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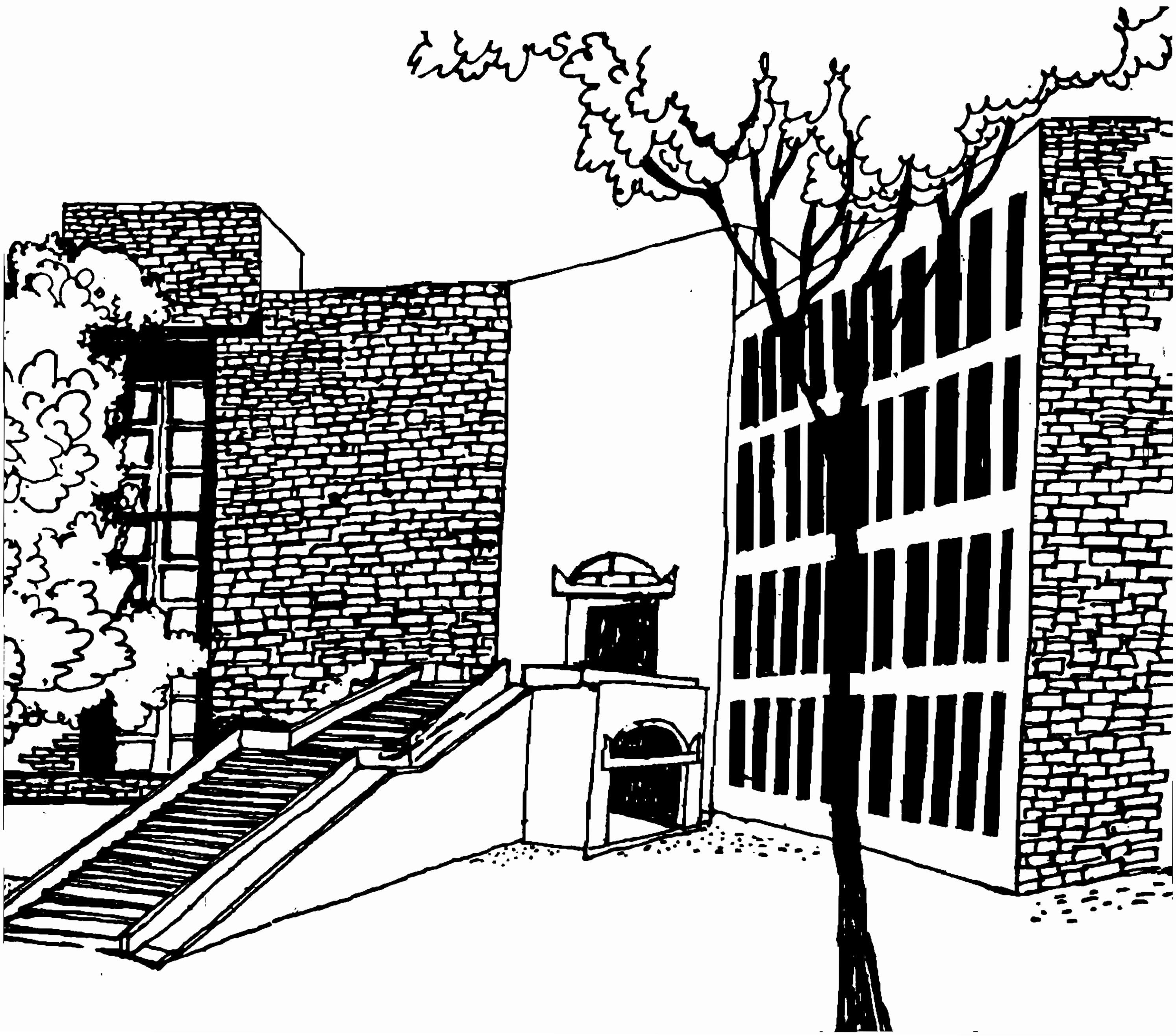

EXISTENCE OF A PARETO OPTIMAL EQUAL LOSS ALLOCATION IN PURE DISTRIBUTION PROBLEMS

By

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Abstract

In this paper we show, that for a large class of pure distribution problems, embedded in a very general framework, a Pareto optimal equal loss allocation exists.

1. Introduction : A pure distribution problem is a planning problem, where a given fixed bundle of commodities is to be allocated among a fixed set of agents. Commodities may include both "goods" (goods and services, facilities, etc.) and "bads" (efforts, taxes, etc.). We shall however concentrate purely on the case of goods. Agents among whom the goods have to be distributed, may be households, social groups, local governments, etc. Since allocation of the goods in our model takes place by means of transfers and/or compulsory contributions, we refer to our problem as a pure distribution problem.

As mentioned by Villar (1992), this is a standard problem in welfare economics, which has been addressed in a number of ways, depending on the equity notions and the class of environments considered. The approach we adopt in this paper is the approach adopted by Villar (1988), Villar (1992) and Nieto (1991) namely: the planner looks for an allocation that satisfies a well defined equity criterion.

The equity criterion we adopt is the equal loss criterion, arising in the literature of axiomatic bargaining, in the works of Chun (1988), Bossert (1992) and Lahiri (1993). The criterion seeks allocations where the looses in utility conceded by the agents in moving from the utopia" point to a Pareto optimal allocation are equal across the agents. The utopia point is the vector of utilities whose ith co-ordinate gives the maximum attainable utility by the ith agent.

In this paper we show, that for a large class of pure distribution problems, embedded in a very general framework, a Pareto optimal equal loss allocation exists.

2. The Model : Consider a pure distribution problem involving n agents and k "goods". Let $N=\{1,\dots,n\}$ be the set of agents and $\mathbb{R}^k_+=\{x=(x^1,\dots,x^k)/x^l \geq 0, l=1,\dots,k\}$ be the consumption set of each agent. A point $x\in(\mathbb{R}^k_+)^n$ denotes an <u>allocation</u>, that can be written as: $x=(x_1,x_2,\dots,x_n)$, where, for each $j=1,2,\dots,n$, $x_j=(x^1_j,\dots,x^k_j)\in\mathbb{R}^k_+$ and denotes the jth agents' bundle when allocation x is obtained.

We consider a very general model of preferences which allows for consumption externalities and non-convexities. Thus, u_j: $(\mathbb{R}^k_+)^n \to \mathbb{R}$ denotes agents j's utility function over allocations. The n-tuple $u=(u_1,\ldots,u_n)$ is called a utility profile.

We make the two following assumptions on the utility function of the agents:

(A 1) $u_j : (\mathbb{R}^k_+)^n \to \mathbb{R}$ is continuous $\forall j \in N$

(A 2) If $x, y \in (\mathbb{R}^k_+)^n$, $x \geq y$ and $x_j = y_j$ then $u_j(x) \leq u_j(y)$

The first assumption is standard. The second assumption says that when an allocation change to a situation where some agents get more goods whilst others get the same, the latter will not be happier. This latter assumption makes it clear that although we allow consumption externalities, we do not impose it (that is, purely self concerned individuals are permitted).

We do not require any convexity requirement on preferences.

Let $w \in \mathbb{R}^k_+$ denote a given bundle of goods to be distributed among the n-agents. The set of feasible allocations can be written as,

$$A(w)=\{x \in (\mathbb{R}^k_+)^n / \Sigma^n_{j=1} x_j \leq w\}$$

A distribution problem, D can be characterized by means of a pair:

$D=[u,w]$.

Let D be a given distribution problem. Then for each $j \in N$, define $M_j(D)=\max\{u_j(x)/x \in A(w)\}$. $M_j(D)$ is called the aspiration level of agent j and the vector $M(D)=(M_1(D),\ldots,M_n(D))$ is called the utopia point of the problem D.

Define, $EL(D)=\{x \in A(w)/M_j(D)-u_j(x)<M_t(D)-u_t(x) \Rightarrow x_j=0\}$

EL (D) consists of feasible points such that all agents suffer equal losses in moving from the utopia point, or else those agents with lower loss receive nothing. Hence, those allocations in $EL(D)$ where all individuals receive some good satisfy the property of strictly equalizing all welfare losses from the utopia point. Conversely, any feasible allocation strictly equalizing all welfare losses from the utopia point belong to $EL(D)$.

An allocation $x^* \in A(w)$ is called <u>weakly Pareto optimal</u> if there is no $x' \in A(w)$ such that $u(x') \gg u(x^*)$.

In the next section we shall prove our main theorem, which shows that there exists a weakly Pareto optimal allocation belonging to $EL(D)$ i.e. equalizing losses, if assumption A 1 and A 2 are satisfied.

3 The Main Theorem :-

<u>Theorem 1</u> :- For any distribution problem $D=[u,w]$ where u satisfies assumptions A 1 and A 2, there exists an $x^* \in A(w)$ which is both weakly Pareto optimal and belongs to $EL(D)$.

<u>Proof</u> :- Let $g: A(w) \to \mathbb{R}$ be defined by

$$g(x)=\max_j \{M_j(D)-u_j(x)\}.$$

g is continuous and $A(w)$ is compact. Hence there exists $x \in A(w)$ which solves:

$$g(x) \to \min$$
$$\text{s.t.: } x \in A(w).$$

Let $\alpha = g(x')$ and $e \in \mathbb{R}^n$ be the vector with all coordinates equal to 1. Define $F:(\mathbb{R}^k_+)^n \to \mathbb{R}$ as follows: $F(x)=u(x)-M(D)+\alpha$. Consider the following problem:

$$
\begin{aligned}
&F(x) \geq 0 \\
&F_i(x) > 0 \Rightarrow x_i = 0 \qquad\qquad (1) \\
&x \in A(w)
\end{aligned}
$$

Associate with (1) the following programming problem:

$\Sigma^{n}_{i=1} \Sigma^{k}_{l=1} x^{l}_{i} \rightarrow \min$

s.t. $F(x) \geqq 0$, $x \in A(w)$.

Clearly x' is feasible for the problem and by assumption (A 1), F is continuous. Thus the constrained set is compact. Let x^* solve the problem.

Suppose $F_i(x^*) > 0$ and $x^*_i > 0$. Construct $z \in (\mathbb{R}^{k}_{+})^{n}$ as follows:

$z_j = x^*_j \; \forall \; j \neq i$

x^*_i with $F_i(z) \geqq 0$.

Such a z exists by the continuity of F_i. Further $F_j(z) \geqq F_j \geqq 0 \; \forall \; j \neq i$ by (A 2) and $\Sigma^{n}_{j=1} \Sigma^{k}_{l=1} z^{l}_{j} < \Sigma^{n}_{j=1} \Sigma^{k}_{l=1} x^{*l}_{i}$ contradicting that x^* solves the above problem. Hence $x^*_i = 0$. Thus $x^* \in EL(D)$.

Let $y \in A(w)$ with $u(y) >> u(x^*)$. Then $M(D)-u(y) << M(D)-u(x^*)$. Thus $\max_j [M_j(D)-u_j(y)] < \max_j [M_j(D)-u_j(x^*)] \leqq \alpha$, contradicting that α is the value of our first programming problem. Thus, x^* is also weakly Pareto optimal.

Q.E.D.

Thus, Theorem 1, shows that we can always distribute a bundle of k-goods among n-agents so that: (i) all individuals getting a positive amount of at least one good sustain the same loss in welfare in moving from the utopia point,; (ii) if there are individuals with a lower welfare loss, they get no goods; hence, effectively they can be ignored (expost) as far as the distribution problem goes; (iii) the resulting allocation is weakly Pareto optimal. All these results have been obtained without any convexity assumption on the preferences and by allowing for a wide range of externalities.

As a corollary to the above theorem we have the following:

<u>Corollary 1</u> :- Let $D=[u,w]$ be a distribution problem and let $\alpha > 0$, $\beta_j \in \mathbb{R}^{k}_{+}$, $j=1,\dots,n$ be real numbers. Suppose $u'_j(x) = \alpha u_j(x) + \beta_j \; \forall$

$x \in (\mathbb{R}^k_+)^n$ and let $D' = [u', w]$ where $u' = (u'_1, \ldots, u'_n)$. Then x^* is weakly Pareto efficient and belongs to $EL(D)$ if and only if it is weakly Pareto efficient and belongs to $EL(D')$.

This Corollary, specifies the appropriate informational requirements on the preferences in order to isolate, weakly Pareto efficient equal loss allocations.

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