOC with the minimum number of free arcs. Further, we can assume that there does not exist another optimal solution $(y', f') \neq (y^*, f^*)$, such that $y'_a \leqslant y_a^*$ and $f'_a \leqslant f_a^*$ for all arcs a. Such optimal solutions will be referred to as *minimal free arc extreme optimal solutions.* Given an optimal solution (y^*, f^*) to OFOC, let $G^* = (V^*, A^*)$ be the graph induced by the arcs with $f_a^* > 0$. Since all costs are non-negative, the graph G^* can be assumed to be acyclic. We next prove that the optimal flow f^* can be decomposed into the sum of flows along $2k+1$ paths $\{P_i\}_{i=1}^{2k+1}$ from s to t, where the first k paths each carry a flow of $C - r$ and the last $k+1$ paths each carry a flow of r.

Proposition 3.2. *Let (y^*, f^*) be a minimal free arc extreme optimal solution to OFOC. There exist $2k+1$ paths $\{P_i\}_{i=1}^{2k+1}$, from s to t, such that*

$$f_a^* = (C - r)\sum_{i=1}^{k} P_i^a + r \sum_{i=k+1}^{2k+1} P_i^a, \tag{2}$$

where $P_i^a = 1$ if $a \in P_i$, 0 otherwise.

Proof. By Proposition 3.1, each arc has a flow $f_a^* \in \{lC+r\}_{l=0}^{k} \cup \{lC-r\}_{l=1}^{k} \cup \{lC\}_{l=1}^{k}$. Construct the graph $G' = (V^*, A')$ as follows. If $f_a^* = lC + r$, construct $l+1$ copies of the arc a in A', l with a flow of C and one with a flow of r. If $f_a^* = lC$, construct l copies each with a flow of C. If $f_a^* = lC - r$, construct l copies of the arc, $l - 1$ with a flow of C and one with a flow of $C - r$.

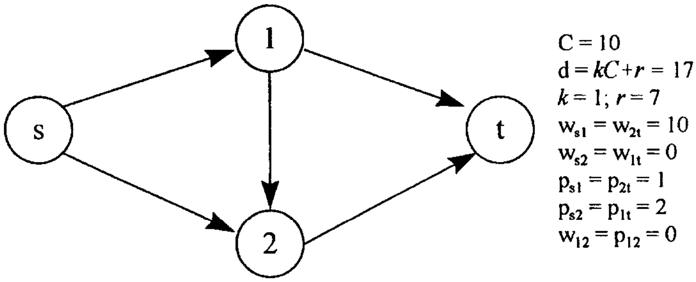

Fig. 2.

Let $n_f^+(i)\{n_f^-(i)\}$ be the number of arcs in G' with a flow of f entering {leaving} node i. By Proposition 3.1, each node in $V^*\setminus\{s,t\}$ satisfies exactly one of the following conditions:

(i) $n_C^+(i) = n_C^-(i)$; $n_r^+(i) = n_r^-(i) \leqslant 1$; $n_{C-r}^+(i) = n_{C-r}^-(i) \leqslant 1$.
(ii) $n_C^+(i) = n_C^-(i) + 1$; $n_r^+(i) + 1 = n_r^-(i) = 1$; $n_{C-r}^+(i) + 1 = n_{C-r}^-(i) = 1$.
(iii) $n_C^+(i) = n_C^-(i) - 1$; $n_r^+(i) = n_r^-(i) + 1 = 1$; $n_{C-r}^+(i) = n_{C-r}^-(i) + 1 = 1$.

For the nodes s and t we have $n_C^-(s) = n_C^+(t) = k$ and $n_r^-(s) = n_r^+(t) = 1$.

From the graph G' construct the graph $\bar{G} = (V^*, \bar{A})$, where each arc with a flow of C in G' is replaced by two parallel arcs, one with a flow of r and the other with a flow of $C - r$. Each arc in $\bar{A}$ has a flow of $C - r$ or r. Define $m_f^+(i)\{m_f^-(i)\}$ to be the number of arcs in $\bar{G}$ with a flow of f entering {leaving} node i. Note that in the graph $\bar{G}$ we have

$$m_{C-r}^+(t) = m_{C-r}^-(s) = k, \qquad m_{C-r}^+(i) = m_{C-r}^-(i) \quad \text{for } i \in V^* \setminus \{s,t\}.$$

A flow of $k(C - r)$ is sent from s to t using only the arcs in $\bar{A}$ with a flow of $C - r$. Thus by Menger's theorem (see [2]), there exist k arc disjoint paths from s to t in $\bar{G}$ using only the arcs with a flow of $C - r$. These correspond to k paths $\{P_i\}_{i=1}^k$ in G^*. Also observe that in the graph $\bar{G}$ we have

$$m_r^+(t) = m_r^-(s) = k + 1, \qquad m_r^+(i) = m_r^-(i) \quad \text{for } i \in V^* \setminus \{s,t\}.$$

A flow of $(k + 1)r$ is sent from s to t using only the arcs in $\bar{A}$ with a flow of r. Once again by Menger's theorem we have $k + 1$ arc disjoint paths from s to t in $\bar{G}$ using only the arcs with a flow of r. These correspond to $k + 1$ paths $\{P_i\}_{i=k+1}^{2k+1}$ in the graph G^*. The paths $\{P_i\}_{i=1}^{2k+1}$ satisfy (2). Since (y^*, f^*) is an extreme solution, each arc in $\bar{A}$ with positive flow must be in one of the paths $\{P_i\}_{i=1}^{2k+1}$. The result thus follows. $\square$

As an example consider the graph in Fig. 2. Assume that $d = 17, C = 10$. Assume that $w_{s1} = w_{2t} = 10$, $w_{s2} = w_{1t} = 0$, $p_{s1} = p_{2t} = 1$, $p_{s2} = p_{1t} = 2$, $w_{12} = p_{12} = 0$.

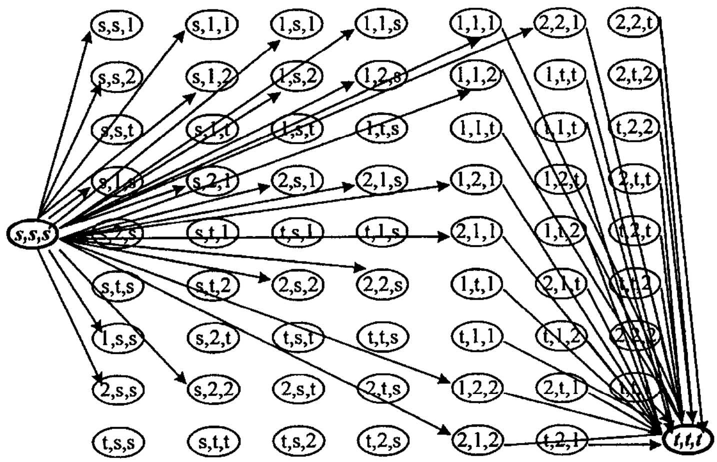

Fig. 3.

Consider the solution (y^*, f^*) where $f^*_{s1} = f^*_{2t} = 10$, $f^*_{s2} = f^*_{1t} = 7, f^*_{12} = 3$. The flow f^* can be decomposed into a flow of 3 units along the path $\{(s,1),(1,2),(2,t)\}$ and a flow of 7 units along the paths $\{(s,1),(1,t)\}$ and $\{(s,2),(2,t)\}$.

3.2. Polynomial algorithm to solve OFOC for bounded $\lfloor d/C \rfloor$

We use the results from Section 3.1 to devise an algorithm to solve OFOC. The complexity of the algorithm is polynomial for bounded k, where $d = kC + r$. The algorithm is based on the decomposition of the flow in the optimal solution into a flow along $2k + 1$ paths. As shown in Proposition 3.2, the paths $\{P_i\}_{i=1}^{k}$, have a flow of $C - r$, while the paths $\{P_i\}_{i=k+1}^{2k+1}$ have a flow of r.

Given the graph $G = (V,A)$, construct an auxiliary graph $H = (N,E)$ to mimic flow along the $2k + 1$ paths. Each node v in N corresponds to a $(2k + 1)$-tuple $(v(1), v(2), ..., v(2k + 1))$, where $v(i) \in V$ for $i = 1, ..., 2k + 1$. The graph H thus contains $|V|^{2k+1}$ nodes. Let s_H be the node in H corresponding to the $(2k + 1)$-tuple $(s, s, ..., s)$, and t_H be the node in H corresponding to the $(2k + 1)$-tuple $(t, t, ..., t)$. In the graph H, the arc directed from node u to node v is included if and only if for each $i \in \{1, ..., 2k + 1\}$, either $u(i) = v(i)$, or $(u(i), v(i))$ is an arc in A. Thus, a path $P_H = \{(s_H, u_1), (u_1, u_2), ..., (u_r, t_H)\}$ in H defines $2k + 1$ paths $\{P_i\}_{i=1}^{2k+1}$ in G, for

$$P_i = \{(s, u_1(i)), (u_1(i), u_2(i)), ..., (u_r(i), t)\}.$$

Note that our definition allows for $u_j(i) = u_{j+1}(i)$ for some i and j.

Consider the graph in Fig. 2. In this case we have $k = 1$. The corresponding graph H thus contains 4^3 nodes, each corresponding to a 3-tuple $(u(1), u(2), u(3))$ for $u(i) \in \{1, 2, 3, 4\}, i = 1, 2, 3$. We show the graph H in Fig. 3 with all 64 nodes. However, for the sake of clarity, we only show the arcs leaving the node in H corresponding to

(s,s,s) or entering the node corresponding to (t,t,t). In the graph H there is an arc directed from the node u to node v where u corresponds to the 3-tuple (s,s,s) and v corresponds to the 3-tuple $(1,2,s)$, since $(u(1),v(1)) = (s,1)$ and $(u(2),v(2)) = (s,2)$ are arcs in G while $u(3) = v(3) = s$. The path in H corresponding to the 3-tuples (s,s,s), $(1,1,s)$,$(2,1,2)$, (t,t,t), defines three paths in G where

$$P_1 = \{(s,1),(1,2),(2,t)\}, \qquad P_2 = \{(s,1),(1,t)\}, \qquad P_3 = \{(s,2),(2,t)\}.$$

For each pair of arcs $e = (u,v) \in E$ and $a \in A$, define

$$n_1^e(a) = |i \in \{1,...,k\} : a = (u(i),v(i))|,$$

$$n_2^e(a) = |i \in \{k+1,...,2k+1\} : a = (u(i),v(i))|.$$

Once again consider the graph G from Fig. 2 and the corresponding graph H. Let $e = (u,v) \in E$, where u corresponds to the 3-tuple $(1,1,s)$ and v corresponds to the 3-tuple $(2,1,2)$. For $a = (1,2) \in A$, we have $n_1^e(a) = 1$ and $n_2^e(a) = 0$, since $(u(i),v(i)) = (1,2)$ only for $i = 1$.

The arc e in E corresponds to a flow of $C-r$ from $u(i)$ to $v(i)$, $i = 1,...,k$, and a flow of r from $u(i)$ to $v(i)$, $i = k+1,...,2k+1$, along the arcs $(u(i),v(i))$, $i = 1,...,2k+1$. If $u(i) = v(i)$, no flow is assumed to have taken place. Define

$$f_a^e = (C-r)n_1^e(a) + rn_2^e(a), \tag{3}$$

for each arc $a \in A$ and $e \in E$. The flow f_a^e corresponds to the total flow along arc a in A defined by arc e in H. Define

$$b_e^p = \sum_{a\in A} p_a f_a^e, \quad b_e^w = \sum_{a \in A} w_a \lceil f_a^e/C \rceil; b_e = b_e^p + b_e^w.$$

b_e^p represents the cost of sending $C-r$ units of flow from $u(i)$ to $v(i)$, $i = 1,...,k$, and r units from $u(i)$ to $v(i)$, $i = k+1,...,2k+1$. b_e^w represents the cost of purchasing sufficient capacity for the flow described above.

Once again returning to the example in Fig. 2, and considering e to be the arc in H from the node corresponding to the 3-tuple $(1,1,s)$ to the node corresponding to the 3-tuple $(2,1,2)$, we have

$$f_{s1}^e = 0, \quad f_{s2}^e = 7, \quad f_{12}^e = 3, \quad f_{1t}^e = 0, \quad f_{2t}^e = 0.$$

This implies that

$$b_e^p = 2 \times 7 + 0 \times 3 = 14, \qquad b_e^w = 0 \times 1 + 0 \times 1 = 0.$$

Consider the shortest path in H from s_H to t_H, using arc costs b_e. We prove that such a shortest path defines the optimal solution to OFOC.

Theorem 3.1. *Given the problem OFOC, let the auxiliary graph H be defined as above. Arc weights b_e are as defined above for $e \in E$. Let P_H^* represent the shortest* *path in H from s_H to t_H. Let $\{P_i^*\}_{i=1}^{2k+1}$ be the $2k+1$ paths in G corresponding to P_H^*. Define the flow vector f^*, where*

$$f_a^* = (C-r)\sum_{i=1}^{k} P_i^*(a) + r\sum_{k+1}^{2k+1} P_i^*(a),$$

in which $P_i^*(a) = 1$ if $a \in P_i^*$, 0 otherwise. Let $y_a^* = \lceil f_a^*/C \rceil$. The vector (y^*, f^*) is an optimal solution to OFOC.

Proof. Consider any path $\bar{P} = \{e_1, e_2, \ldots, e_q\}$ from s_H to t_H in H. Let

$$b(\bar{P}) = \sum_{e \in \bar{P}} b_e.$$

Let $\{\bar{P}_i\}_{i=1}^{2k+1}$, be the $2k+1$ paths in G corresponding to $\bar{P}$. Define the flow vector $\bar{f}$, where

$$\bar{f}_a = (C-r)\sum_{i=1}^{k} \bar{P}_i(a) + r\sum_{k+1}^{2k+1} \bar{P}_i(a),$$

where $\bar{P}_i(a) = 1$, if $a \in \bar{P}_i$, 0 otherwise. Let $\bar{y}_a = \lceil \bar{f}_a/C \rceil$.

Note that

$$\bar{f}_a = \sum_{i=1}^{q} f_a^{e_i},$$

where $f_a^{e_i}$ is defined for each arc a in A as in (3). Note that

$$\sum_{i=1}^{q} b_{e_i}^p = \sum_{i=1}^{q}\sum_{a\in A} p_a f_a^{e_i} = \sum_{a\in A}\sum_{i=1}^{q} p_a f_a^{e_i} = \sum_{a\in A} p_a \bar{f}_a$$

and

$$\sum_{i=1}^{q} b_{e_i}^w = \sum_{i=1}^{q}\sum_{a\in A} w_a \lceil f_a^{e_i}/C \rceil \geqslant \sum_{a\in A} w_a \lceil (\sum_{i=1}^{q} f_a^{e_i})/C \rceil.$$

Thus, for each path $\bar{P}$ in H from s_H to t_H, there exists a corresponding solution to OFOC whose cost is no more than the length of the path.

Now we prove that given an extreme optimal solution to OFOC, there exists a path in H from s_H to t_H with length no more than the cost of the optimal solution.

Let (y', f') be an extreme optimal solution to OFOC. Since all costs are non-negative, we can assume that the subgraph $G' = (V', A')$ of G, induced by the arcs with $f_a{}' > 0$ is acyclic.

By Proposition 3.2, the flow f' can be decomposed into $2k+1$ paths $\{P_i{}'\}_{i=1}^{2k+1}$, in G'. Let $P_i{}' = \{(v_i^1, v_i^2), (v_i^2, v_i^3), \ldots, (v_i^{l_i}, v_i^{l_i+1})\}$, $i = 1, \ldots, 2k+1$, where $v_i^1 = s$ and $v_i^{l_i+1} = t$. Since G' is acyclic, each path P_i' has at most $|V'|$ nodes, i.e., $l_i + 1 \leqslant |V'|$. If $l_i + 1 < |V'|$, define $v_i^j = t$ for $l_i + 2 \leqslant j \leqslant |V'|$. In the auxiliary graph

H, consider the path $P' = \{e_1', e_2', ..., e_{|V'|}'\}$, where e_j' is the arc in H directed from the node corresponding to the $2k+1$-tuple $(v_1^j, v_2^j, ..., v_{2k+1}^j)$ to the node corresponding to the $2k+1$-tuple $(v_1^{j+1}, v_2^{j+1}, ..., v_{2k+1}^{j+1})$, where the nodes v_i^j are as defined above.

The above procedure defines the path P' in H given paths $\{P_i'\}_{i=1}^{2k+1}$, in G'. Note that by construction, given any arc $a \in A'$, there is exactly one arc e_j' in P', such that $f_a^{e_j'} > 0$, where $f_a^{e_j'}$ is as defined in (3). Note that

$$\sum_{j=1}^{|V'|} b_{e_j'}^p = \sum_{j=1}^{|V'|} \sum_{a \in A} p_a f_a^{e_j'} = \sum_{a \in A} \sum_{j=1}^{|V'|} p_a f_a^{e_j'} = \sum_{a \in A} p_a f_a'$$

and

$$\sum_{i=j}^{|V'|} b_{e_j'}^p = \sum_{j=1}^{|V'|} \sum_{a \in A} w_a \lceil f_a^{e_j'}/C \rceil = \sum_{a \in A} w_a \sum_{j=1}^{|V'|} \lceil f_a^{e_j'}/C \rceil = \sum_{a \in A} w_a \left\lceil \left(\sum_{j=1}^{|V'|} f_a^{e_j'} \right) /C \right\rceil .$$

The last equality holds since there is exactly one arc e_j' in P' such that $f_a^{e_j'} > 0$. Thus the length of the path P' is no more than the value of the optimal solution (y', f').

We have thus shown that the shortest path in H from s_H to t_H must have length equal to the value of the optimal solution to OFOC. The flows and capacities corresponding to this path define the optimal solution. The result thus follows. □

Once again consider the graph G in Fig. 2. In the corresponding graph H, the shortest path from (s,s,s) to (t,t,t) is given by the path corresponding to the node sequence $(s,s,s), (1,1,s), (2,1,2), (t,t,t)$. The length of this path in H is given by $20+14+34 = 68$. This path corresponds to three paths in G with

$$P_1 = \{(s,1),(1,2),(2,t)\}, \qquad P_2 = \{(s,1),(1,t)\}; P_3 = \{(s,2),(2,t)\}.$$

A flow of $C - r = 3$ is sent along P_1 and a flow of $r = 7$ is sent along each of P_2 and P_3. The total cost of this flow is also 68.

3.3. An asymptotically optimal heuristic

We provide a simple heuristic that is shown to be asymptotically optimal. This is similar to the heuristic given by Magnanti and Mirchandani for the case without flow costs. The heuristic solution is obtained as follows:

(1) In the graph G find the shortest path from s to t using arc costs $Cp_a + w_a$. Send kC units of flow through this shortest path.
(2) Find the shortest path in G from s to t using arc costs $rp_a + w_a$. Send r units of flow through this path.
(3) Let f_a^{h} be the resulting flow on each arc a. Define $y_a^{\mathrm{h}} = \lceil f_a^{\mathrm{h}}/C \rceil$. $(y^{\mathrm{h}}, f^{\mathrm{h}})$ is the heuristic solution.

Proposition 3.3. *Let Z^{h} be the cost of the heuristic solution and Z^* be the cost of the optimal solution. We have*

$$\frac{Z^{\mathrm{h}}}{Z^*} \leqslant (k+1)/k.$$

Proof. Consider the problem where kC units are to be sent from s to t. The optimal solution to this problem is to find the shortest path from s to t with arc costs $Cp_a + w_a$ and send kC units along this path. Let $\bar{Z}$ be the cost of this solution. Note that

$$\bar{Z} \leqslant Z^* \leqslant Z^{\mathrm{h}} \quad \text{and} \quad Z^h \leqslant \bar{Z}\frac{k+1}{k}.$$

The result thus follows. □

Note that as k increases the heuristic solution is asymptotically optimal.

Remark 3.1. Note that the heuristic gives the optimal solution if $d = kC$. In general, if we are seeking a solution no worse that $(1+\delta)Z^*$ and $\delta \geqslant 1/k$, use the heuristic to obtain a suitable approximate solution. If $\delta < 1/k$, use the algorithm in Section 3.2 to obtain an exact solution.

4. The two-facility one-commodity problem

In this section we consider the two-facilty one commodity problem discussed earlier. For the case where flow costs p_a are not all zero, TFOC is clearly NP-hard since it contains OFOC as a special case if we set $w_a^1 = w_a^2$, i.e., it costs the same to buy 1 unit or C units of capacity. We can thus restrict attention to the special case where all flow costs are 0. Note that in this case we may as well assume that d is integer, since if d is fractional the total cost of sending d or $\lceil d \rceil$ units of flow is the same.

TFOC without flow costs is very similar in structure to OFOC with flow costs. In fact, all the results from Sections 2 and 3 can be extended to TFOC, with minor modifications. Rather than repeat all the proofs, we simply discuss the minor modifications that can be used to obtain all the results.

Proposition 4.1. *The problem TFOC is* NP*-hard for the case when all flow costs are 0.*

Proof. The proof is similar to that of Proposition 2.1. Consider an instance of Minimum Cover as in the proof of Proposition 2.1. Construct the graph G_F as described. On the graph G_F, consider the problem TFOC, where $kC+1$ units of flow is to be sent from s to t for $C > 10n$. Arcs a along the paths P_j have $w_a^1 = w_a^2 = M$. The arcs of the form $(s, 0_1^1)$ and $\{(0_i^2, 0_{i+1}^1)\}_{i=2}^n$ have $w_a^1 = 4M/C, w_a^2 = 4M$. All other arcs have $w_a^1 = w_a^2 = 0$. The flow cost p_a is 0 on all arcs.

The rest of the proof is identical to that of Proposition 2.1. TFOC on G_F has a solution of value $(2n+1)kM + 4(n+1)M/C$ or less if and only if, there exists a solution to minimum cover. The result thus follows. □

To obtain an algorithm for TFOC we need structural properties similar to those obtained in Section 3.1. The key result to obtain is one similar to Proposition 3.1, since the rest would then follow as in Section 3. We need to modify some of the definitions in Section 3.1 to obtain such a result. A solution to TFOC is a vector (y_1^*, y_2^*, f^*), where $y_1^*(a)$ corresponds to the number of units of facility 1 purchased (of capacity 1 each) for arc a, $y_2^*(a)$ corresponds to the number of units of facility 2 purchased (of capacity C each) for arc a, and $f^*(a)$ corresponds to the flow on arc a. Given a solution (y_1^*, y_2^*, f^*), define an arc a to be a free arc if $f^*(a) < \lceil f^*(a)/C \rceil C$. Note that this definition is consistent with the definition of free arcs in Section 3.1.

Proposition 4.2. *Let (y_1^*, y_2^*, f^*) be an optimal solution to TFOC with the minimum number of free arcs. All the free arcs defined by (y_1^*, y_2^*, f^*) lie on a path (ignoring direction) from s to t. Free arcs directed forward along this path have a flow from $\{lC+r\}_{l=0}^{k}$, and those directed backwards have a flow from $\{lC-r\}_{l=1}^{k}$.*

Proof. The proof is similar to that of Proposition 3.1. As in the proof of Proposition 3.1 we can show that for any node set $X \subseteq V, s \in X, t \in V \setminus X$, at least one of the sets $\delta^+(X)$ or $\delta^-(X)$ contains a free arc. This implies at least one path from s to t (ignoring direction) using only free arcs.

Assume that there are two such paths P_1 and P_2 defined by the free arcs. Without loss of generality we can assume that the two paths have no arc in common. If this is not the case we can restrict attention to the distinct set of arcs in the two paths. Define

$$A_1 = \{a \in P_1 \cup P_2 : y_1^*(a) > 0\},$$
$$A_2 = P_1 \cup P_2 \setminus A_1.$$

Note that for $a \in A_1$, we can assume that $y_1^*(a) = f^*(a) - Cy_2^*(a)$ since $w^1(a) \geqslant 0$. For each arc in A_2, we have $f^*(a) < Cy_2^*(a)$.

Let $P_i^{\mathrm{f}}(P_i^{\mathrm{b}})$ be the arcs in P_i in the forward (backward) direction. For each arc a in $P_1 \cup P_2$ (with a flow of $f^*(a)$), let $W^I(a, f^*(a))$ be the cost (in terms of the cost of capacity to be purchased) of increasing flow by one unit from $f^*(a)$ to $f^*(a)+1$, and $W^D(a)$ be the savings (in terms of the cost of capacity to be purchased) from decreasing flow by one unit, from $f^*(a)$ to $f^*(a)-1$. If $a \in A_2$, then $W^I(a, f^*(a)) = 0$. If $a \in A_1$, then $W^I(a, f^*(a)) \leqslant w_1(a)$ and $W^D(a, f^*(a)) = w_1(a)$. Therefore,

$$W^I(a, f^*(a)) \leqslant W^D(a, f^*(a)) \quad \text{for all arcs } a \in P_1 \cup P_2.$$

This implies that either

$$\sum_{a \in P_1^{\mathrm{f}} \cup P_2^{\mathrm{b}}} W^I(a, f^*(a)) \leqslant \sum_{a \in P_1^{\mathrm{b}} \cup P_2^{\mathrm{f}}} W^D(a, f^*(a))$$

or

$$\sum_{a\in P_1^{\mathrm{b}}\cup P_2^{\mathrm{f}}} W^I(a,f^*(a)) \leqslant \sum_{a\in P_1^{\mathrm{f}}\cup P_2^{\mathrm{b}}} W^D(a,f^*(a)).$$

Without loss of generality assume that

$$\sum_{a\in P_1^{\mathrm{f}}\cup P_2^{\mathrm{b}}} W^I(a,f^*(a)) \leqslant \sum_{a\in P_1^{\mathrm{b}}\cup P_2^{\mathrm{f}}} W^D(a,f^*(a)). \tag{4}$$

Further note that if $\lfloor f_1/C \rfloor = \lfloor f_2/C \rfloor$ and $\lfloor f_1/C \rfloor C < f_1 < f_2 < \lceil f_1/C \rceil C$, we have

$$W^I(a,f_1) \geqslant W^I(a,f_2), \quad \text{and} \quad W^D(a,f_1) \geqslant W^D(a,f_2). \tag{5}$$

Define

$$\alpha = \min\{y_1^*(a) + Cy_2^*(a) - f^*(a) : a \in P_1^{\mathrm{b}} \cup P_2^{\mathrm{F}}\},$$

$$\beta = \min\{f^*(a) - (\lceil f^*(a)/C \rceil - 1)C : a \in P_1^{\mathrm{f}} \cup P_2^{\mathrm{f}}\}.$$

Consider the solution $(\bar{y}_1, \bar{y}_2, \bar{f})$, where,

$$\bar{f}_a = \begin{cases} f_a^* - \min\{\beta,\alpha\} & \text{for } a \in P_1^f \cup P_2^b, \\ f_a^* + \min\{\beta,\alpha\} & \text{for } a \in P_2^f \cup P_1^b, \\ f_a^* & \text{otherwise.} \end{cases}$$

For each arc a, $\bar{y}_1(a)$ and $\bar{y}_2(a)$ are defined so as to minimize the cost of installing sufficient capacity on arc a for a flow of $\bar{f}(a)$. From (4) and (5) we thus have

$$\sum_{a\in A}(w^1(a)\bar{y}_1(a) + w^2(a)\bar{y}_2(a)) \leqslant \sum_{a\in A}(w^1(a)y_1^*(a) + w^2(a)y_2^*(a)).$$

Thus $(\bar{y}_1, \bar{y}_2, \bar{f})$ is also an optimal solution to TFOC. However, it has one fewer free arc than (y_1^*, y_2^*, f^*), contradicting our assumption. Thus, the free arcs define exactly one path from s to t.

The rest of the proof is identical to the proof of Proposition 3.1. □

We can thus prove an equivalent of Proposition 3.2. Define *minimal free arc extreme optimal solutions* as in Section 3.1.

Proposition 4.3. *Let (y_1^*, y_2^*, f^*) be a minimal free arc extreme optimal solution to TFOC. There exist $2k+1$ paths $\{P_i\}_{i=1}^{2k+1}$, from s to t, such that*

$$f_a^* = (C-r)\sum_{i=1}^{k} P_i^a + r\sum_{i=k+1}^{2k+1} P_i^a,$$

where $P_i^a = 1$ if $a \in P_i$, 0 otherwise.

Given the graph $G = (V,A)$, we can construct an auxiliary graph $H = (N,E)$ exactly as described in Section 3.2. Define f_a^e as in (3). Define b_e^a to be the minimum cost of installing sufficient capacity (using both types of facilities) on arc a to support a flow of f_a^e. Define

$$b_e = \sum_{a \in A} b_e^a.$$

Using a proof identical to that of Theorem 3.1, we can thus prove that the shortest path in H from s_H to t_H, using arc cost b_e, defines the optimal solution to TFOC.

Theorem 4.1. *Given the problem TFOC, let the auxiliary graph H be defined as above. Arc weights b_e are as defined above for $e \in E$. Let P_H^* represent the shortest path in H from s_H to t_H. Let $\{P_i^*\}_{i=1}^{2k+1}$ be the $2k+1$ paths in G corresponding to P_H^*. Define the flow vector f^*, where*

$$f_a^* = (C-r)\sum_{i=1}^{k} P_i^*(a) + r \sum_{k+1}^{2k+1} P_i^*(a),$$

in which $P_i^*(a) = 1$ if $a \in P_i^*$, 0 otherwise. Let $y_1^*(a), y_2^*(a)$ be the minimum cost capacity to support a flow of f_a^* on arc a. The vector (y_1^*, y_2^*, f^*) is an optimal solution to OFOC.

From Theorem 4.1 it thus follows that for a fixed $\lfloor d/C \rfloor$, TFOC can be solved in polynomial time if all flow costs are 0. Magnanti and Mirchandani have provided a heuristic as in Section 3.3 that is asymptotically optimal for TFOC if all flow costs are 0. This heuristic also provides the optimal solution if $d = kC$.

5. Extended formulations for OFOC and TFOC

In this section, we use the characterization of extreme optimal solutions from Section 3 to obtain extended formulations for OFOC and TFOC. In each case we show that the LP-relaxation of the extended formulation gives a better lower bound for the integer optimum, compared to the natural formulation, with a family of "cut set" inequalities added. This is valuable because the LP-relaxation of the extended formulation can be solved in polynomial time, while the separation problem for the "cut set" inequalities is hard. In each case, we also characterize objective function coefficients for which the LP-relaxation of the extended formulation gives integer optima.

5.1. Extended formulation for OFOC

We first consider a natural formulation for OFOC. For each arc (i,j), let f_{ij} be the flow and y_{ij} the batches of capacity installed (each batch provides C units of capacity). OFOC can be formulated using the natural formulation **NFO** (see also [6]) below:

$$\text{Min} \quad \sum_{(i,j)\in A} w_{ij}y_{ij} + \sum_{(i,j)\in A} p_{ij}f_{ij}$$

$$\text{s.t.} \quad \sum_j f_{ji} - \sum_j f_{ij} = \begin{cases} -kC - r & \text{for } i = s, \\ kC + r & \text{for } i = t, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

$$Cy_{ij} - f_{ij} \geqslant 0, \quad y, f \geqslant 0; y \text{ integer.} \tag{7}$$

Define the polytopes

$$LPO_1 = \{(y,f) \geqslant 0 | (y,f) \text{ satisfies (6), (7)}\},$$

$$IPO = \{(y,f) \in LPO_1, y \text{ integer}\}.$$

We describe a set of strong valid inequalities for *IPO* that are similar to, and extend inequalities described by Magnanti and Mirchandani. Given $X \subset V$, define $\delta^+(X)$ ($\delta^-(X)$) to be the arcs in the cut directed out of (into) X. Given $X \subset V$, $s \in X$, $t \in V \setminus X$, partition the arcs in $\delta^+(X)$ into the sets B_1 and B_2. For any arc set $D \subset \delta^-(X)$, define the *cut set* inequality

$$\sum_{a\in B_1} f_a + r\sum_{a\in B_2} y_a + (C-r)\sum_{a\in D} y_a - \sum_{a\in D} f_a \geqslant r\lceil d/C \rceil. \tag{8}$$

For any arc set $S \subset A$, and any vector $x \in R^A$, define $x_S = \sum_{a\in S} x_a$. Given a vector (y,f) satisfying inequalities (8), and a set $S \subseteq A$, define $k_S = \lfloor (f_S - 1)/C \rfloor$ and $r_S = f_S - Ck_S$. We also assume hereafter that $k = \lfloor (d-1)/C \rfloor$ where $d = kC + r$. Notice that therefore $r > 0$. We now prove that the cut set inequalities are valid for *IPO*.

Theorem 5.1. *The cut set inequalities* (8) *are valid for IPO.*

Proof. To prove validity, we have to show that $f_{B_1} + ry_{B_2} + (C-r)y_D - f_D \geqslant r\lceil d/C \rceil$. There are two cases to consider.

Case 1: $f_D = 0$. Since the net flow across any cut is d units, $f_{B_2} \geqslant d - f_{B_1}$. Since $Cy_{B_2} \geqslant f_{B_2} \geqslant d - f_{B_1}$, $y_{B_2} \geqslant \lceil f_{B_2}/C \rceil \geqslant \lceil (d - f_{B_1})/C \rceil = \lceil (Ck + r - Ck_{B_1} - r_{B_1})/C \rceil$. Since we assume that $r > 0$ and $r_{B_1} \leqslant C$, we have

$$y_{B_2} \geqslant \begin{cases} k - k_{B_1} & \text{if } r \leqslant r_{B_1}, \\ k - k_{B_1} + 1 & \text{if } r > r_{B_1}. \end{cases}$$

Therefore, if $r \leqslant r_{B_1}$, we have

$$\begin{aligned} f_{B_1} + ry_{B_2} &= Ck_{B_1} + r_{B_1} + ry_{B_2} = (C-r)k_{B_1} + r_{B_1} + rk_{B_1} + ry_{B_2} \\ &\geqslant (C-r)k_{B_1} + r + rk \geqslant r\lceil d/C \rceil, \end{aligned}$$

where the last inequality follows from the fact that $k+1 = \lceil d/C \rceil$. Similarly, if $r > r_{B_1}$, we have

$$\begin{aligned} f_{B_1} + ry_{B_2} &= (C-r)k_{B_1} + r_{B_1} + rk_{B_1} + ry_{B_2} \\ &\geqslant (C-r)k_{B_1} + r_{B_1} + r(k+1) \geqslant r\lceil d/C \rceil. \end{aligned}$$

Case 2: $f_D > 0$. In this case, the flow across the arcs $B_1 \cup B_2$ is at least $d + f_D$. Thus, $y_{B_2} \geqslant \lceil f_{B_2}/C \rceil \geqslant \lceil (d + f_D - f_{B_1})/C \rceil = \lceil (Ck + r + Ck_D + r_D - Ck_{B_1} - r_{B_1})/C \rceil$. Therefore,

$$y_{B_2} \geqslant \begin{cases} k + k_D - k_{B_1} & \text{if } r + r_D - r_{B_1} \leqslant 0, \\ k + k_D - k_{B_1} + 1 & \text{if } 0 < r + r_D - r_{B_1} \leqslant C, \\ k + k_D - k_{B_1} + 2 & \text{if } C < r + r_D - r_{B_1}. \end{cases}$$

Since $f_D > 0$, $y_D \geqslant k_D + 1$. Therefore, the left-hand side of the inequality $K = f_{B_1} + ry_{B_2} + (C-r)y_D - f_D \geqslant (C-r)k_{B_1} + r_{B_1} + rk_{B_1} + ry_{B_2} + C - rk_D - r - r_D$. If $r + r_D - r_{B_1} \leqslant 0$, then $K \geqslant (C-r)k_{B_1} + rk + C \geqslant r(k+1)$. If $0 < r + r_D - r_{B_1} \leqslant C$, then $K \geqslant (C-r)k_{B_1} + rk + r \geqslant r(k+1)$. If $C < r + r_D - r_{B_1}$, then $K \geqslant (C-r)k_{B_1} + rk + r + C + r_{B_1} - r_D \geqslant r(k+1)$ where the last inequality follows from the fact that $C \geqslant r_D$. □

Define the polytope

$$LPO_2 = \{(y,f) \in LPO_1 | (y,f) \text{ satisfies } (8)\}.$$

In general, optimizing over LPO_2 is hard because the separation problem for the cut set inequalities is hard.

We now define an extended formulation for OFOC, based on the characterization of extreme optimal solutions in Proposition 3.1. By Proposition 3.1, all free arcs in an extreme optimal solution lie on a single path from s to t. Free arcs directed forward along this path have a flow from $\{lC + r\}_{l=0}^{k}$, free arcs directed backward on this free path have a flow of $\{lC - r\}_{l=1}^{k}$. All other arcs have flow that is a multiple of C.

Define variables h_{ij} which takes the value l if the flow on arc (i,j) equals lC, $lC + r$, or $(l+1)C - r$. In other words, h_{ij} takes the value l if the flow on arc (i,j) is at least lC but less than $(l+1)C$. The variable $e_{ij}(g_{ij})$ takes the value 1 if the flow on arc (i,j) is of the form $lC + r$ $(lC - r)$. The variable y_{ij} is as defined for the naural formulation. OFOC can now be formulated using the extended formulation **EFO** shown below:

$$\text{Min} \quad \sum_{(ij) \in A} w_{ij} y_{ij} + \sum_{(ij) \in A} p_{ij}(Ch_{ij} + re_{ij} + (C-r)g_{ij})$$

$$\text{s.t.} \quad \sum_j (e_{ji} - g_{ji} - e_{ij} + g_{ij}) = \begin{cases} -1 & \text{for } i = s, \\ 1 & \text{for } i = t, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

$$\sum_j (h_{ji} + g_{ji} - h_{ij} - g_{ij}) = \begin{cases} -k & \text{for } i = s, \\ k & \text{for } i = t, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

$$y_{ij} - e_{ij} - h_{ij} - g_{ij} \geqslant 0, \qquad e_{ij}, h_{ij}, g_{ij}, y_{ij} \geqslant 0, \text{ integer} \tag{11}$$

Define the polytopes,

$$EPO = \{(y,e,g,h) \geqslant 0 | (y,e,g,h) \text{ satisfies } (9),(10),(11)\},$$

$$EIPO = \{(y,e,g,h) \in EPO | e,g \in \{0,1\}, y,h \text{ integer}\}.$$

Let ϕ denote the linear transformation defined by $f_a = re_a + (C-r)g_a + Ch_a$. The next result shows that $IPO = \phi(EIPO)$. It is stated here without proof, since the proof is fairly straightforward and uses the fact that any extreme vector $(y,f) \in IPO$ if and only if there exists a vector $(y,e,g,h) \in EIPO$, where $f_a = re_a + (C-r)g_a + Ch_a$.

Theorem 5.2. *Any vector $(y,f) \in IPO$ if and only if there exists a vector $(y,e,g,h) \in EIPO$ where $f_a = re_a + (C-r)g_a + Ch_a$.*

The next result shows that $\phi(EPO) \subseteq LPO_2$, i.e., the linear transformation of the polytope EPO (from the LP-relaxation of the extended formulation) is contained in the polytope LPO_2 (from the LP-relaxation of the natural formulation and all cut set inequalities).

Lemma 5.1. *Given any vector $(y,e,g,h) \in EPO$, the vector $(y,f) \in LPO_2$, where $f_a = re_a + (C-r)g_a + Ch_a$.*

Proof. Consider any vector $(y,e,g,h) \in EPO$. Define $f_a = re_a + (C-r)g_a + Ch_a$ for each arc a. Since e,g,h satisfy constraints (9) and (10),

$$r\left(\sum_j (e_{ji} - g_{ji} - e_{ij} + g_{ij})\right) = \begin{cases} -r & \text{for } i = s, \\ r & \text{for } i = t, \\ 0 & \text{otherwise,} \end{cases}$$

$$C\left(\sum_j (h_{ji} + g_{ji} - h_{ij} - g_{ij})\right) = \begin{cases} -Ck & \text{for } i = s, \\ Ck & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

Adding the two sets of equations, we obtain

$$\sum_j (f_{ji} - f_{ij}) = \begin{cases} -d & \text{for } i = s, \\ d & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that (y,f) satisfies all equations in (6).

Since $Cy_{ij} - Ce_{ij} - Cg_{ij} - Ch_{ij} \geqslant 0$, and $Ce_{ij} + Cg_{ij} + Ch_{ij} \geqslant f_{ij}$, it follows that $Cy_{ij} - f_{ij} \geqslant 0$. Therefore, (y, f), satisfies all inequalities (7).

The equality constraints in **EFO** imply that $\sum_j (h_{sj} + e_{sj}) = \sum_j (h_{jt} + e_{jt}) = k + 1$. Therefore, $e_{B_1} + h_{B_1} + e_{B_2} + h_{B_2} - e_D - h_D \geqslant k + 1 = \lceil d/C \rceil$ across any cut $X \subset V$ such that $s \in X$, $t \in V \backslash X$. Observe that the vector (y, e, g, h) satisfies the following:

$$\begin{aligned} r(e_{B_1} + h_{B_1} + e_{B_2} + h_{B_2} - e_D - h_D) &\geqslant r\lceil d/C \rceil, \\ f_{B_1} &= Ch_{B_1} + re_{B_1} + (C - r)g_{B_1}, \\ ry_{B_2} &\geqslant re_{B_2} + rh_{B_2} + rg_{B_2}, \\ (C - r)y_D &\geqslant (C - r)(e_D + h_D + g_D), \\ re_D + Ch_D + (C - r)g_D &= f_D. \end{aligned}$$

This implies that

$$\begin{aligned} &f_{B_1} + ry_{B_2} + (C - r)y_D - f_D \geqslant r\lceil d/C \rceil \\ &\quad + (C - r)(h_{B_1} + g_{B_1} + e_D) + rg_{B_2} \geqslant r\lceil d/C \rceil. \end{aligned}$$

Thus, the vector (y, f) satisfies all the cut set inequalities (8). The result thus follows. □

Therefore, EPO provides a LP-relaxation for OFOC that is at least as strong as the one provided by LPO_2 in terms of the lower bound.

Theorem 5.3. *Comparing EPO and LPO_2 we have*

$$\begin{aligned} &\min \left\{ \sum_{a \in A} w_a y_a + \sum_{a \in A} p_a (Ch_a + re_a + (C - r)g_a) | (y, e, g, h) \in EPO \right\} \\ &\quad \geqslant \min \left\{ \sum_{a \in A} w_a y_a + \sum_{a \in A} p_a f_a | (y, x) \in LPO_2 \right\}. \end{aligned}$$

The LP-relaxation of the extended formulation is thus at least as strong as the LP-relaxation of the natural formulation, even after the addition of all cut set inequalities (8).

We now establish conditions on the cost function coefficients that guarantee integer solutions for OFOC. Let $a(i,j)$ $(b(i,j))$ denote the minimum distance from node i to node j if arc costs are set to $w_{ij} + Cp_{ij}$ $(w_{ij} + rp_{ij})$ and let $P^{\alpha}(i,j)$ $(P^{\beta}(i,j))$ be the shortest such path. The next result gives a condition under which optimizing over EPO results in a integer optimal solution.

Theorem 5.4. *If for every arc (i,j), $a(s,j) - a(s,i) + b(s,i) - b(s,j) \leqslant w_{ij} + (C - r)p_{ij}$, then the the optimal solution over EPO (the linear programming relaxation of EFO) is an integer.*

Proof. The dual **DEFO** of the LP-relaxation of EFO is

(DR)
$$
\begin{aligned}
&\operatorname{Max}(\alpha_t - \alpha_s)k + \beta_t - \beta_s \\
&\text{s.t. } \alpha_j - \alpha_i - \gamma_{ij} \leqslant Cp_{ij}, \\
&\beta_j - \beta_i - \gamma_{ij} \leqslant rp_{ij}, \\
&\alpha_j - \alpha_i - \beta_j + \beta_i - \gamma_{ij} \leqslant (C-r)p_{ij}, \\
&\gamma_{ij} \leqslant w_{ij}, \\
&\alpha_i, \beta_i, \gamma_{ij} \geqslant 0.
\end{aligned}
$$

Set $\alpha_i = a(s,i)$ and $\beta_i = b(s,i)$ for every node i. Set $\gamma_{ij} = w_{ij}$ for every arc. Send kC units of flow from s to t on the shortest $P^\alpha(s,t)$ path and r units of flow on the shortest $P^\beta(s,t)$ path. Set $y_{ij} = l$ if flow on arc (i,j) equals lC and $y_{ij} = l+1$ if flow equals $lC + r$. This gives a primal feasible solution. It is easy to verify that the α_i, β_i and γ_{ij} values satisfy dual feasibility. Clearly, if flow value e_{ij} or f_{ij} on any arc is positive, the corresponding dual constraint is satisfied at equality since flow is along shortest paths. Hence, complementary slackness conditions are satisfied and therefore, we have an optimal solution. □

As a corollary to this result we obtain

Corollary 5.1. *If $w_{ij}/p_{ij} = q$ for all arcs (i,j), then the optimal solution over EPO is integer.*

Proof. Let $Q_\alpha(i)$ ($Q_\beta(i)$) denote the set of arcs on the shortest $P^\alpha(s,i)$ ($P^\beta(s,i)$) path from node s to node i. Then $a(s,i) = \sum_{(kl)\in Q_\alpha(i)}(w_{kl} + Cp_{kl}) = (q+C)\sum_{(kl)\in Q_\alpha(i)} p_{kl}$. Similarly, $b(s,i) = (q+r)\sum_{(kl)\in Q_\beta(i)} p_{kl}$. Let δ_i denote the shortest distance from node s to node i using p_{kl} as arc lengths. Then, $a(s,j) - a(s,i) + b(s,i) - b(s,j) = (C-r)(\delta_j - \delta_i) \leqslant (C-r)p_{ij} \leqslant w_{ij} + (C-r)p_{ij}$. The result thus follows by Theorem 5.4. Notice that we can send the entire flow d on the shortest $s-t$ path with p_{ij} as arc costs. □

5.2. Extended formulation for TFOC

We first consider a natural formulation for TFOC as in Magnanti and Mirchandani. For each arc (i,j), let f_{ij} be the flow, y^1_{ij} the batches of facility 1 (each batch has 1 unit), and y^2_{ij} the batches of facility 2 (each batch has C units) installed. TFOC can be formulated using the natural formulation NFT shown below:

$$
\operatorname{Min} \sum_{(i,j)\in A} w^1_{ij} y^1_{ij} + \sum_{(i,j)\in A} w^2_{ij} y^2_{ij} + \sum_{(i,j)\in A} p_{ij} f_{ij}
$$

$$
\text{s.t.} \quad \sum_j f_{ji} - \sum_j f_{ij} = \begin{cases} -kC - r & \text{for } i = s, \\ kC + r & \text{for } i = t, \\ 0 & \text{otherwise.} \end{cases} \tag{12}
$$

$$
Cy^2_{ij} + y^1_{ij} - f_{ij} \geqslant 0, \tag{13}
$$

$y^1, y^2, f \geqslant 0$; y^1, y^2 integer.

Define the polytopes

$$LPT_1 = \{(y^1, y^2, f) \geqslant 0 | (y^1, y^2, f) \text{ satisfies } (12), (13)\},$$

$$IPT = \{(y_1, y_2, f) \in LPT_1, y^1, y^2 \text{ integer}\}.$$

We describe *cut set* inequalities similar to inequality (8). Given $X \subset V, s \in X, t \in V \backslash X$, let B denote the set of arcs in the cut $\delta^+(X)$. For any arc set $D \subseteq \delta^-(X)$, define the cut set inequality

$$\sum_{a \in B}(y_a^1 + r y_a^2) + \sum_{a \in D}(y_a^1 + (C - r)y_a^2) - \sum_{a \in D} f_a \geqslant r\lceil d/C \rceil. \tag{14}$$

The validity of the cut set inequalities (14) can be proved as for the cut set inequalities (8).

Theorem 5.5. *The cut set inequalities* (14) *are valid for IPT.*

Define the polytope

$$LPT_2 = \{(y^1, y^2, f) \in LPT_1 | (y^1, y^2, f) \text{ satisfies } (14)\}.$$

For the extended formulation, define variables h_{ij}^1 (h_{ij}^2) which take the value l if the flow on arc (i,j) equals lC, $lC + r$, or $(l + 1)C - r$ and the capacity is provided by type 1 (type 2) facilities. Thus if flow on arc (i,j) is $kC + r$ and $y_{ij}^2 = l$ and $y_{ij}^1 = (k - l)C + r$, then $h_{ij}^1 = l$ and $h_{ij}^2 = k - l$. The variable $e_{ij}^1(e_{ij}^2)$ takes the value 1 if the flow on arc (i,j) is $lC + r$ and the capacity for the last r units is provided by type 1 (type 2) facilities. Similarly $g_{ij}^1(g_{ij}^2)$ takes the value 1 if the flow on arc (i,j) is $lC - r$ and the capacity for the last $C - r$ units of flow is provided by type 1 (type 2) facilities. TFOC can be formulated using the extended formulation **EFT** shown below:

$$\text{Min} \sum_{(ij) \in A} \sum_{u=1}^{2} w_{ij}^u y_{ij}^u$$

$$\text{s.t.} \sum_{j} \sum_{u=1}^{2} (e_{ji}^u - g_{ji}^u - e_{ij}^u + g_{ij}^u) = \begin{cases} -1 & \text{for } i = s, \\ 1 & \text{for } i = t, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

$$\sum_{j} \sum_{u=1}^{2} (h_{ji}^u + g_{ji}^u - h_{ij}^u - g_{ij}^u) = \begin{cases} -k & \text{for } i = s, \\ k & \text{for} i = t, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

$$y_{ij}^1 - r e_{ij}^1 - C h_{ij}^1 - (C - r) g_{ij}^1 \geqslant 0, \tag{17}$$

$$y_{ij}^2 - e_{ij}^2 - h_{ij}^2 - g_{ij}^2 \geqslant 0, \tag{18}$$

$$e_{ij}^u, h_{ij}^u, g_{ij}^u, y_{ij}^u \geqslant 0, \text{ integer.}$$

Define the polytopes

$$\begin{aligned} EPT &= \{(y^u, e^u, g^u, h^u; u = 1,2) \\ &\quad \geqslant 0 | (y^u, e^u, g^u, h^u; u = 1,2) \text{ satisfies } (15)-(18)\}, \\ EIPT &= \{(y^u, e^u, g^u, h^u; u = 1,2) \in EPT | e^u, g^u \in \{0,1\}, y, h \text{ integer}\}. \end{aligned}$$

Let θ denote the linear transformation defined by $f_a = \sum_{u=1}^{2}(Ch_a^u + re_a^u + (C-r)g_a^u)$. The next result (stated without proof) shows that $IPT = \theta(EIPT)$.

Theorem 5.6. *Any vector $(y, f) \in IPT$ if and only if there exists a vector $(y, e, g, h) \in EIPT$ where $f_a = re_a + (C-r)g_a + Ch_a$.*

For any subset of arcs $S \subset A$, and any quantity x_a^u, define $x_S^u \equiv \sum_{a \in S} x_a^u$ for $u = 1,2$. The next result shows that $\theta(EPT) \subseteq LPT_2$.

Lemma 5.2. *Given any vector $(y^u, e^u, g^u, h^u; u = 1,2) \in EPT$, the vector $(y^1, y^2, f) \in LPT_2$, where $f_a = \sum_u (re_a^u + (C-r)g_a^u + Ch_a^u)$.*

Proof. One can show that the vector (y^1, y^2, f) satisfies (12) and (13) similar to the proof of Lemma 5.1. We now prove that the vector (y^1, y^2, f) satisfies all cut set inequalities (14).

The equality constraints in **EFT** imply that $\sum_j (h_{sj}^1 + h_{sj}^2 + e_{sj}^1 + e_{sj}^2) = \sum_j (h_{jt}^1 + h_{jt}^2 + e_{jt}^1 + e_{jt}^2) = k + 1$. Therefore, $h_B^1 + h_B^2 + e_B^1 + e_B^2 - h_D^1 - h_D^2 - e_D^1 - e_D^2 \geqslant k + 1 = \lceil d/C \rceil$, across any cut $X \subset V$ such that $s \in X$, $t \in V \backslash X$. Observe that the vector $(y^u, e^u, g^u, h^u; u = 1,2)$ satisfies the inequalities

$$\begin{aligned} y_B^1 &\geqslant re_B^1 + Ch_B^1 + (C-r)g_B^1, \\ ry_B^2 &\geqslant r(e_B^2 + h_B^2 + g_B^2), \\ y_D^1 &\geqslant re_D^1 + Ch_D^1 + (C-r)g_D^1, \\ (C-r)y_D^2 &\geqslant (C-r)(e_D^2 + h_D^2 + g_D^2), \\ r(h_B^1 + h_B^2 + e_B^1 + e_B^2 - h_D^1 - h_D^2 - e_D^1 - e_D^2) &\geqslant r\lceil d/C \rceil, \end{aligned}$$

and the equalities

$$\begin{aligned} re_D^1 + Ch_D^1 + (C-r)g_D^1 &= f_D^1, \\ re_D^2 + Ch_D^2 + (C-r)g_D^2 &= f_D^2. \end{aligned}$$

This implies that

$$y_B^1 + ry_B^2 + y_D^1 + (C-r)y_D^2 - f_D \geqslant r\lceil d/C \rceil.$$

The result thus follows. $\square$

Therefore, EPT provides a LP-relaxation for TFOC that is at least as strong as the one provided by LPT_2 in terms of the lower bound.

Theorem 5.7. *Comparing EPT and LPT_2 we have*

$$\min\left\{\sum_{a\in A}\sum_{u=1}^{2} w_a^u y_a^u \mid (y^u, e^y, g^u, h^u; u = 1,2) \in EPT\right\}$$
$$\geqslant \min\left\{\sum_{a\in A}\sum_{u=1}^{2} w_a^u y_a^u \mid (y^1, y^2, f) \in LPT_2\right\}.$$

If $w_{ij}^2 \geqslant Cw_{ij}^1$, then we need not consider facility 2 on arc (i,j). Therefore, we assume that $w_{ij}^2 < Cw_{ij}^1$. Hence, the minimum cost of sending lC units on any arc (i,j) is always lw_{ij}^2. Let $a(i,j)$ $(b(i,j))$ be the shortest distance from i to j if we set arc costs to $w_{kl}^2(\min\{rw_{kl}^1, w_{kl}^2\})$, and let $w_{ij}(r) = \min\{rw_{kl}^1, w_{kl}^2\}$. We now give a sufficient condition under which optimizing over EPT results in an integer solution.

Theorem 5.8. *If for every arc $(i,j), a(s,j) - a(s,i) + b(s,i) - b(s,j) \leqslant \min\{(C-r)w_{ij}^1, w_{ij}^2\}$ then the optimal solution over EPT is integer.*

Proof. Set $\alpha_i = a(s,i)$ and $\beta_i = b(s,i)$ for every node i. Set $\gamma_{ij}^1 = w_{ij}^1$ and $\gamma_{ij}^2 = w_{ij}^2$ for every arc. It is easy to verify that this gives a dual feasible solution. Send kC units of flow on the shortest $P^\alpha(s,t)$ path and r units of flow on the shortest $P^\beta(s,t)$ path from s to t. If flow on arc (i,j) equals lC, set $y_{ij}^2 = h_{ij}^2 = l$. Suppose flow on arc (i,j) equals $lC + r$. If $rw_{ij}^1 \leqslant w_{ij}^2$, set $y_{ij}^1 = r$, $e_{ij}^1 = 1$ and $y_{ij}^2 = h_{ij}^2 = l$. If $rw_{ij}^1 > w_{ij}^2$, set $y_{ij}^2 = l+1$, $h_{ij}^2 = l$ and $e_{ij}^2 = 1$. This gives a primal feasible solution. If flow value e_{ij}^1, e_{ij}^2, h_{ij}^1 or h_{ij}^2 on any arc is positive, then the corresponding dual constraint is satisfied at equality since flow is along shortest paths. Complementary slackness conditions are therefore satisfied and hence, we have an optimal solution. □

As a corollary to this result we obtain

Corollary 5.2. *If $w_{ij}^2/w_{ij}^1 = q$ for all arcs (i,j), then the optimal solution over EPT is integer, with no reverse arcs on the free path.*

Proof. We assume that $q < C$, since otherwise, we need not consider facility two in any optimal solution. Let $Q_\alpha(i)$ denote the set of arcs on the shortest $P^\alpha(s,i)$ path from node s to node i. Then $a(s,i) = \sum_{(kl)\in Q_\alpha(i)} w_{kl}^2 = q\sum_{(kl)\in Q_\alpha(i)} w_{kl}^1$. There are two cases to consider.

Case 1: Suppose $q \leqslant r$. Then $w_{ij}(r) = w_{ij}^2$ for all arcs. Therefore, the shortest $P^\alpha(s,i)$ path is the same as the shortest $P^\beta(s,i)$ path.
Hence,

$$a(s,j) - a(s,i) + b(s,i) - b(s,j) = 0 \leqslant \min\{(C-r)w_{ij}^1, w_{ij}^2\}.$$

Case 2: If $r < q$, then $w_{ij}(r) = rw_{ij}^1$. Let δ_i denote the shortest distance from s to i if we set all arc costs to w_{ij}^1.

Table 1
Graphs used in computational tests

Graph name	No. of nodes	No. of arcs
Graph 1	27	102
Graph 2	30	120
Graph 3	40	160
Graph 4	50	200

Then,

$$\begin{aligned} a(s,j) - a(s,i) + b(s,i) - b(s,j) &= q(\delta_j - \delta_i) + r(\delta_i - \delta_j) \\ &= (q - r)(\delta_j - \delta_i) \leqslant (q - r)w_{ij}^1 \\ &\leqslant \min\{(C - r)w_{ij}^1, w_{ij}^2\}. \end{aligned}$$

The result thus follows by Theorem 5.8. $\square$

6. Computational results

In the previous section we saw a theoretical justification for using extended formulations for both OFOC and TFOC. In this section we present computational results showing the efficacy of the extended formulations in practice.

At the outset of the computational tests, we were seeking answers to the following questions:

(1) How much more effective is the extended formulation compared to the natural formulation?

(2) How effective is the extended formulation in solving OFOC and TFOC?

To try and obtain answers to these questions, we solved a total of 378 problems (189 for each of OFOC and TFOC), using each of the two formulations (natural and extended). All problems were generated from four basic graphs described in Table 1. Graph 1 comes from a real world problem while graphs 2, 3, and 4 are randomly generated. Given the graphs the graphs and costs, new problems are generated by randomly generating a new source and sink. Figs. 4 and 5 contain results for OFOC while Figs. 6 and 7 contain results for TFOC. Each point in the graphs represents an average over three problems.

All computational tests are on an HP Apollo 715/50 and the LP-solver used is CPLEX version 2.1.

For OFOC, Graph 1 was used to generate three problems each for flow values of $kC + r$ for $k \in \{1,3,6,9\}$ and $r \in \{1,2,\dots,9\}$. In all the problems we use a batch size of $C = 10$. Each of the three problems were generated by randomly generating a new source and sink. Optimal solutions for the LP-relaxation of the natural formulation (Z1) as well as the extended formulation (Z2) were recorded for each problem. The graph in Fig. 4 records the average percent gap (over three problem instances) between

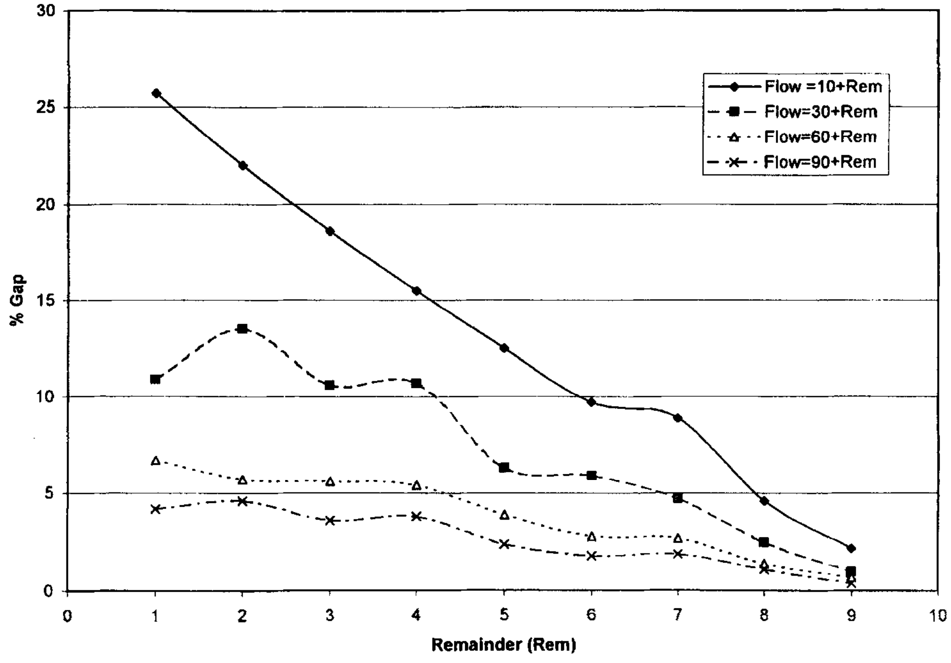
30
25
20
15
10
5
0
% Gap
Flow =10+Rem
Flow=30+Rem
Flow=60+Rem
Flow=90+Rem
0
1
2
3
4
5
6
7
8
9
10
Remainder (Rem)

Fig. 4.

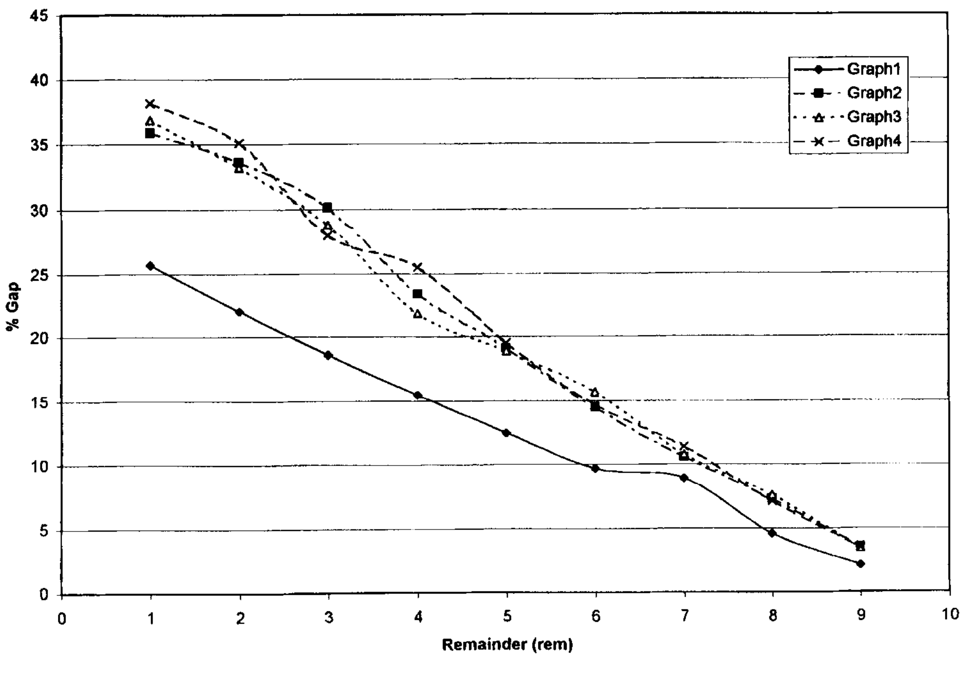
45
40
35
30
25
20
15
10
5
0
% Gap
Graph1
Graph2
Graph3
Graph4
0
1
2
3
4
5
6
7
8
9
10
Remainder (rem)

Fig. 5.

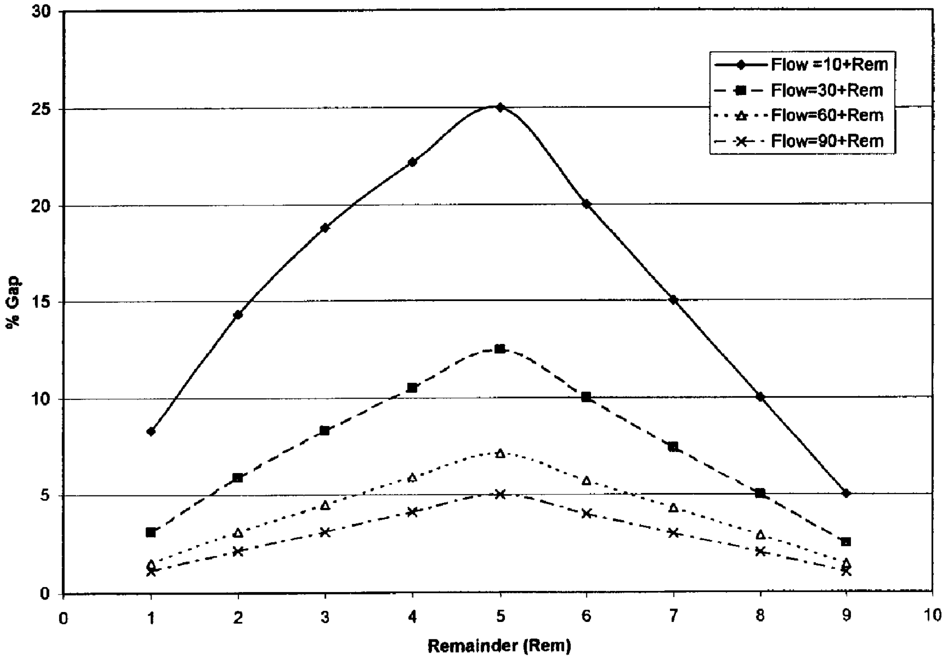
Flow =10+Rem
Flow=30+Rem
Flow=60+Rem
Flow=90+Rem
% Gap
Remainder (Rem)
0
5
10
15
20
25
30
0
1
2
3
4
5
6
7
8
9
10

Fig. 6.

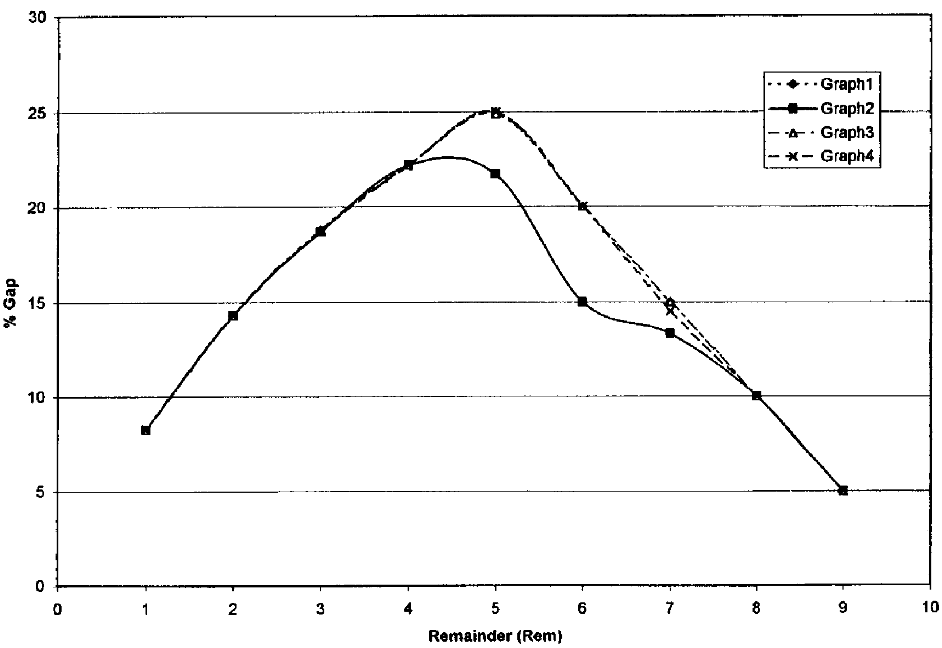
Graph1
Graph2
Graph3
Graph4
% Gap
Remainder (Rem)
0
5
10
15
20
25
30
0
1
2
3
4
5
6
7
8
9
10

Fig. 7.

the two solutions given by $100(Z2 - Z1)/Z2$ for different flow values. Fig. 5 records the average percent gap for the four graphs in Table 1 for flow values of $10 + r$, for $r \in \{1,2,\ldots,9\}$.

From our computational test solving OFOC we make the following observations:

1. For each problem instance attempted (189 in all), the LP-relaxation of the extended formulation gives an integer optimal solution for OFOC. This is not the case for even one problem instance using the natural formulation. The percent gaps ($100(Z2-Z1)/Z2$) observed are as large as 38% between the optimal solution of the LP-relaxation of the natural and extended formulation (see Figs. 4 and 5).
2. The largest amount of time taken to obtain the integer solution for OFOC using the extended formulation is under 6 s.
3. The gap between the natural and extended formulations tends to be large (over 30%) for small remainders (i.e., flow $= kC + r$ where $r = 1$) and declines significantly (to as low as 3%) when the remainder is close to batch size of capacity (i.e., flow $= kC + r$ for $r = 9$ where $C = 10$).
4. The gap between the natural and extended formulations declines (see Fig. 4) as the total flow increases. For example, the gap is 25.7% when flow is 11 ($=1 \times 10 + 1$) and declines to 4.2% when flow is 91 ($=9 \times 10 + 1$), even though the remainder is the same in both cases.

To test performance of the two formulations for TFOC, we used the same four graphs listed in Table 1. Facility 1 was assumed to provide a capacity of 1 unit while facility 2 was assumed to provide a capacity of 10 units. The costs for facility 2 are as in the problem instances for OFOC. The costs of facility 1 are randomly generated such that between 4 and 6 units of facility 1 cost the same as 1 unit of facility 2. Flow costs are assumed to be zero in each case. Figs. 6 and 7 contain the average percent gap (over three problem instances for each point) between the LP-relaxation of the natural and extended formulations for TFOC. From the computational results we make the following observations:

1. For each problem instance (189 in all), the LP-relaxation of the extended formulation gives an integer optimal solution for TFOC. This is not the case for even one problem instance using the natural formulation. The percent gap between the LP-relxation to the natural and extended formulations are as large as 25% (see Figs. 6 and 7).
2. The largest amount of time taken to obtain the optimal solution using the extended formulation was under 8 s.
3. The percent gap between the natural and extended formulation tends to be large (about 25%) for remainders around 5 (see Fig. 6) and declines significantly (to about 2–3%) as the remainder declines to 1 or increases to 9 (recall that facility 2 has a batch size of 10).
4. The gap between the natural and extended formulations declines (see Fig. 6) as the total flow increases. For example, the gap is 25% when flow is 15 ($=1 \times 10 + 5$) and declines to 5% when flow is 95 ($=9 \times 10 + 5$).

From our computational experiments we conclude that the extended formulations given in Section 5 are very effective in solving OFOC and TFOC and are far superior to the natural formulations. In each instance attempted by us, the extended formulation results in an integer optimal solution without resorting to branch-and-bound. This supports our claim that the extended formulations are effective for solving OFOC and TFOC.

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