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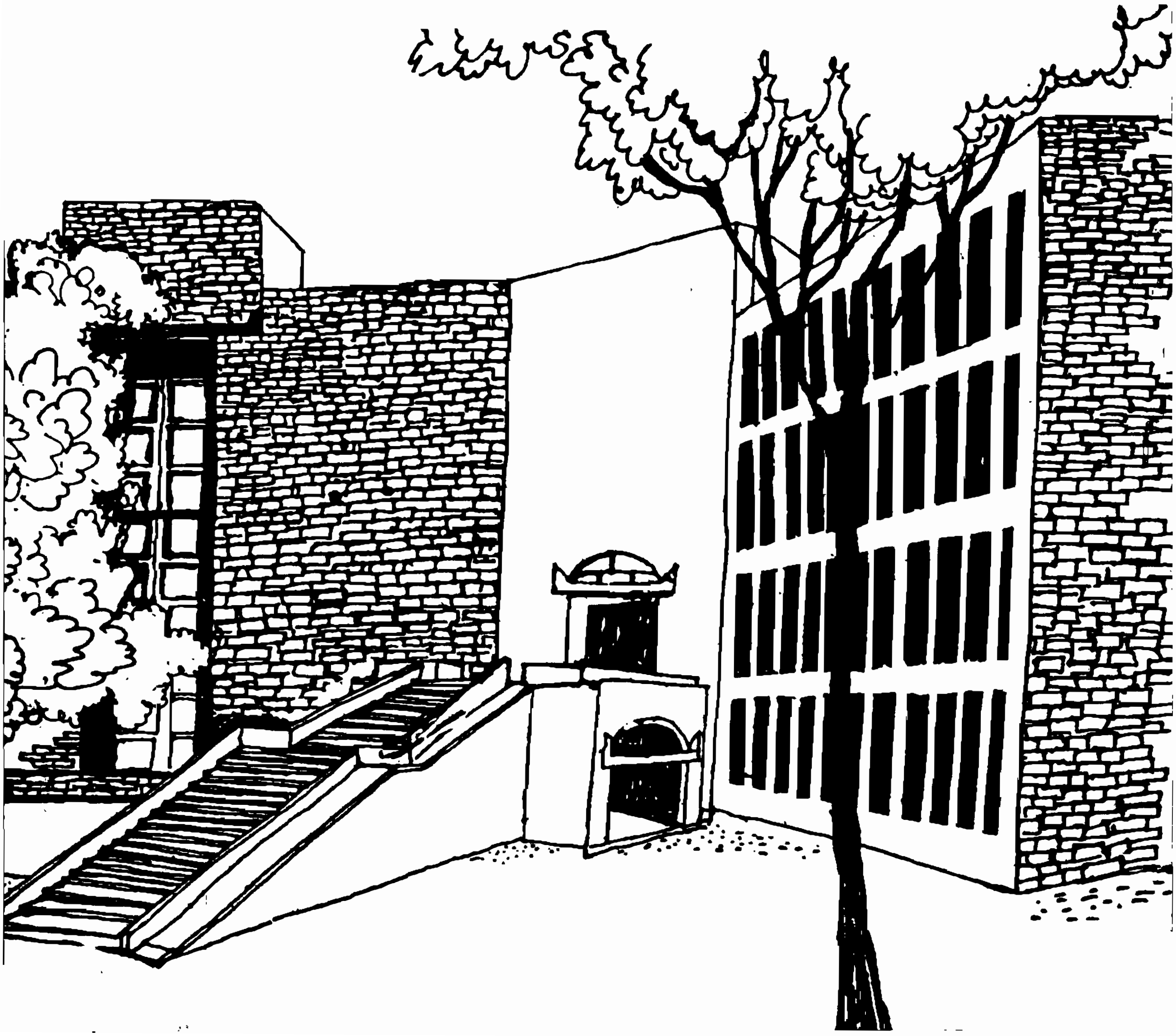

AXIOMATIC CHARACTERIZATION OF THE CAO CHOICE FUNCTION FOR MULTIATTRIBUTE CHOICE PROBLEMS

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Abstract

In this paper we provide an axiomatic characterization of a choice function due to Cao (1981) for multiattribute choice problems.

1. Introduction :- The purpose of this paper is to provide an axiomatic characterization of the choice function due to Cao (1981) for multiattribute choice problems. The main property used in characterizing this solution is termed restricted convexity, which is a modification of the convexity assumption used by Myerson (1981) to characterize the utilitarian choice function.

2. Multiattribute Choice Problems :- A multiattribute choice problem is an ordered pair (S,c) where $0 \in S \subseteq \mathbb{R}^n_+$ and $c \in \mathbb{R}^n_+$ for some $n \in \mathbb{N}$ (the set of natural numbers). The set S is called the set of feasible attribute vectors and the point c is called a target point.

We shall consider the following class $\mathcal{Q}$ of admissible multiattribute choice problems: $(S,c) \in \mathcal{Q}$ if and only if

(i) S is compact and convex

(ii) S satisfies minimal transferability: $x \in S$, $x_i > 0 \Rightarrow \exists\, y \in S$ with $y_i < x_i$ and $y_j > x_j$ for $j \neq i$.

(iii) S is comprehensive: $x \in S$, $0 \leqq y \leqq x \Rightarrow y \in S$.

(Here for x, $y \in \mathbb{R}^n$, $x \geqq y$ means $x_i \geqq y_i\ \forall i \in \{1,\dots,n\}$; $x > y$ means $x \neq y$ and $x \geqq y$; $x >> y$ means $x_i > y_i\ \forall\, i=1,\dots,n$).

A <u>domain</u> is any subset D of $\mathcal{Q}$.

A (multiattribute) <u>choice function</u> on D is a function $F: D \to \mathbb{R}^n_+$ such that $\forall (S,c) \in D$, $F(S,c) \in S$.

Let $F: D \to \mathbb{R}^n_+$ be a choice function. Three important properties often required of a choice function are the following:

(P.1) <u>Efficiency</u> :- $\forall (S,c) \in D$, $x \in S$, $x \geqq F(S,c) \Rightarrow x = F(S,c)$

(P.2) <u>Symmetry</u> :- If $\forall$ permutation $\sigma: N \to N$, $\sigma(S)=S$ and $\sigma(c)=c$, then $F_i(S,c)=F_j(S,c)\ \forall\, i,j \in \{1,\dots,n\}$. Here for $x \in \mathbb{R}^n_+$, $\sigma(x)$ is the vector in $\mathbb{R}^n_+$ whose ith coordinate is $x_{\sigma(i)}$ and $\sigma(S)=\{\sigma(x): x \in S\}$.

(P.3) <u>Scale Independence</u> :- $\forall (S,c) \in D$, $\alpha \in \mathbb{R}^n_{++}$, $(\alpha.S, \alpha.c) \in D \Rightarrow F(\alpha.S, \alpha.c) = \alpha.F(S,c)$.

Here $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n_+ / x_i > 0\ \forall\ i=1,\dots,n\}$; $\alpha.x = (\alpha_1 x_1, \dots, \alpha_n x_n) \in \mathbb{R}^n_+$ for $x \in \mathbb{R}^n_+$ and $\alpha.S = \{\alpha.x / x \in S\}$.

A fourth property that we shall invoke in the subsequent analysis is the following modified version of convexity, the latter itself being due to Myerson (1981):

(P.4) Restricted Convexity :- $\forall (S,c), (S',c) \in D$ if for $\beta \in [0,1]$ $(\beta S+(1-\beta)S',c) \in D$, where $\beta S+(1-\beta)S'=\{\beta x+(1-\beta)y : x \in S, y \in S'\}$, then $F(\beta S+(1-\beta)S',c)=\beta F(S,c)+(1-\beta)F(S',c)$.

The domain we shall be considering in our analysis is an important subdomain of the following:

$\mathcal{L}_u = \{(S,c) \in \mathcal{L} / c=u(S)$ where $u_i(S)=\max\{x_i / x \in S\}\}$.

For obvious reasons problems (S,c) in $\mathcal{L}_u$ will be denoted by S.

Given $S \in \mathcal{L}_u$ let $P(S)=\{x \in S / y \in S, y \geq x \Rightarrow y=x\}$.

Our analysis will concentrate on the following domain:

$\mathcal{L}^0_u = \{S \in \mathcal{L}_u / x \in P(S), y \in P(S), \beta \in (0,1) \Rightarrow \beta x+(1-\beta)y \notin P(S)\}$

On $\mathcal{L}^0_u$ as in Cao (1981) we define the following choice function:

$$C(S) = \arg\max_{x \in S} \Sigma^n_{i=1} x_i / u_i(S) \quad \text{if } S \neq \{0\}$$

$$= 0 \quad \text{if } S = \{0\}$$

It is easy to verify that this choice function is well defined.

3. The Characterization Theorem :-

Theorem :- The only choice function on $\mathcal{L}^0_u$ to satisfy properties (P.1), (P.2), (P.3), (P.4) is C.

Proof :- That C satisfies the above properties is easy to verify. Hence let $F:\mathcal{L}^0_u \rightarrow \mathbb{R}^n_+$ be a choice function satisfying (P.1) to (P.4). If $S=\{0\}$ then $F(S)=0=C(S)$. By property (P.3), we can restrict ourselves to a domain $D=\{S \in \mathcal{L}^0_u / S \neq \{0\}, u(S)=e\}$ where $e \in \mathbb{R}^n_+$ is the vector with all coordinates equal to one. For $S \in D$, let $Q(S)=\{p \in \Delta^{n-1} / \Sigma^n_{i=1} p_i F_i(S) < \arg\max_{x \in S} (\Sigma^n_{i=1} p_i x_i)\}$

Here $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ / \Sigma^n_{i=1} x_i = 1\}$.

Towards a contradiction assume $\bigcup_{S \in D} Q(S) = \Delta^{n-1}$. Each Q(S) is open and Δ^{n-1} is compact. Hence there exists a finite integer $K \in \mathbb{N}$

and sets $S_1, \ldots, S_K \in D$ such that $\Delta^{n-1} = \bigcup_{k=1}^{K} Q(S_k)$.

Consider $S = \Sigma^{K}_{k=1} \frac{1}{K} S_k \in D$. By (P.1), $F(S) \in P(S)$. Thus there exists $p \in \Delta^{n-1}$ such that $p.x \leq p.F(S)\ \forall\ x \in S$. This we obtain by applying the supporting hyperplane theorem to S (which is a convex set) at F(S).

Now $p \in Q(S_k)$ for some $k \in \{1, \ldots, K\}$. Thus $\exists z \in S_k$ such that $\Sigma^{n}_{i=1} P_i F_i(S_k) < \Sigma^{n}_{i=1} P_i z_i$

Consider the point $\frac{1}{K}[z + \Sigma_{j \neq k} F(S_j)] \in S$.

$$\Sigma^{n}_{i=1} \frac{P_i}{K} [z_i + \Sigma_{j \neq k} F_i(S_j)] > \Sigma^{n}_{i=1} \frac{P_i}{K} \Sigma^{K}_{j=1} F_i(S_j)$$

$$= \Sigma^{n}_{i=1} P_i F_i(S)$$

by restricted convexity. This is a contradiction. Hence $\{Q(S') : S' \in D\}$ do not cover Δ^{n-1}. Hence there exists $p \in \Delta^{n-1}$ such that $\Sigma^{n}_{i=1} P_i F_i(S) = \arg\max_{x \in S} (\Sigma^{n}_{i=1} F_i x_i)\ \forall\ S \in D$. By symmetry $p = \frac{1}{n} e$, which proves the theorem.

Q.E.D.

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