Assessing NSE's Daily Zero Coupon Yield Curve Estimates: A Comparison with Few Competing Alternatives

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Abstract

This study assesses and compares, on select criteria of evaluation, the time series of daily term structure estimates provided by the National Stock Exchange (NSE) [using the Nelson-Siegel (1987; NS) methodology] with author's own estimates of NS, Svensson (1992; SV) and Cox-Ingersoll-Ross (1985; CIR). While no model comes across as best on all criteria of evaluation, NS as estimated by NSE (and in this study) turn out to be the worst of the lot. Wherever CIR comes out to be better than SV, however, the difference is only marginal. As none of the models came out to be best on all days, given the numerical tractability of parsimonious models and availability of relatively cheap computing resources, it is suggested that NSE report estimates based on 3/4 competing specifications and not just using NS, which should be phased out. A suitable alternative exists in SV.

[Preliminary Draft. Please Don't Quote]

I. Introduction

This study assesses and compares, on select criteria of evaluation, the time series of daily term structure estimates provided by the National Stock Exchange (NSE[1]) with that from two competing alternatives.

Although NSE clearly provides a useful service to the Indian financial markets by providing daily estimates of the term structure, one wonders why NSE has chosen Nelson-Siegel (1987; NS). The attractiveness of NS lies in its parsimony and its ability to provide economically sensible estimates of the term structure. However, there exist equally attractive parsimonious models for term structure estimation and *a priori* there doesn't seem to be any justification for choosing NS over others. In fact, just from a pure theoretical standpoint, NS may not be an especially good choice. Bjork and Christensen (1996) and Filipovic (1999) show that within the Heath-Jarrow-Morton (1992) framework (with either deterministic or stochastic volatility), there exists *no* non-trivial interest rate model that is consistent with the NS family[2].

Even so, one would like to believe that NSE has compared/compares the performance of competing models over a period of time and has found NS to be better than the others. As things stand, though, NSE neither provides the exact estimation strategy for its term structure estimation nor results on comparative performance of NS against other competing (parsimonious) specifications. Normative though it may sound, this is not a good thing.

Although term structure modelling using (over-parameterized) splines is also popular, this study doesn't purport to involve itself with the argument of over-parameterized vs. parsimonious models. The objective here is simply to provide an understanding as to how NS fares against competing specifications, and whether NSE should consider other models (too).

In particular, this study compares the performance of NSE's NS estimates with that of Svensson (1992; SV), Cox-Ingersoll-Ross (1985; CIR) and author's own estimates of NS[3]. Note that while NS and SV are atheoretical models designed to fit the observed

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[1] The premier stock exchange of India; also hosting the biggest (both wholesale and retail) debt market; see http://www.nse-india.com for details

[2] Author thanks Prof. J. R. Varma for pointing out this result

[3] NSE provides neither the exact estimation strategy nor the objective function for its NS estimates

yields, CIR provides implications for a term structure based on a theoretical model derived from intertemporal description of a competitive economy.

Since the literature is replete with the description/applications of term structure methods studied here (see Bliss, 1997, Ioannides, 2003 and others for a survey and original papers for the details), after describing the methodology of estimation, we jump straight into comparison.

II. Methodology

Estimating a term structure, Bliss (1997) notes, requires decision on the following three aspects:

1. A Pricing Function
2. A Discount/Rate Function
3. Estimation Technique

Since the main objective of the study is to compare across various term structure specifications, an important decision aspect becomes:

4. Criteria of Evaluation

2.1 The Pricing Function

In the literature (see, for example, Bolder and Streliski, 1999, Bliss, 1997, Darbha, Roy and Pawaskar, 2003 etc.) it is standard to specify the price of a default-risk free bond, in absence of arbitrage, as:

$$P = \sum_{m=1}^{M} c_m \delta_m \qquad \textbf{[1]}$$

where M is the time to maturity of the bond, c_m is the cash flow received at time m, and δ_m is what is called the 'discount function' in the term structure literature. The above equation relates the discounted cash flows from the bond in discrete time periods to the price of the bond. It is a rather straight forward matter to convert 'discount function' to a 'rate function' using the following equation:

$$r(m) = -\frac{\ln(\delta_m)}{m} \qquad \textbf{[2]}$$

Since, conditions for perfect markets don't exist in reality, and cash flows are received only at discrete times, in practice one needs to give a stochastic form to equation **[1]**, such as:

$$P = f\ [c_m, r(m)] + \varepsilon \qquad \textbf{[3]}$$

where ε is the 'error' term and accounts for whatever is not captured in the function f about how bonds are priced. Bliss (1997) uses the term "omitted pricing factors" for

"...factors which have been omitted from the bond pricing equation which nonetheless impact the pricing of bonds[4]"

2.2 The Discount Rate Function

The next decision in the exercise of term structure estimation involves the selection of a form for the discount rate function. By selection, as stated earlier, the discount rate forms studied here are the following:

1. Nelson-Siegel (1987)
2. Svensson (1992)
3. Cox-Ingersoll-Ross (1985)

While the first two in the above list are empirical curve fitting exercise, the last results from general equilibrium models of a competitive economy. What follows is a brief description of models to be estimated in this study. For details the reader is referred to the aforementioned articles.

2.2.1 NS

NS assume that the instantaneous forward rate is the solution to a second order differential equation with two equal roots. The forward rate function used by NS is:

$$f(m\ ;b) = \beta_0 + \beta_1 \exp(-m/\tau_1) + \beta_2 \frac{m}{\tau_1} \exp(-m/\tau_1) \qquad \textbf{[4]}$$

[4] R. R. Bliss (1997), "Testing Term Structure Estimation Methods", Advances in Futures and Options Research, 9

where $b \equiv (\beta_0, \beta_1, \beta_2, \tau)$ is the vector of parameters to be estimated. The spot rate function can in turn be derived by integrating the above equation. This gives:

$$s(m\,;b) = \beta_0 + (\beta_1 + \beta_2)\frac{1 - \exp(-m/\tau_1)}{m/\tau_1} - \beta_2 \exp(-m/\tau_1) \qquad \textbf{[5]}$$

The spot rate function has four parameters. While β_0 and $\beta_0 + \beta_1$ are implied long-rate and short-rate respectively, β_2 gives the medium term component of the yield curve, and along with τ defines the shape of the curve. The possible shapes of the term structure that result as parameters vary can be found in NS, SV and Bolder and Streliski (1999) and won't be discussed here.

2.2.2 SV

SV adds a fourth term to the forward rate function given by NS, with two additional parameters, (β_3, τ_2), thereby adding to the flexibility of the shape of the term structure (possibility of a second 'hump' – or what is often referred to as an *S*-shaped curve in the literature – with β_3 and the other time decay parameter, τ_2). The corresponding functions are then given as:

$$f(m\,;b) = \beta_0 + \beta_1 \exp(-m/\tau_1) + \beta_2 \frac{m}{\tau_1}\exp(-m/\tau_1) + \beta_3 \frac{m}{\tau_2}\exp(-m/\tau_2) \qquad \textbf{[6]}$$

$$s(m\,;b) = \beta_0 + (\beta_1 + \beta_2)\frac{1 - \exp(-m/\tau_1)}{m/\tau_1} - \beta_2 \exp(-m/\tau_1) + \beta_3 \frac{1 - \exp(-m/\tau_2)}{m/\tau_2} - \beta_3 \exp(-m/\tau_2) \qquad \textbf{[7]}$$

2.2.3 Empirical Implications of the Cox-Ingersoll-Ross Model

The dynamics of the interest rate process in the CIR model is given as[5]:

$$dr_t = \kappa\,(\theta - r)\,dt + \sigma\sqrt{r}\,dz \qquad \textbf{[8]}$$

[5] κ is the mean reversion coefficient, θ is the mean of the process, *r* is the instantaneous short rate, σ^2 is the scale factor for variance of *r*, and λ is the price of risk associated with *r*.

CIR, just like other affine models, in absence of arbitrage, results in the following pricing equation:

$$P\,[r,\,t,\,T] = A\,[t,\,T]\;e^{-B\,[t,\,T]\,r} \qquad \textbf{[9]}$$

where for $\tau = T - t$

$$A\,[t,T] = \left[\frac{\phi_1 \exp\,(\phi_2\tau)}{\phi_2\left(\exp\,(\phi_1\tau) - 1\right) + \phi_1}\right]^{\phi_3} \qquad \textbf{[10]}$$

$$B\,[t,T] = \frac{\exp(\phi_1\tau) - 1}{\phi_2\left(\exp(\phi_1\tau) - 1\right) + \phi_1} \qquad \textbf{[11]}$$

where

$$\phi_1 = \left((\kappa + \lambda)^2 + 2\sigma^2\right)^{1/2} \qquad \textbf{[12]}$$

$$\phi_2 = \left(\kappa + \lambda + \phi_1\right) / \; 2 \qquad \textbf{[13]}$$

$$\phi_3 = 2\kappa\theta / \sigma^2 \qquad \textbf{[14]}$$

Value of a coupon bond can then be written as:

$$V\,[t,c,d] = \sum c_i P(r,t,d_i) \qquad \textbf{[15]}$$

where *d* is the vector of coupon payment dates.

Then, given the prices of the traded bonds, one can estimate the parameters ϕ_1, ϕ_2, ϕ_3 and *r* (though for actual dynamics it is not possible[6] to separately identify the parameters, θ, κ and λ). The long-rate and volatility of the short-rate are given as a function of the parameters ϕ_1, ϕ_2, ϕ_3 as follows:

$$r_L = \left(\phi_1 - \phi_2\right)\phi_3 \qquad \textbf{[16]}$$

$$\sigma^2 = 2\left(\phi_1\phi_2 - \phi_2^2\right) \qquad \textbf{[17]}$$

[6] for risk-neutral dynamics with λ, the parameters of the process can be uniquely identified from equations 5.13 – 5.15

Before moving further, it must be acknowledged that the theoretical CIR model describes the process of *real* rates, as opposed to nominal rates. However, that said, it is still attractive for modelling nominal rates because the structure of the model precludes negative interest rates.

It is also intuitive because, like NS and SV, the model implies that the long rate $(m \to \infty)$ converges to a constant. Now, although, volatility of the yield of the longest maturity bond traded in the money market is clearly not zero, the fact that it converges to a constant makes it appealing.

Here it becomes important to state that it is not intended here to test the theoretical CIR model, as in whether the restrictions it imposes are empirically fulfilled or not. Attempt is simply to fit the data to the functional form on the lines of Brown and Dybwig (1986), Barone, Cuoco, and Zautzik (1991), and Brown and Schaefer (1994) and others, and see how well a parsimonious single factor model compares with atheoretical NS and SV functional forms.

2.3 Estimation

The optimization problem is to minimize the weighted sum of square of (price) errors, i.e. the objective function is:

$$min \sum_{i}^{N} \left(\omega_i \varepsilon_i \right)^2 \qquad \textbf{[18]}$$

subject to non-negativity constraints imposed on the short-rate, the long rate $(m \to \infty)$ and on the τ *s;* where $\varepsilon_i = P_i - \hat{P}_i$, and

$$\omega_i = \frac{1/d_i}{\sum_{j}^{N} 1/d_j} \qquad \textbf{[19]}$$

where d_i is the Macaulay duration of the i^{th} bond[7].

[7] This weighing scheme corrects for the heteroskedasticity problem in the error terms which occurs if the price errors are used instead of yield error. See Coleman, Fisher and Ibbotson (1995), Bliss (1997) and Bolder and Streliski (1999) for a discussion. Using duration weighted loss function is also a proxy to

The loss function above has been specified as a function of price errors. An alternative exists in taking the yield errors. However, since it is the bond *prices* that are traded in the market, and not the yields, it makes sense to specify a loss function in terms of the variable which is directly observed / traded in the market. Further, the weighting scheme used – other than taking care of heteroskedasticity – also takes care of minimizing yield errors indirectly. Recall that duration is a function of first derivative of price w.r.t yield, and the weighting scheme is inverse of duration.

The basic process of determining the optimal parameters involves initialization of the parameters, finding pricing errors based on 'starting' values and minimization of the objective function.

2.4 Criteria of Evaluation/Comparison of NSE's Term Structure Estimates

This study uses the following criteria for evaluation. The justification being that all together, they encompass the important criteria of goodness of fit, robustness of parameters, value of the loss function, in-sample and out-of-sample characteristics, and economic interpretation.

1. Objective Function Value
2. In-sample and Out-of-sample Mean Absolute Price Error (MAPE) and Standard of Price Error (STDPE) - monthly
3. In-sample and Out-of-sample Mean Absolute Yield Error (MAYE) and Standard Deviation of Yield Error (STDYE) - monthly
4. Behaviour of monthly MAPE and MAYE with residual time to maturity
5. Behaviour of monthly in-sample and out-of-sample errors MAPE and MAYE with (alternative measures of) liquidity
6. Time series behaviour of implied short and long rates
7. Summary Statistics for Parameter Estimates and Convergence Properties (Time series behaviour of the parameters Iterations, function count evaluations and time required for convergence)
8. Behaviour of key spot and forward rates (maturity: 1 month to 8 years)

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minimize yield errors when price errors are used in the loss function. Subramanian (2001) uses a liquidity (instead of duration) weighted loss function

Before presenting the results, however, given the nature of the exercise, a note on the selection of bonds and broad estimation strategy is warranted.

On Selection of Out-of-sample Bonds, Data and Estimation Strategy

Data for the study pertains to the daily trades in the Wholesale Debt Market (WDM) segment of the NSE. It was noticed from the WDM database that around 30-60 different bonds are traded each day and since each day not all the traded bonds are the same, to ensure that results are robust to selection of bonds in the sample, data for each day was segregated into in-sample and out-of-sample parts. To remove any biases in selection, 15% of bonds traded (rounded off to the nearest whole number) were selected each day at random to assess the out-of-sample characteristics of the estimated term structure.

Issues in estimation of the term structure for India are discussed in detail in Darbha, Roy and Pawaskar (2003b) and as far as selection of bonds and estimation strategy is concerned this study follows their approach, i.e. all bonds traded during the day are included for estimation, value weighted prices are used while calculating pricing errors, and – as mentioned earlier – errors are weighted by inverse of duration. Also, only bonds with $T + 0$ and $T + 1$ settlement dates have been taken for the purpose of estimation,[8] to ensure that the estimated term structure best captures the expectations on the trade date.

For NS, NSE's NS estimates for the first day of Jun, 2003[9] have been used to initialize the search. From then on, previous (or next) day's estimates were used as the starting values for the next (or previous) day. To ensure sensible results (the objective function is highly non-linear in parameters), the long rate was constrained to lie between 0 and 20% and the short-rate to be greater than - 4%[10]. As required by construction, τ was constrained to be greater than zero. Entire estimation exercise was carried out on MATLABv6.5 (using the constrained minimization search procedure *fmincon*[11]).

[8] accounting for more than 90% of the number of trades on most days; dates where none of the bonds settled on $T + 0$ or $T + 1$ dates, bonds settling on $[T + 2]^{th}$ date would also be included
[9] Though, to an extent arbitrary, the choice of the day was dictated by the number of trades during the day and distance from the first day and last days of the sample.
[10] exactly = long rate for 2nd Jun, 2003 - 10% ~ - 4%
[11] employs Nelder-Mead routine to search the parameter space

Same starting values as NS were used for SV with β_3 being initialized same as β_2 and τ_2 initialized as (approx.) 1 more than τ_1[12]. To enable identification, difference between τ_1 and τ_2 was constrained to be least 0.25. As in NS, both τ_1 and τ_2 were constrained to be greater than zero.

For CIR, trial and error along with the results reported by Brown and Dybwig (1986) for the United States were used as the bases to select the minimum and maximum values for ϕ_1, ϕ_2 and ϕ_3. Short rate was constrained to be greater than 1% and less than 20%. Further, the initial values were chosen to be such that ϕ_1 was greater than ϕ_2. All parameter values were constrained to be greater than zero during estimation.

The sample period under study is April, 1999 to May, 2005, comprising 1774 trading days. For 24 days out of these, comparison could not be possible. Details are available on request. As **Figure 1** below shows, the trading activity has clearly increased substantially over time (note that the ordinate shows logarithm of trading activity).

Figure 1

In what follows, results are discussed treating all the models on equal footing – which is to say, not especially discussing NS_NSE only.

[12] Some degree of trial and error was involved in selection of the initial values for SV

III. Results and Discussion

3.1 Objective Function Value

The loss function reflects the (weighted) price errors in the units of Rupees squared. More informative, however, would be the percentage error in basis points. Even though the errors have been weighted by inverse of duration, a rough idea of this can be had by taking the square root of the objective function value.

Table 1

Summary statistics for 100 X (square root of the objective function)

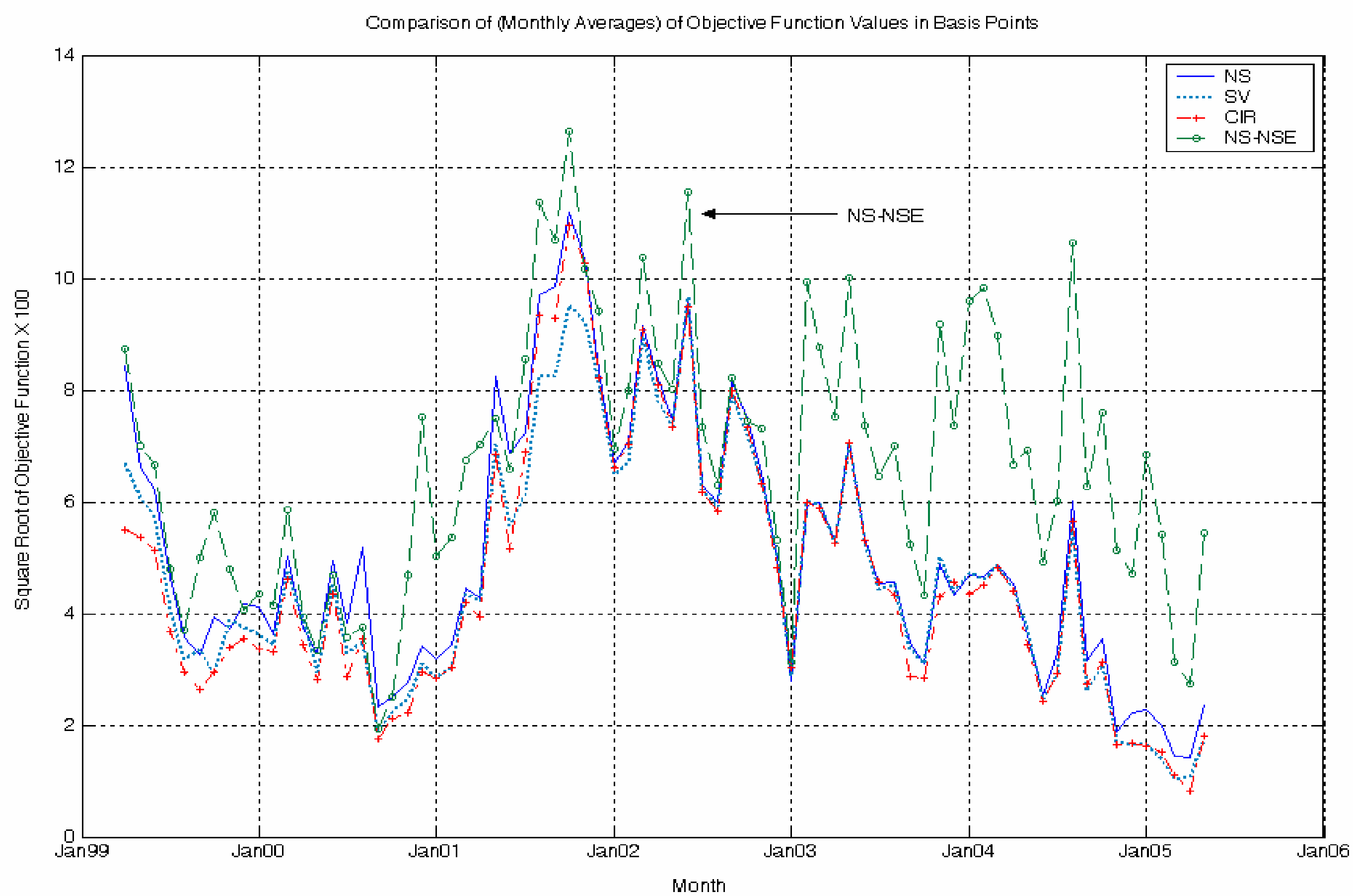
	Min	Max	Mean	Std. Dev.	CoV
NS	0.18	37.92	4.87	3.66	0.75
SV	0.13	38.11	4.51	3.47	0.77
CIR	0.15	37.94	4.44	3.62	0.81
NS_NSE*	0.23	39.85	6.33	4.45	0.70

* - Results are only indicative for NS_NSE. The objective function/estimation strategy used by NSE is not known. Also, there are no out-of-sample bonds in NSE's estimation.

Table 1 table presents summary statistics for 100 times the square root of the objective function value. Clearly, given just the values corresponding to the entire sample, there is little to choose from between NS, SV or CIR, with both mean and standard deviation of similar order for all three. Similar pattern is seen in the plot of the monthly averages of the objective function (see **Figure 2**) for the three models.

Also shown in both the table (and the plot) are the values of the objective function values corresponding to NSE's estimates of parameters. It is clear that the loss function value corresponding to the NSE parameter values are considerably away from the 'optimum'. However, it must be added that the results here are only indicative, as not only the exact objective function / estimation strategy used by NSE is not known, but also because in-sample and out-of-sample parts have been estimated separately in this study.

Figure 2


Comparison of (Monthly Averages) of Objective Function Values in Basis Points
Square Root of Objective Function X 100
Month
NS
SV
CIR
NS-NSE
NS-NSE
0
2
4
6
8
10
12
14
Jan99
Jan00
Jan01
Jan02
Jan03
Jan04
Jan05
Jan06

3.2 In-sample and Out-of-sample MAPE/STDPE

Tables 2a and **2b** present respectively summary statistics on in-sample and out-of-sample MAPE and STDPE. Though the coefficient of variation of NS_NSE is the least, its mean is the highest, with standard deviation not less by much when compared with others. Clearly, there is little telling us here either as to which model is the 'best'.

Table 2a

Summary statistics for 100 X In-sample MAPE

	Min	Max	Mean	Std. Dev.
NS	0.00+	993.42	65.33	79.14
SV	0.00+	984.15	61.45	73.37
CIR	0.00+	985.00	58.90	70.50
NS_NSE*	0.00+	866.16	65.58	68.44

* - Results are only indicative for NS_NSE. The objective function/estimation strategy used by NSE is not known. Also, there are no out-of-sample bonds in NSE's estimation.

+: $< 1 \text{ X } 10^{-4}$

Table 2b

Summary statistics for 100 X Out-of-sample MAPE

	Min	Max	Mean	Std. Dev.
NS	0.00+	896.73	68.27	81.80
SV	0.00+	679.74	63.95	74.81
CIR	0.00+	724.25	63.14	74.43
NS_NSE*	0.00+	629.83	65.47	67.55

* - Results are only indicative for NS_NSE. The objective function/estimation strategy used by NSE is not known. Also, there are no out-of-sample bonds in NSE's estimation.

+: $< 1 \text{ X } 10^{-4}$

It is clear that while some bonds are priced almost accurately (minimum MAPE ~ 0), the maximum pricing error for the entire sample period nears almost Rs. 10 for the three models. This lack of in-sample fit is not entirely unexpected given that models used are all parsimonious. The fact that out-of-sample values are similar in order to the in-sample ones is an indication that results are indeed robust to the bonds left out of sample. What is worrying, however, is that standard deviation is of the order greater than the mean for each model, suggesting that some bonds are priced with considerably large errors. This is discouraging from the point of view of derivatives pricing.

Figure 3 present monthly averages of MAPE/STDPE for the four models. The plots resemble the summary statistics in **Tables 2a/2b**, with no model out-doing any other during the entire sample period. Though, NS does seem to be the worst amongst the lot in both in-sample and out-of-sample results, but not by far. What is interesting is that pricing errors are of the lower in the beginning of the sample than during any other time during sample, despite the fact that liquidity was, at best, increasing at around the year 2000, and reached the order of the what it was now only around 2003. This is, perhaps, an indication that liquidity does not necessarily imply small pricing errors. More would be seen when we study the behaviour of errors with liquidity.

Although comparison with NSE results is also shown, it must be reiterated that NSE doesn't segregate the trades data into in-sample and out-of-sample parts. In this study, daily out-of-sample bonds list that was used for NS/SV/CIR has been used to get in-sample and out-of-sample results for NS_NSE using NSE's NS parameters.

3.3 In-sample and Out-of-sample MAYE / STDYE

Although, as argued before, price errors are used to estimate the models, it may be worthwhile to see how well the models compare on yield to maturity errors.

As it turns out, MAYE and STDYE illustrate the difference across the models much better. As seen both in the plot of monthly averages (**Figure 4**) and the table of summary statistics (**Tables 3a** and **3b**), SV and CIR clearly score over both NS and NS_NSE (with not only the highest mean but also a considerably higher standard deviation). Again, as noticed in the case of price errors, in-sample and out-of-sample yield errors are also of the similar order, implying that results are robust to the selection of bonds in estimation.

Figure 3 (In-sample and Out-of-sample Price Errors)

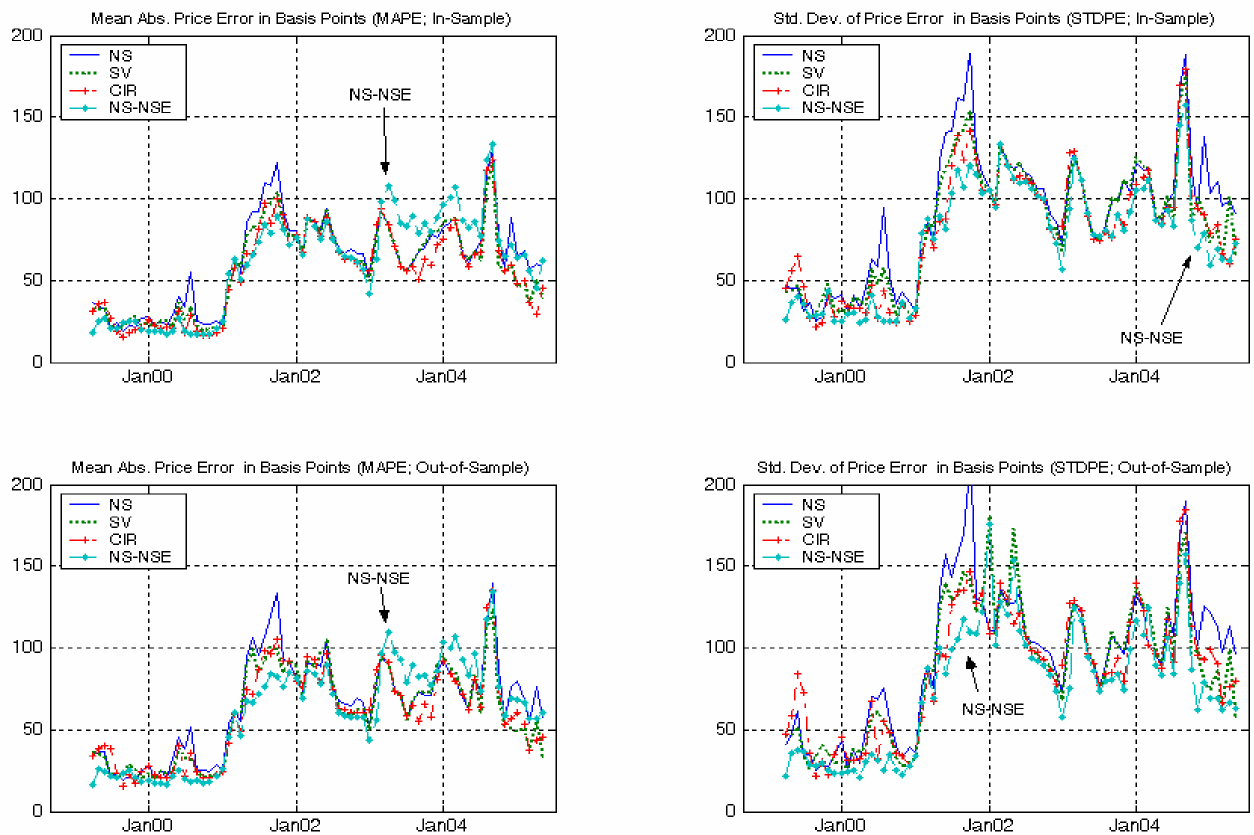


Table 3a

Summary statistics for 100 X In-sample MAYE

	Min	Max	Mean	Std. Dev.
NS	0.00+	261.51	12.14	13.37
SV	0.00+	291.65	11.07	11.86
CIR	0.00+	276.34	10.76	11.36
NS_NSE*	0.00+	295.62	17.06	23.67

* - Results are only indicative for NS_NSE. The objective function/estimation strategy used by NSE is not known. There are no out-of-sample bonds in NSE's results either

+: $< 1 \times 10^{-4}$

Table 3b

Summary statistics for 100 X Out-of-sample MAYE

	Min	Max	Mean	Std. Dev.
NS	0.00+	281.94	12.81	14.75
SV	0.00+	231.19	11.79	13.12
CIR	0.00+	278.65	11.78	13.40
NS_NSE*	0.00+	284.19	17.02	23.88

* - Results are only indicative for NS_NSE. The objective function/estimation strategy used by NSE is not known. There are no out-of-sample bonds in NSE's results either

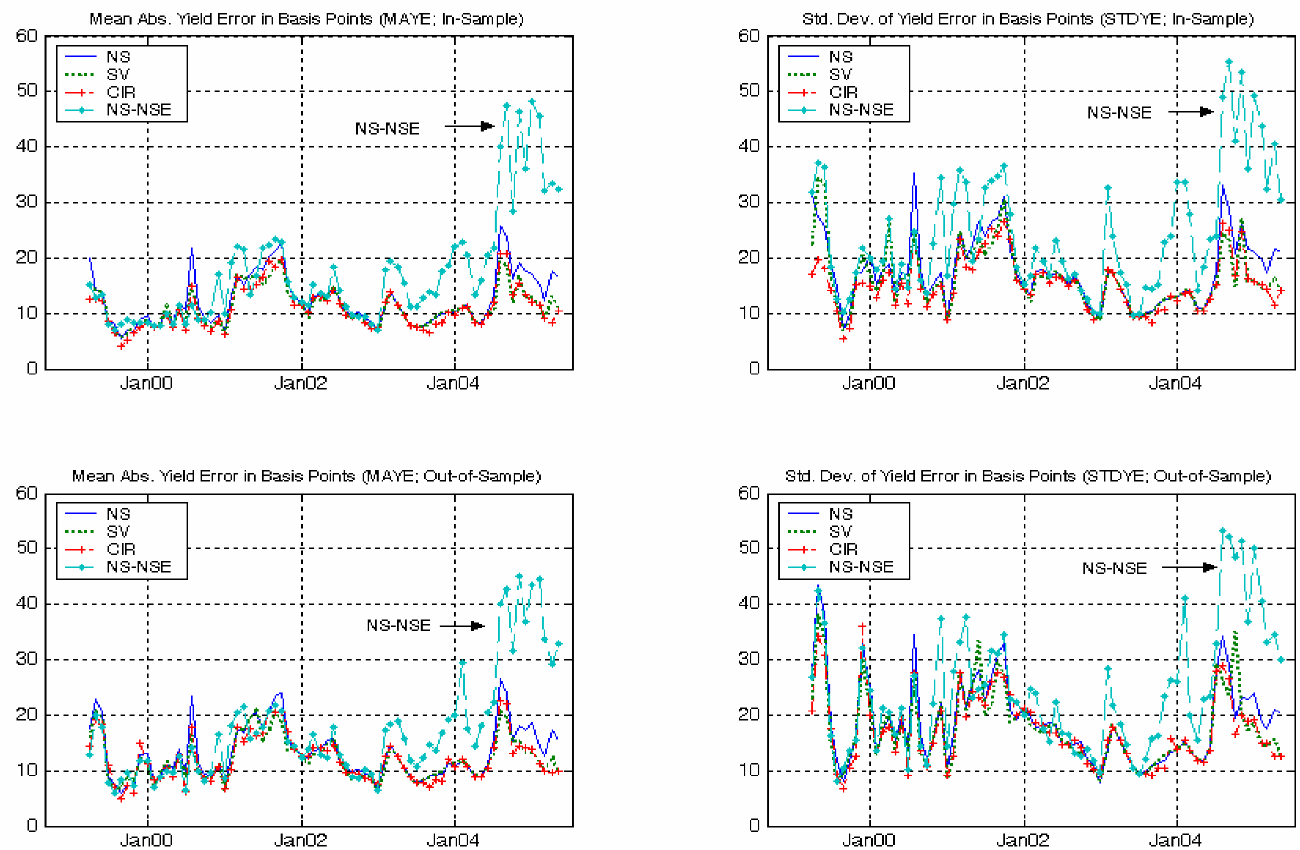
+: $< 1 \times 10^{-4}$

The fact that monthly averages of yield errors mostly lie between 10 – 25 basis points (with mean as low as around 10-12 basis points for SV and CIR) is encouraging.

Although monthly averages of price and yield errors are informative and shed light on the performance of the models, clubbing errors of bonds with different durations and taking their time averages hides the duration effect in bond pricing.

The problem is that since prices of short-maturity bonds are relatively insensitive to small changes in yields, despite weighting by inverse of duration, there may be a tendency to over-fit the short-end. Also, there may be security-specific factors for long-ma long-maturity bonds which are not captured in the specified loss function.

Figure 4 (In-sample and Out-of-sample Yield Errors)



A reason for better fit for short-maturity bonds could also be that omitted pricing factors are less important for these bonds. Comparing an error of 10 basis points for a T-Bill with that for a long dated bond may be worse than comparing apples and oranges.

In what follows, we study the behaviour of both price and yield errors with residual time to maturity and liquidity.

3.4 MAPE with Residual Time to Maturity and Liquidity

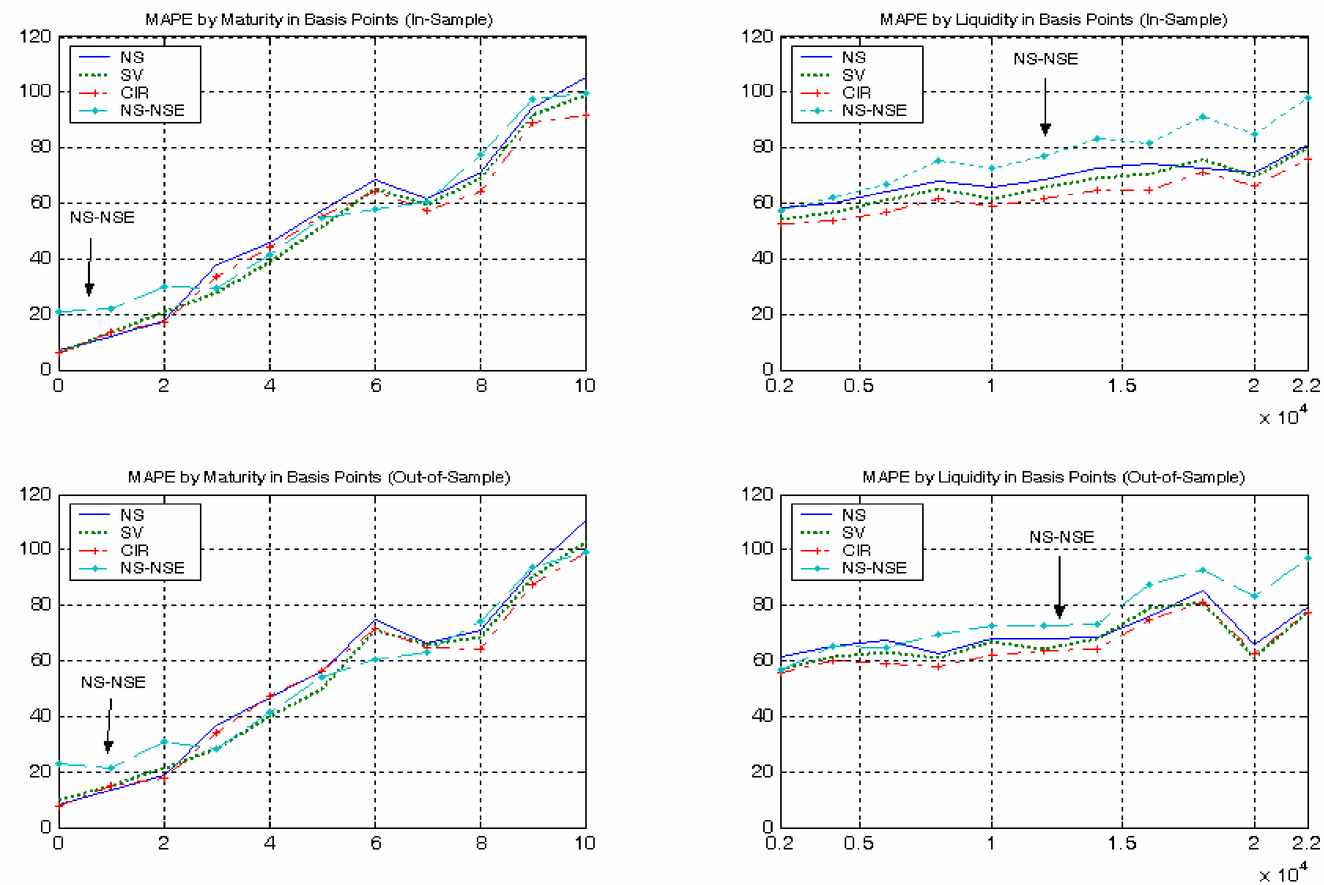
Figure 5 shows both in-sample and out-of-sample behaviour of price errors with maturity and liquidity (as defined by money value of trades in a bond). The graphs are to be read as follows. For errors with maturity (quadrants 1 and 3), points 0 – 2 on the abscissa refer to bonds with maturity between 0 and 2 years. Point '10' on the abscissa refers to all bonds with time to maturity greater than 10 years. Read similarly for with-liquidity plots (quadrants 2 and 4).

The duration effect in price errors is clearly visible for all models. Although the price errors for each model behave more-or-less similarly, bonds with maturity less than 2 years are priced with most error by NS_NSE. For maturities larger than that there is little to choose from amongst the four models. As noted/guessed earlier, there is hardly any relation between liquidity and the price errors.

3.5 MAYE with Residual Time to Maturity and Liquidity

Figure 6, akin to the earlier plot, shows MAYE with residual time to maturity and liquidity. Clearly, the short-end yields are priced the worst, with NS_NSE 'ahead' of all others here. At higher maturities, yield errors are insensitive to both residual time to maturity and liquidity.

Figure 5 (MAPE by Maturity/Liquidity)



Behaviour of MAPE/MAYE with time to maturity suggests that the short-end indeed is being over-fitted, resulting in high yield errors and good price-fit at the short-end.

This suggests that parsimonious term structures could, however, for be fairly useful pricing short-term interest rate derivatives, with both in-sample and out-of-sample pricing error less than 20 basis points. Although most (path-dependent) interest rate derivatives require the probability *distribution* of short rates, these models could provide a starting point by looking at dynamics of the rate derived from these models.

Liquidity based on the value of trades doesn't seem to have any relationship with the behaviour of errors. For the Indian data this was found by Darbha, Roy and Pawaskar (2003) too. They point out,

"…that, while the extent of trading in a security facilitates price discovery, thereby reducing pricing errors, volume per se is an inadequate proxy for capturing liquidity differences in a cross-section of securities, at least in the Indian context. This finding is important since…it is common in the existing literature to use volume as a proxy for liquidity…[13]*"*

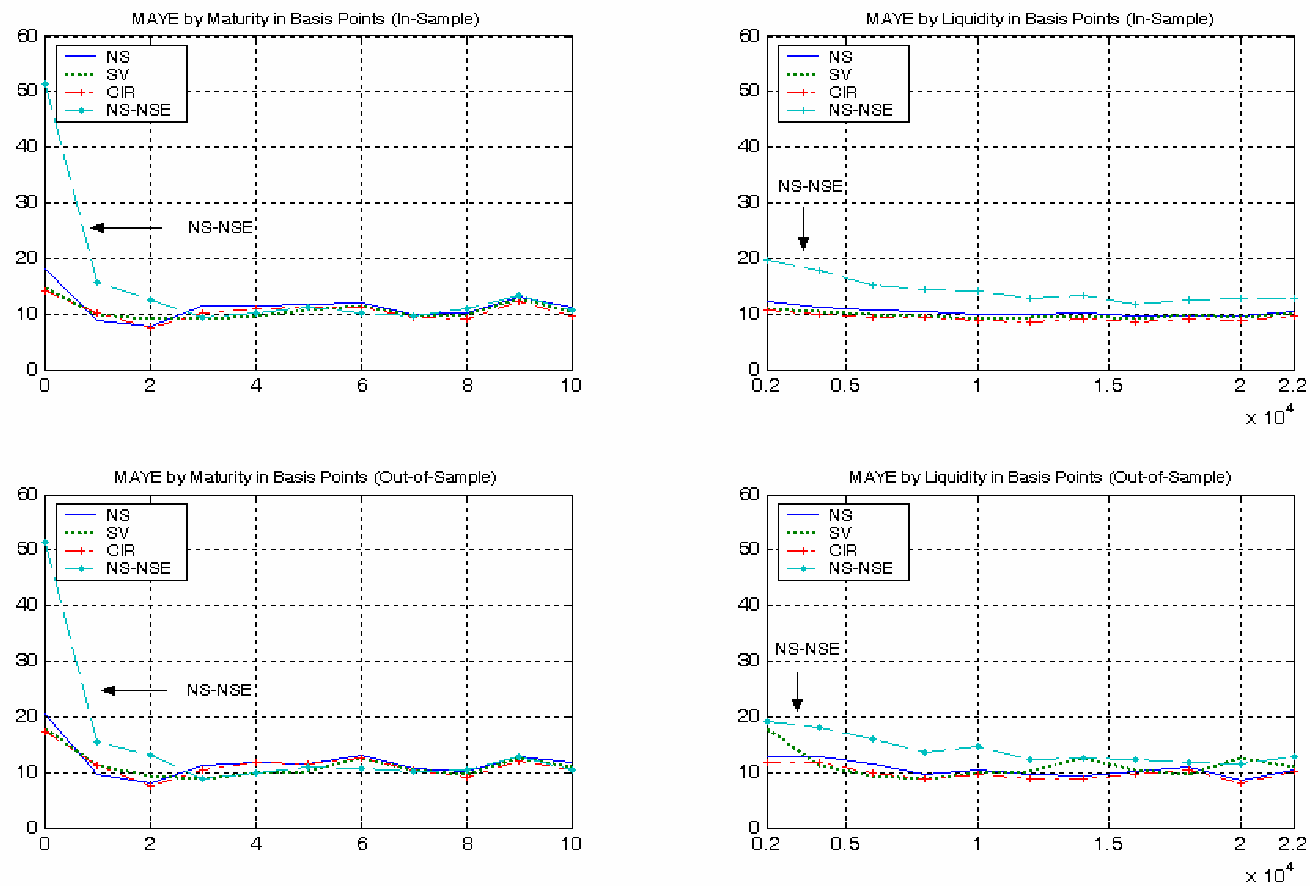
They go on to add that

"…amounts outstanding (issue size) differs significantly across securities; consequently, similar trading volumes imply entirely different degrees of liquidity for bonds with significantly different outstanding amounts.[14]*"*

[13] Darbha, G., Roy, S. D. and V. Pawaskar "Idiosyncratic Factors in Pricing Sovereign Bonds: An Analysis Based on GoI Bond Market", *Journal of Emerging Market Finance*, p. 22

[14] *ibid*

Figure 6 (MAYE by Maturity/Liquidity)



Although, interesting in their own right and informative from the point of view of derivatives pricing too, the next three criteria looked at – the behaviour of parameters, the short and the long rate and the behaviour of forward and spot rates for various maturities – are more directly relevant from the point of view of monetary policy analysis. Or, rather it should be said that a term structure suitable for monetary policy purposes should be acceptable on these counts.

3.6 Time series Behaviour of Implied Short and Long Rates

For models to be useful, the short-rate derived from these models should have a high correlation with the very-short (say, one-day) rate prevailing in the money market.

For NS and SV, the short and the long-rate are given by the parameters $\beta_0 + \beta_1$ and β_0 respectively. For CIR equation **[17]** gives the long rate.

Figure 7a shows the evolution of implied short and long rate from the three models. While the implied short-rates from the three models broadly follow the similar pattern, at the beginning of the sample long rate from CIR is highly erratic.

Inability of CIR to produce 'smooth' series for the long rate is not altogether surprising given that it is a nonlinear function of the parameters ϕ_1, ϕ_2, ϕ_3 (equation **[17])**. After around the middle of the sample, CIR's estimates are closer to NS and SV estimates suggesting that the parameter values have a higher correlation than for the period prior to that. Also, long rates from the three models seem to move together after around the middle of the sample, suggesting that relatively low trading activity in the market may be causing the daily long-term rate to behave erratically and also differently across models.

For a comparison of yield spread (= long rate – short rate) from the three models and comparison with a representative market short rate see **Figure 7b**. Ignoring the erraticity in the long rate in CIR in the beginning of the sample and in NS_NSE during the end of the sample, the term structure clearly has been upward sloping throughout the sample period. None of the models quite capture the high volatility (‘jumps’) in the MIBID/MIBOR series. This is not very surprising because the pricing model estimated in this study captures only the cash-flow and coupon effects and not short-term liquidity mis-matches – which often is a key main reason for high short-term volatility of the over-night rate – and security-specific properties (see Darbha, Roy and Pawaskar, 2003).

Summary statistics (**Table 4a**) also reveal that while CIR, NS and SV are virtually similar in the properties of the implied short-rate, MIBID/MIBOR is clearly more volatile with its highest coefficient of variation. Long rate from comes SV across as the most volatile while from CIR as the least.

Correlation statistics (**Table 4b)** with the market short-rate MIBID/MIBOR are reasonable with results from all models again very similar – an indication that on most days there are a sufficient number of bonds traded with near-zero residual term to maturity allowing each model able to capture the short-rate fairly closely. It is also an indication that the yield curve at the very short-end is close to flat.

Table 4a

Implied Short and Long Rates from the three models

	Min	Max	Mean	Std. Dev.	CoV
Short-Rate					
NS	4.03	10.86	6.78	2.13	0.31
SV	3.96	11.76	6.81	2.14	0.31
CIR	3.33	11.52	6.72	2.10	0.31
NS_NSE	3.35	11.26	7.27	2.05	0.28
MIBID/MIBOR	3.10	23.75	6.69	2.24	0.34
Long-Rate					
NS	6.48	19.08	10.15	3.09	0.30
SV	6.39	19.15	10.02	2.66	0.27
CIR	0	12.75	7.64	1.88	0.25
NS_NSE	3.4	16.31	10.44	2.31	0.22

Figure 7a (Short-rate/Long-rate Comparison)

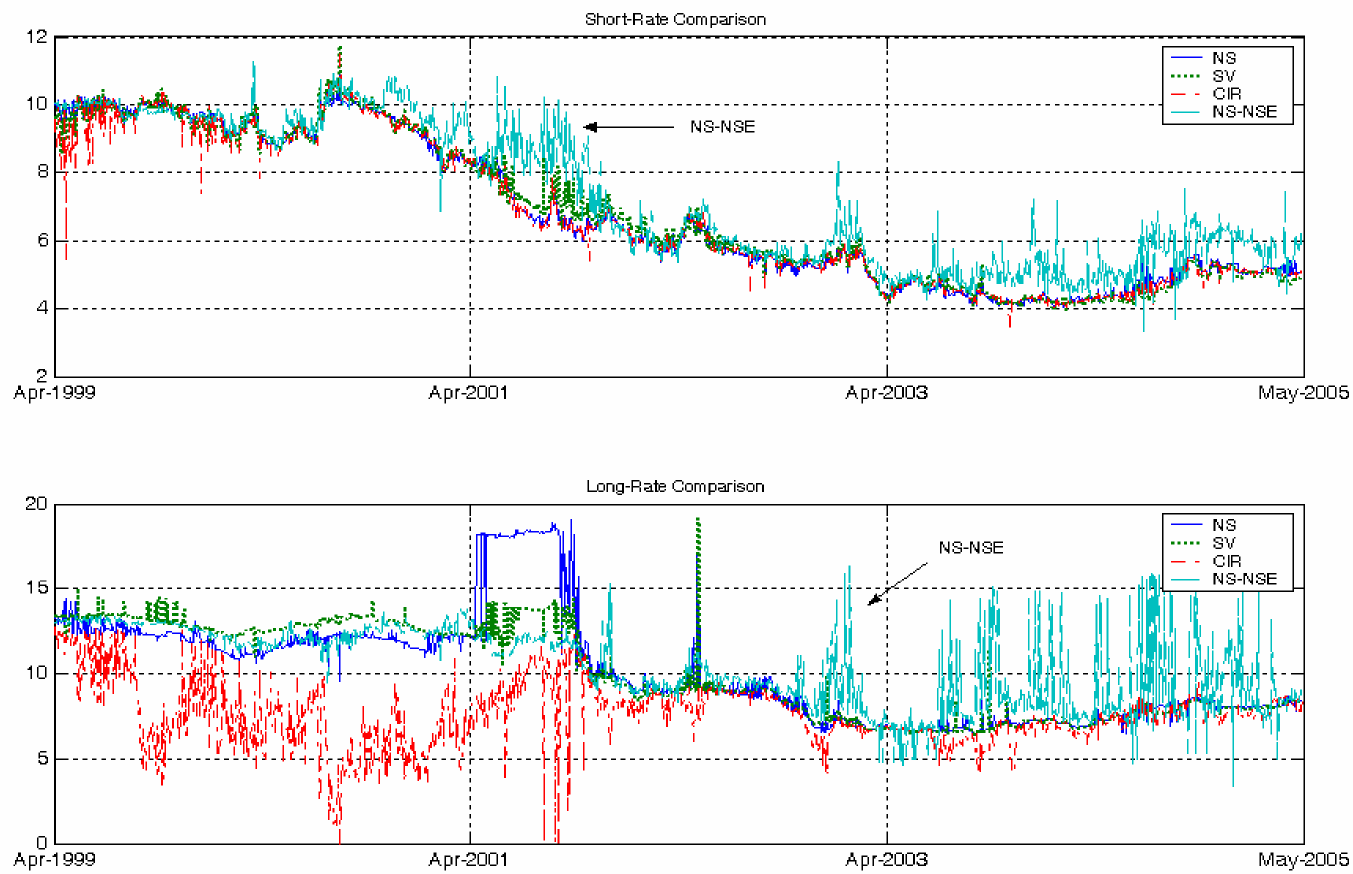


Figure 7b (Comparison of Yield Spreads and Comparison of Short rate with MIBID/MIBOR)

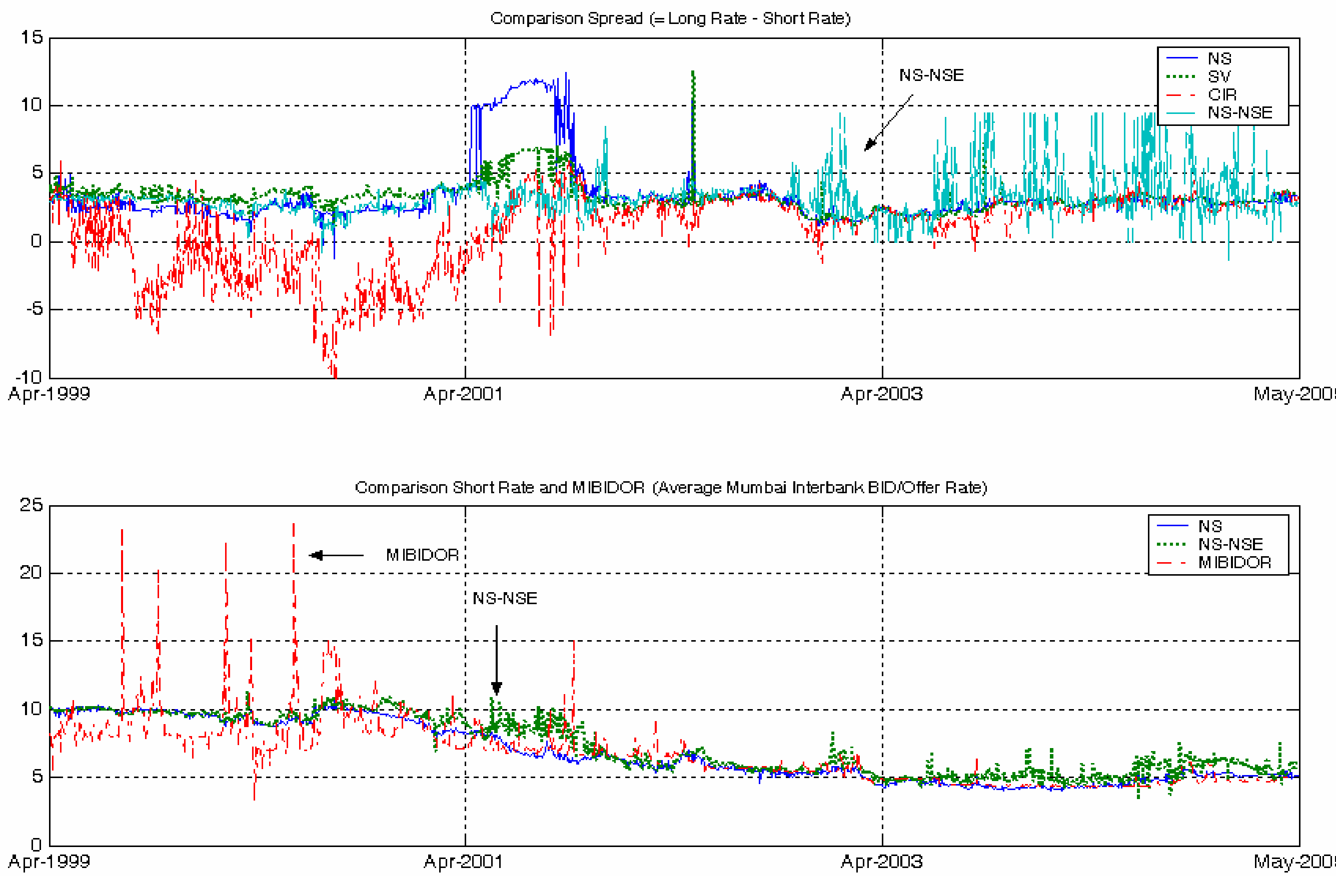


Table 4b

Correlation with MIBID/MIBOR

NS	SV	CIR	NS_NSE
0.83	0.85	0.85	0.81

3.7 Summary Statistics for Parameter Estimates and Convergence Properties

Table 5 shows the summary statistics for parameters of the three models. Barring parameters ϕ_1 and ϕ_2, the coefficient of variation (CoV; standard deviation compared to the mean) for CIR parameters is the least. Also, loosely speaking, it is the parameters β_0 and β_1 of NS/SV and parameter, r of CIR which are directly comparable, for both relate to the short rate. CIR again comes across as slightly better than the other two. However, while convergence properties of CIR (number of iterations and function count evaluations) as reflected by CoV are better, on an average NS converges fastest. Even amongst NS, SV and NS_NSE - the three models with directly comparable parameters – there is little to choose from them as the 'best'. All three are computationally unproblematic, and as far as estimation is concerned converge fairly fast.

However, all the three models being highly non-linear, the problem of local versus global optima remains. Bolder and Streliski [1999] discuss this issue in detail and try to deal with it using a thorough local grid search procedure. One of the main issues is initialization of the parameters. As stated earlier, in this study NSE's NS estimates for first trading day of June, 2003 have been used to get the starting value. For the first estimation, locally the starting values were varied and results were found to be quantitatively insensitive. From that day on, previous (or next) day's converged parameter values were taken as the starting value. For CIR, values of the implied short-rate and long-rate suggested appropriate starting values – again based on results for first trading day of June, 2003.

In a forthcoming companion paper we study the objective function for the three specifications and deal with the estimability issue in detail.

Table 5

Summary Statistics for Parameter Estimates and Convergence Properties

	Min	Max	Mean	Std. Dev.	CoV
			NS		
β_0	6.48	19.08	10.15	3.09	0.30
β_1	-12.43	1.29	-3.37	2.25	0.67
β_2	-10.49	11.07	1.22	4.53	3.70
τ	2.71	25.27	11.06	8.89	0.80
Iter	1	30	4.74	5.08	1.07
FnCount	11	187	33.58	30.73	0.92
			SV		
β_0	6.39	19.15	10.02	2.66	0.27
β_1	-12.56	-1.05	-3.21	1.04	0.32
β_2	-11.52	6.8	-1.59	1.50	0.94
β_3	-11.00	2.04	-1.21	1.10	0.91
τ_1	0.00	16.45	2.92	1.00	0.34
τ_2	1.61	16.7	5.14	1.63	0.32
Iter	1	80	5.59	6.33	1.13
FnCount	15	604	51.78	50.62	0.98
			CIR		
ϕ_1	0.00	3.05	0.14	0.17	1.17
ϕ_2	0.00	3.00	0.09	0.16	1.74
ϕ_3	0.00	2.7	1.64	0.28	0.17
r	0.03	0.12	0.07	0.02	0.31
Iter	1	40	6.82	4.58	0.67
FnCount	11	294	61.33	30.42	0.50
			NS_NSE		
β_0	3.40	16.31	10.44	2.31	0.22
β_1	-9.40	1.38	-3.18	1.50	0.47
β_2	-15.2	9.61	-3.25	3.14	0.97
τ	0.65	15.50	5.05	3.86	0.76

Note: - CoV: Coefficient of Variation; Iter: Iterations; FnCount: No. of Function Count Evaluations

3.8 Behaviour of Key Forward and Spot Rates (maturity: 1 month to 10 years)

The final criterion that is looked at is the behaviour of forward and spot rates for maturities 1 to 10 years. The idea is that if these term structure models are going to be of any use to the central banker or the financial markets, they should, in the least result in fairly 'smooth' series of forward rate and spot rates for reasonable times to maturity. The argument being: agents would not be expected to change expectations regarding future inflation abruptly on a day-to-day basis.

Given the parameters, forward rate for NS and SV are directly given by equations **[5]** and **[7]** respectively. For CIR a functional form for instantaneous forward rate can be derived from equation **[10]** noting that

$$f[r,t,T] = -\frac{\partial P(r,t,T)}{\partial T} = -\frac{\partial P(r,t,T)}{\partial \tau} \qquad \textbf{[20]}$$

where $\tau = T - t$

Then, using the same notation as used in the set of equations **[9]** to **[18]**, the instantaneous forward rate function for time $\tau = T - t$ at time *t* for CIR can be derived as:

$$f[r,t,T] = \phi_2\phi_3\left(\frac{\phi_1 \exp(\phi_1\tau)}{D} - 1\right) + r\left(\frac{\phi_1^2}{D^2}\right)\exp(\phi_1\tau) \qquad \textbf{[21]}$$

where $D = \phi_2[\exp(\phi_1\tau) - 1] + \phi_1$

Figures 8a and **8b** present results for forward and spot rates respectively for maturities 1 month – 10 years. It is apparent that not only has the term structure been upward sloping for almost the entire sample period under study, the forward rate series for all the models do not have many 'jumps'. Unlike the long rate series which as we noticed showed erraticity for all the models, long maturity forward and spot rate series are fairly smooth, suggesting that while evaluating term structure models, it may be worthwhile to have a look at both the long rate series and long maturity forward/spot rate series before making a call on the usefulness of a term structure.

Table 6a

Summary Statistics for Stability of Forward Rates for Some Key Maturities

Maturity	Min	Max	Mean	Std. Dev.	CoV
NS					
1 month	4.06	10.86	6.81	2.13	0.31
3 month	4.12	10.86	6.89	2.13	0.31
1	4.33	10.96	7.19	2.12	0.30
3	4.74	11.58	7.89	2.16	0.27
5	5.03	12.54	8.47	2.23	0.26
7	5.47	13.42	8.93	2.33	0.26
10	5.83	14.49	9.46	2.51	0.27
SV					
1 month	4.00	11.69	6.84	2.13	0.31
3 month	4.07	11.56	6.9	2.12	0.31
1	4.23	11.08	7.18	2.11	0.29
3	4.64	11.97	7.9	2.17	0.27
5	4.99	12.94	8.5	2.26	0.27
7	5.5	13.42	8.95	2.37	0.27
10	5.9	13.71	9.39	2.51	0.27
CIR					
1 month	3.46	11.50	6.77	2.10	0.31
3 month	3.70	11.45	6.85	2.11	0.31
1	4.36	11.32	7.21	2.12	0.29
3	4.76	11.90	7.93	2.18	0.28
5	5.09	12.27	8.48	2.25	0.27
7	5.40	12.96	8.92	2.34	0.26
10	5.67	14.00	9.40	2.46	0.26
NS_NSE					
1 month	3.73	11.05	7.24	2.02	0.28
3 month	4.26	10.69	7.21	1.99	0.28
1	4.47	10.81	7.25	2.00	0.28
3	4.45	11.53	7.86	2.18	0.28
5	4.87	12.14	8.48	2.35	0.28
7	5.33	12.55	8.93	2.42	0.27
10	5.84	12.98	9.38	2.41	0.26

Table 6b

Summary Statistics for Stability of Spot Rates for Some Key Maturities

Maturity	Min	Max	Mean	Std. Dev.	CoV
			NS		
1 month	4.04	10.86	6.80	2.13	0.31
3 month	4.07	10.86	6.83	2.13	0.31
1	4.20	10.86	6.99	2.13	0.30
3	4.45	11.11	7.36	2.13	0.29
5	4.68	11.41	7.70	2.15	0.28
7	4.86	11.69	7.98	2.19	0.27
10	5.10	12.37	8.35	2.25	0.27
			SV		
1 month	3.98	11.72	6.83	2.14	0.31
3 month	4.01	11.66	6.86	2.13	0.31
1	4.15	11.39	6.99	2.12	0.30
2	4.38	10.95	7.36	2.13	0.29
5	4.62	11.34	7.70	2.16	0.28
7	4.83	11.86	8.00	2.20	0.27
10	5.11	12.38	8.35	2.26	0.27
			CIR		
1 month	3.39	11.51	6.75	2.1	0.31
3 month	3.52	11.49	6.79	2.1	0.31
1	4	11.39	6.97	2.11	0.30
3	4.47	11.13	7.38	2.13	0.29
5	4.67	11.34	7.71	2.16	0.28
7	4.83	11.63	8.00	2.20	0.27
10	5.07	12.07	8.35	2.25	0.27
			NS_NSE		
1 month	3.54	11.15	7.25	2.04	0.28
3 month	3.9	10.96	7.24	2.01	0.28
1	4.33	10.62	7.22	1.98	0.27
3	4.59	10.89	7.43	2.04	0.27
5	4.71	11.26	7.73	2.12	0.27
7	4.88	11.57	8.01	2.19	0.27
10	5.09	11.9	8.36	2.25	0.27

This is best noticed in **Figures 8c – 8e**, where short rate is compared with 1 month forward/spot rate and the long rate with 10 year forward/spot rate. While it is hard to distinguish between the short rate and the 1 month rate across the models, it's not the case when one compares the long rate and the 10 year rate. Thus, while the short rate may be directly taken from these models to proxy a short-term interest rate, given the volatility of the long rate, it may be a good idea to extract the *exact* long maturity rate one needs rather than using the long rate. This is also seen in the summary statistics presented in **Tables 6** and **6b** for forward and spot rates. The standard deviation of forward and spot rates is less than that of β_0, the parameter measuring rate at infinity.

Figures 8a and **8b** in conjunction with results in **Tables 6a** and **6b** clearly show that not only has the term structure been upward sloping throughout the sample period, it has also been quite flat – especially at the long(er) end. For some part of the sample, however, plot for NS_NSE suggests a downward sloping term structure. However, the (abrupt) nature of change from upward sloping to downward sloping curve for those days in the sample could just be instability in day-to-day parameter estimation. This is likely as is shown in **Table 7a**, which shows statistics on day-to-day variation in parameters. Clearly, NS_NSE day-to-day estimates are most volatile amongst all the four. A better picture may be obtained by looking at the *percentage change* in parameter values over day. To ensure that the results are not affected by outliers and noise, Table 7b presents weekly moving average for percentage changes corrected for outliers. Clearly, the standard deviation of all the parameters of NS_NSE is many times that of corresponding values for NS and SV.

Table 7a

Standard Deviation of Day-to-Day Parameter Variation

	β_0 / ϕ_1	β_1 / ϕ_2	β_2 / ϕ_3	β_3 / r	τ_1	τ_2
NS	0.68	0.66	1.16	-	0.83	-
SV	0.53	0.52	0.74	0.61	0.73	0.81
CIR	0.18	0.18	0.14	0.003	-	-
NS_NSE	1.57	1.54	2.22	-	3.08	-

Table 7b

Standard Deviation of Weekly Moving Average of Daily Percentage Variation (corrected for outliers)

	β_0 / ϕ_1	β_1 / ϕ_2	β_2 / ϕ_3	β_3 / r	τ_1	τ_2
NS	0.01	0.04	0.34	-	0.02	-
SV	0.009	0.04	0.19	1.33	0.08	0.04
CIR	0.52	2.43	0.12	0.008	-	-
NS_NSE	0.04	0.34	14.2	-	0.21	-

IV. Conclusion: NS_NSE the worst, SV the better of all three

To summarize, both NS and NS_NSE come out to be inferior to CIR and SV on all criteria of evaluation discussed in this study. While SV doesn't come out to be the best on all criteria, wherever CIR comes out to be better of the two, the difference is at best marginal. Also, while in CIR the long rate estimate comes to be highly erratic in the beginning of the sample, in SV the series seems 'well behaved' for the entire sample, when compared with both NS and NS_NSE. Add to that the stability in parameter values and convergence characteristics, SV is clearly the better of all three three, if not categorically the best.

From the theoretical point of view the result it is encouraging because Filipovic (2000) finds that there exist consistent Ito processes with augmented NS, which is exactly what SV is.

This paper shows that no parsimonious model is best on *all* days. If indeed NSE must chose a parsimonious model, then the best it can do is report results for competing specifications. This study has discussed three such. There are others too, including modifications to the one discussed in this study and other single/multi-factor equilibrium models.

It is understandable that reporting results for *all* competing specification may be impracticable. However, given the computational resources these days and numerical tractability of parsimonious models, as this study shows, reporting results on, say, best

3/4 among the lot is eminently achievable. Perhaps NSE should even consider reporting results from a suitably specified over-parameterized splines model too (say, on the lines of Fisher, Nychka and Zervos, 1995 or Waggoner, 1998), if only for the sake of completeness.

Once the results are available in the public domain, it is for users to see/evaluate for themselves which model is the best, and if over time a particular model comes out to be inferior to all others NSE may drop it from the 'list' – as NS_NSE should be. A suitable alternative exists in SV.

Figure 8a (Comparison of Key Forward Rates)

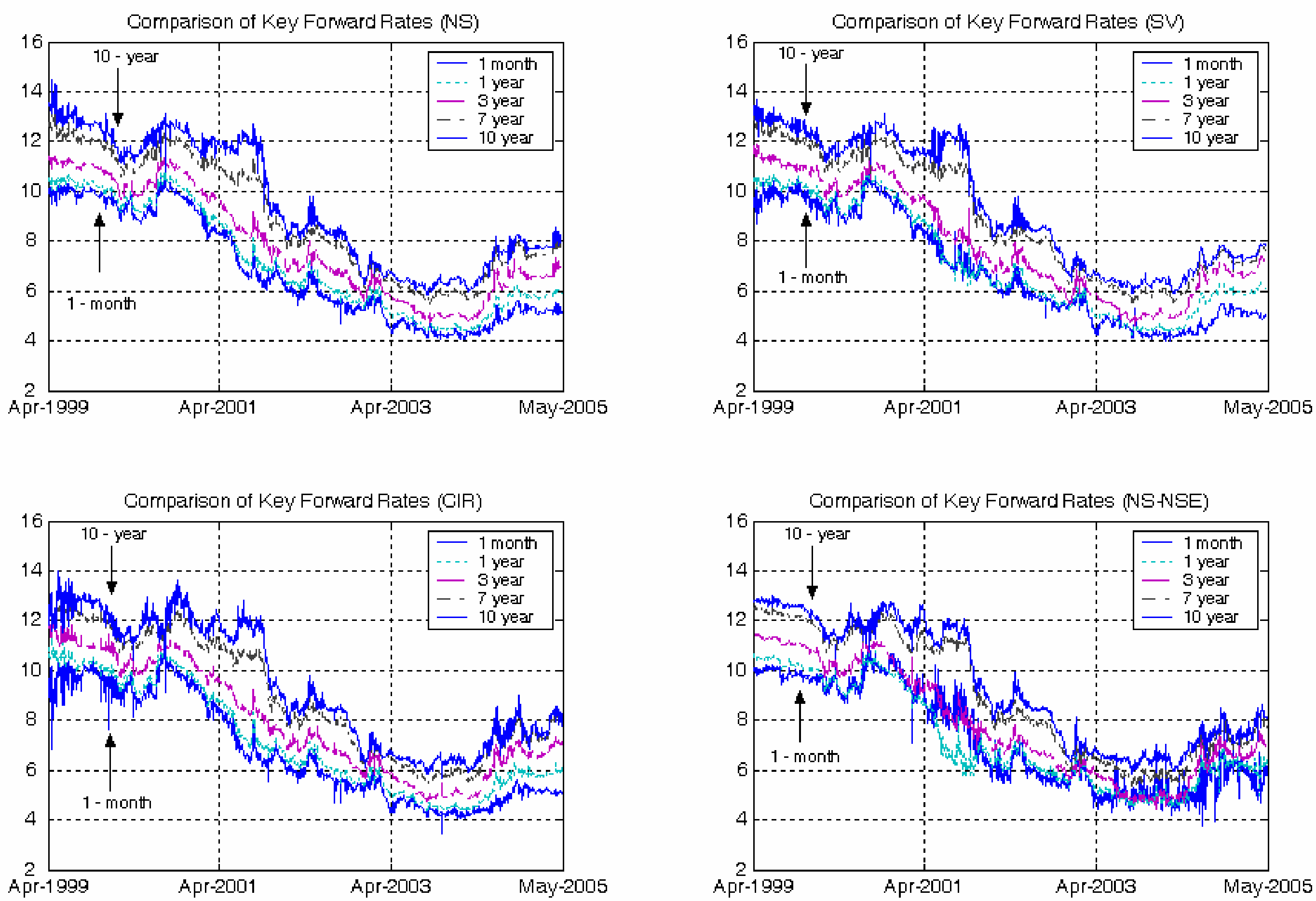


Figure 8b (Comparison of Key Spot Rates)

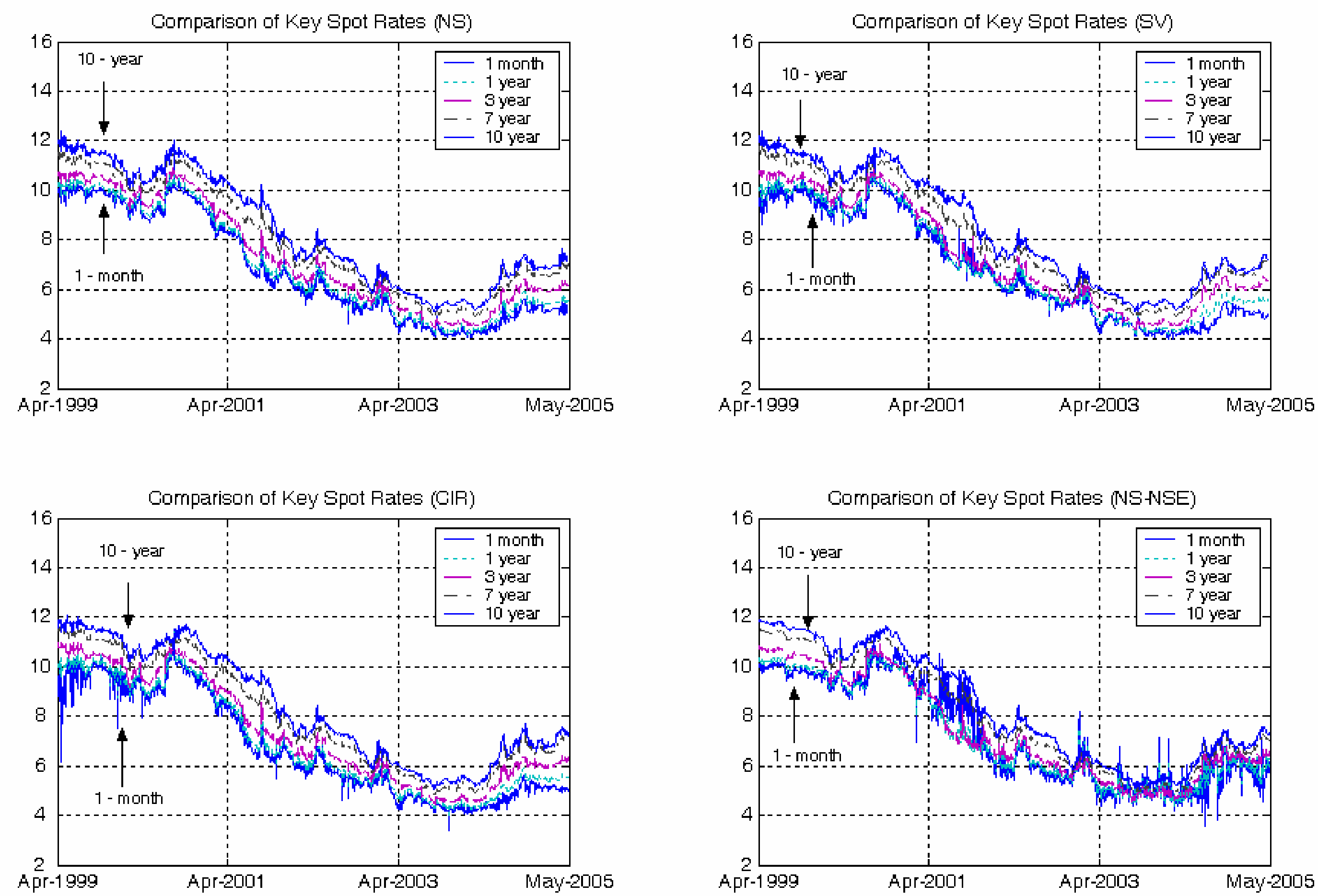
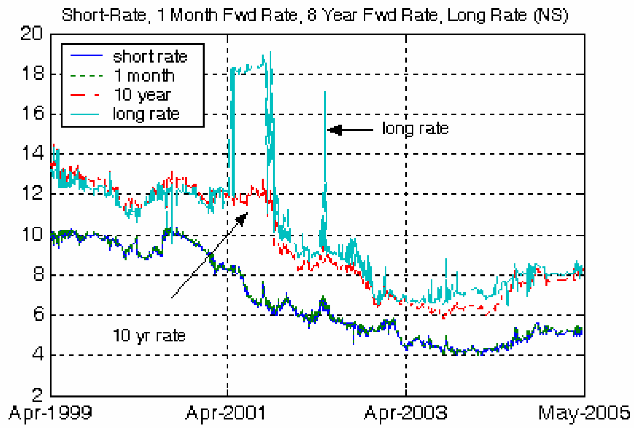
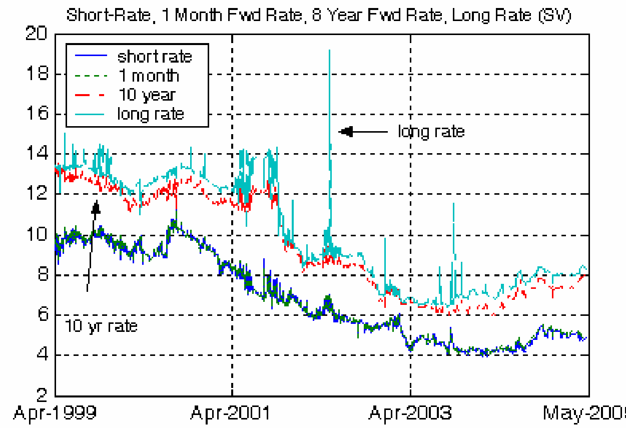
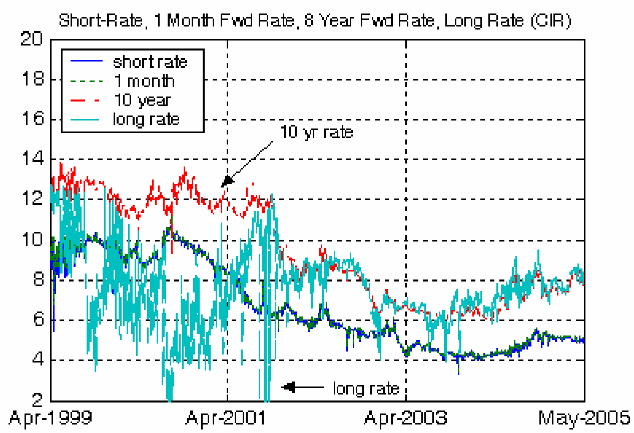


Figure 8c (Comparison of Short-rate and 1-Month Fwd Rate & 10-Year Fwd Rate and Long-rate)







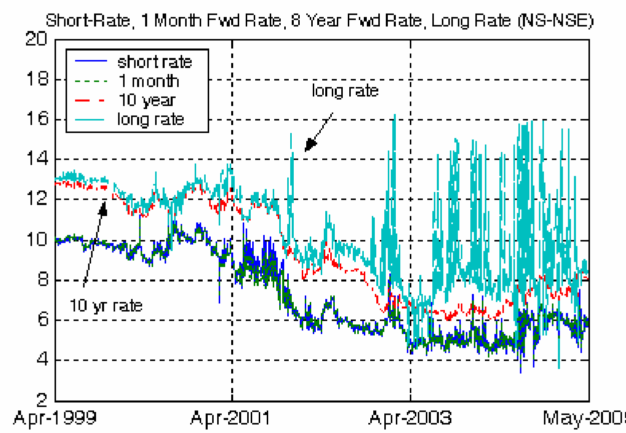


Figure 8d (Comparison of Short-rate and 1-Month Spot Rate & 10-Year Spot Rate and Long-rate)

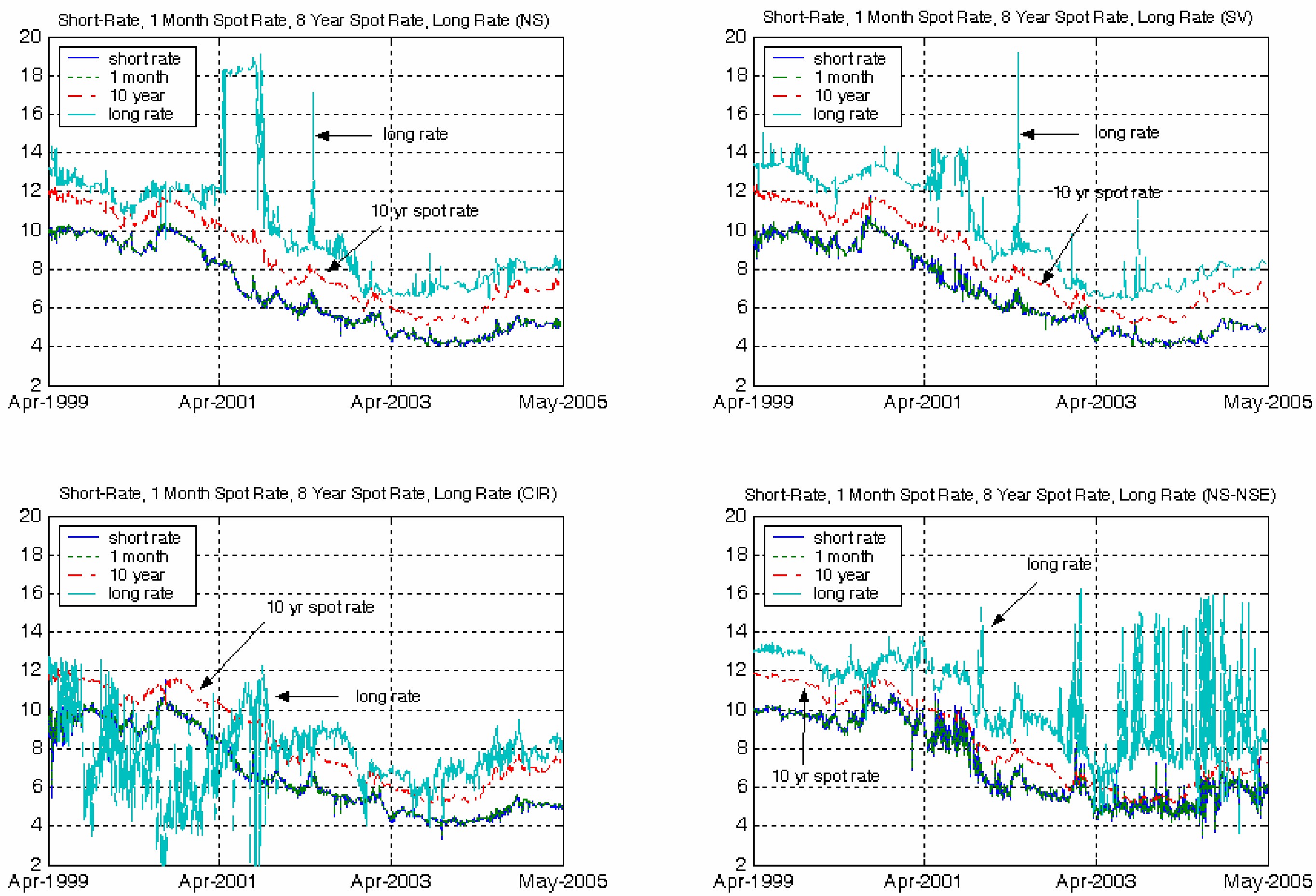
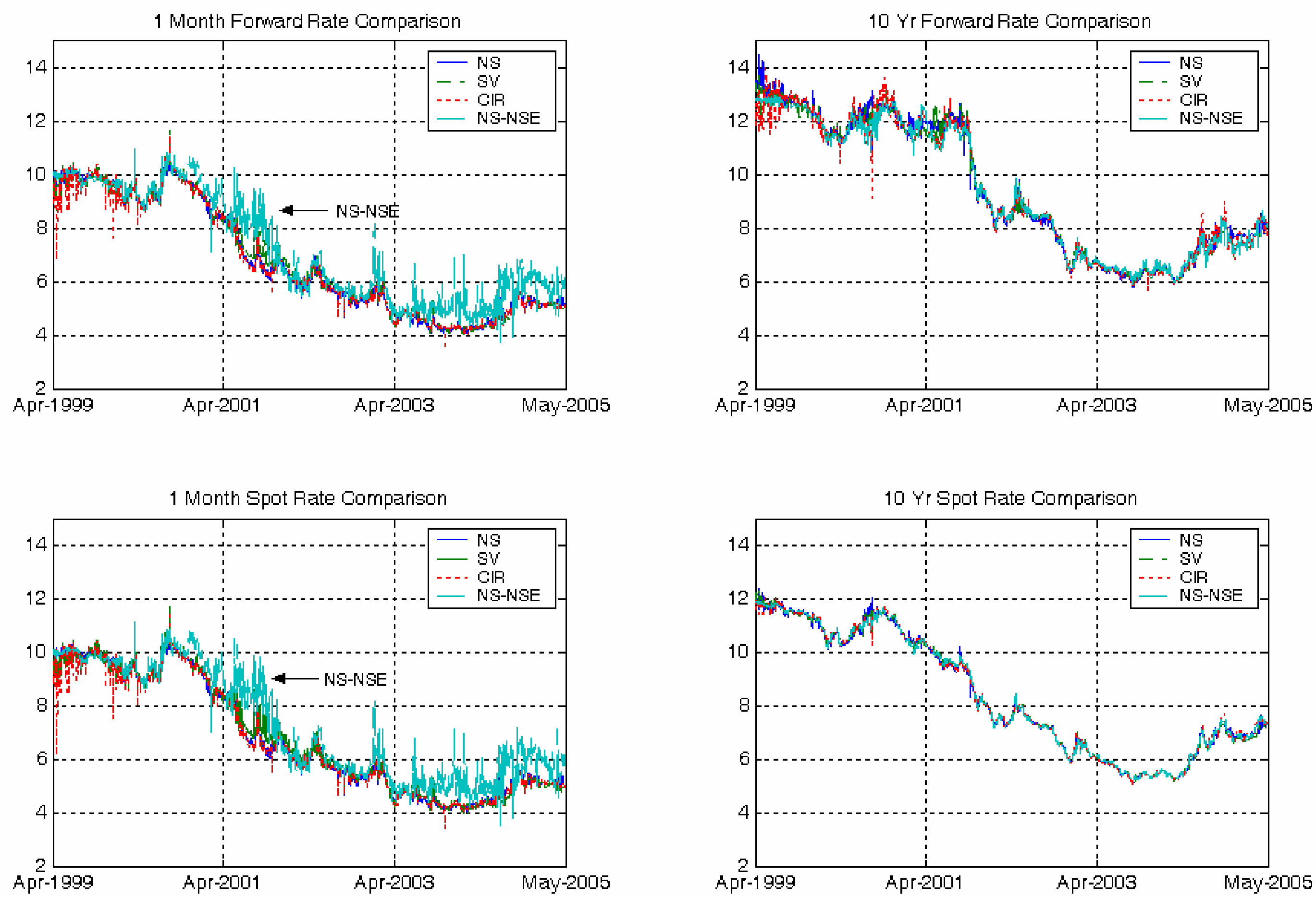


Figure 8e (Comparison of Short-rate and 1-Month Spot Rate & 10-Year Spot Rate and Long-rate – across models)



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