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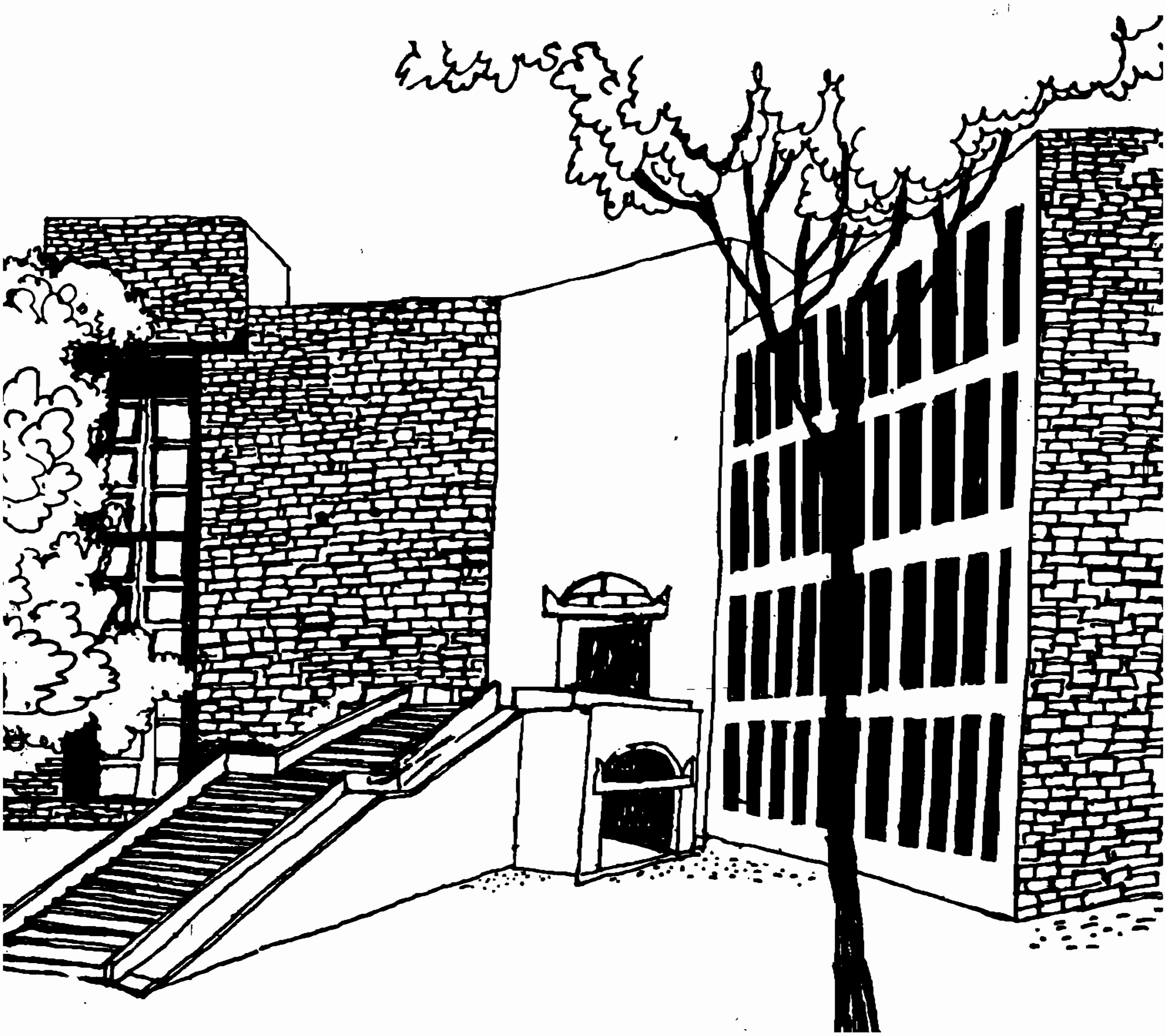

AXIOMATIC CHARACTERIZATIONS OF THE CEA SOLUTION FOR RATIONING PROBLEMS

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
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Abstract

Situations abound in the real world, where aggregate demand for a commodity exceeds aggregate supply. When such situations of excess demand occur, what is required is some kind of rationing. The literature on rationing problems has an interesting origin in the Babylonian Talmud.

The purpose of this paper is to characterize axiomatically and analyze the constrained equal award solution for rationing problems.

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1. Introduction:- Situations abound in the real world, where aggregate demand for a commodity exceeds total supply. In economics, the most common way in which such situations are seen to occur is when government intervenes by pegging the price of a commodity at a level below the market equilibrium price (i.e. the price at which quantity demanded is equal to quantity supplied). In management, the usual way in which such anomalies occur is in the context of supply chain management: there is a distributor of a commodity who is made available the total supplies by a producer; the distributor supplies the commodity to a finite number of retailers; if the orders placed by the retailers add up to a quantity greater than the supply available with the distributor, we are essentially facing a situation of excess demand once again. The excess demand problem in economics has been highlighted and surveyed lucidly, by Silvestre (1986). The excess demand problem in management is a part of a well established lore on frequent stock outs arising in distribution networks. In fact, the problem has such urgency, that computer games have been devised to highlight the merits of the problem.

When such situation of excess demand occur, what is required is some kind of rationing. The literature on rationing problems has an interesting origin in the Babylonian Talmud (: 2000 year old document, which forms the basis of Jewish civil, criminal and religious laws). There, considerable attention has been devoted to the study of a bankruptcy problem: a man dies leaving behind an estate, which is insufficient to meet all his debts. How should the estate be divided among the claimants? The obvious requirement is that the method of division be perceived as being fair.

Recent attempts at giving solutions to this old bankruptcy problem a game theoretic interpretation, can be traced to the paper by O'Neill [1982]. The study of a particular solution known as the contested garment solution received fresh analytical impetus in the work of Aumann and Maschler [1985].

In Moulin [1985, 1988] and Young [1987a, 1987b, 1988, 1993], the mathematical framework of bankruptcy or rationing problems is given the opposite interpretation of cost-sharing or taxation problems. Whereas in rationing problems we are interested in some measure of individual loss i.e. unsatisfied demands, in cost sharing the relevant index is net income that remains after taxation. Both these variables have identical mathematical form. However, in cost sharing if we are interested in maximizing the minimum net income, in rationing we would be interested in minimizing the maximum loss. We obtain a simple algorithm in Lahiri [1997], which gives an explicit solution for the relevant min-max problem.

One of the most popular methods of allocating resources under rationing is the constrained equal awards method, also called the uniform rule by Benassy [1982]. This rule, gives each low demander what he/she demands; all high demanders are given an equal amount, which nevertheless exceeds what any low demander gets. Dagan [1996a] has a useful analysis of this rule. We provide an axiomatic characterization of the constrained equal awards solution using a kind of strategy proofness assumption and show that this rule is the only one to satisfy 'the desired axiom (along with another mild property). Results along similar lines for this and other solutions can be found in Dagan and Volij [1993, 1997].

The above mentioned analysis takes place in a fixed population framework i.e. the agent set or the set of demanders is considered fixed. We subsequently move over to a variable population framework and invoke properties like population monotonicity and Consistency Population monotonicity says that with the arrival of a new agent, no existing agent can get more. Consistency says that if some agents leave with their share of the allocation, then the rule should give the earlier shares to the remaining agents, when what has to be allocated now is what remains after the departing agents have been given their shares. Our results are adaptations of results in Dagan [1996a] and Thomson [1995]. Their results were obtained for games of fair division with single peaked preferences. The basic difference between our framework and the literature on fair division with single peaked preferences are that our preferences have the diagrammatic representation of an isosceles triangle above the horizontal axis. Further, we restrict ourselves to only excess demand situations. With these restrictions, the proofs used by Dagan and Thomson fail to work, since they avail of the larger domain on which their solutions are defined.

In a related paper [Lahiri (forthcoming), we take up the case of the proportional solution and provide an axiomatic characterization of the same using a reduced game property and a property called restricted scale invariance for two agents. In the bargaining games context, reduced games properties have been discussed in Peters, Tijs and Zarzuelo [1994] and Lahiri [1998].

2. The Fixed Population Model:- Consider a set of agents indexed by i=1, 2,...,n where n is a natural number greater than or equal to two. Let N = {1, 2, ..., n} denote the set of agents. A rationing (bankruptcy) problem is an ordered pair

$(d, S) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$ such that $S < \sum_{i=1}^{n} d_i$.

Let B^N denote the set of all rationing problems(for N).

An allocation for $(d, S) \in B^N$ is a vector $x \in \mathbb{R}^n_+$ such

$x_i \leq d_i \; \forall i \in N \; and \; \sum_{i \in N} x_i = S$.

A solution is a function $F: B^N \rightarrow \mathbb{R}^n_+$ such that F(d,S) in an allocation for (d,S) whenever $(d, S) \in B^N$.

Given $(d, S) \in B^N$, the effective demand vector (for (d,S)) denoted d^s is the vector whose i^{th} component

$d_i^s = \min \{ d_i, S \}$

Obviously, since S is what all there is for distribution any claim greater that S is as good as demanding the entire supply. Hence our definition of effective demand.

Given $(d, S) \in B^N$, the point of minimal expectation

$v^{(d,S)}$ (denoted merely by v whenever there is no scope for confusion) is the vector whose i^{th} coordinate v_i is equal to $\max\left\{0, S - \sum_{j \neq i} d_j\right\}$ i.e. what every one else willingly concedes to i.

<u>Observation 1</u>: $v_i \leq d_i \ \forall \ i \in N$

<u>Proof of observation</u>: Suppose $v_i > d_i$ for some $i \in N$

Clearly $d_i > 0 \rightarrow v_i = S - \sum_{j \neq i} d_j$

$\therefore S - \sum_{j \neq i} d_j > d_i$

$\rightarrow S > \sum_{j=1}^{n} d_j$ which is a contradiction. Hence the observation.

<u>Q.E.D.</u>

<u>Observation 2</u>:- Given $(d, S) \in B^N$ if x is any allocation for (d,S), then $x_i \geq v_i \ \forall \ i \in N$.

<u>Proof of observation</u>: Suppose $0 \leq x_i < v_i$ for some $i \in N$.

Then clearly $v_i = S - \sum_{j \neq i} d_j$.

$\therefore x_i < S - \sum_{j \neq i} d_j$.

$\therefore x_i + \sum_{j \neq i} d_j < S$

But $x_j \leq d_j \ \forall \ j \in N$

$\therefore S = \sum_j x_j \leq x_i + \sum_{j \neq i} d_j < S$ which is a contradiction.

This proves the observation.

Q.E.D.

Observations 3:

For all $(d, S) \in B^N$, $\forall \ i \in N$

$$v_i = \max \left\{ 0 , S - \sum_{j \neq i} d_j^S \right\}.$$

Proof:- Let $i \in N$, $k \neq i$, $k \in N$.

If $d_k > S$ then $S - \sum_{j \neq i} d_j < S - d_k < 0$.

$\therefore v_i = 0$.

Since $d_k^s = S$ and $S - \sum_{j \neq i} d_j^s < S - d_k^s = S - S = 0$, $\max \left\{ 0, S - \sum_{j \neq i} d_j^s \right\} = 0$.

$$\therefore v_i = \max \left\{ 0, S - \sum_{j \neq i} d_j^s \right\}.$$

On the other hand if $d_k \leq S \ \forall \ k \in N$, $k \neq i$, then

$d_k^s = d_k \ \forall \ k \in N$, $k \neq i$, so that

$$\sum_{j \neq i} d_j^S = \sum_{j \neq i} d_j$$

This proves the observation in either case.

Q.E.D.

Observation 4:- Given $(d, S) \in B^N$, $\sum_{i \in N} v_i \leq S$

Proof of observation:- Let $x \in \mathbb{R}_+^n$ with $x_i = \frac{d_i}{\sum_{j \in N} d_j} S$.

It is easy to check that x is an allocation for (d,S). Thus the set of allocations for (d,S) is nonempty. Since $v_i \leq x_i \; \forall \, i \in N$ by observation 3, we have, $\sum_{i \in N} v_i \leq S$.

Q.E.D.

3. The Constrained Equal Awards Solution:-

The Constrained Equal Awards solution $CEA : B^N \rightarrow \mathbb{R}_+^n$ is defined as follows: $CEA(d, S) = x$ *where* $x_i = \min\{\lambda, d_i\}, i \in N$ and $\sum_{i=1}^{n} x_i = S$.

It is well known that for each $(d, S) \in B^N$, a unique $\lambda \geq 0$ exists which defines CEA(d,S).

We now state two properties which the constrained equal award solution satisfies.

Equal Treatment (ET):- Given $(d,S)\epsilon B^N, d_i=d_j \rightarrow F_j(d,S)=F_i(d,S)$.

Equal Treatment is standard and simple. It says, if two people make the same demands then they get identical awards. As a postulate of impartiality, nothing could be more meaningful.

Independence of Irrelevant Inflations (I^3):- Given $(d,S),(d',S)\epsilon B^N$ if $d_i=d_i' \forall i \neq k, d_k \leq d_k'$ and $F_k(d,S) < d_k$ then $F_k(d,S) = F_k(d',S)$

Insensitivity to Irrelevant Inflations is a veiled strategy proofness type of condition which says that unilateral upward deviations do not affect the outcome, of the deviating agent provided one's demand is not met originally. It is not as mild a property as equal treatment; yet it provides the required force to characterize the CEA solution. It should be noted, that the solution for a deviating individual is insensitive to inflation of demand by the individual, if the award for the individual was originally less than what was originally demanded. This is the gist of the I^3 property.' (I^3) along with (ET) does not appear to characterize the CEA solution uniquely. If we strengthen (ET) slightly to a Weak Monotonicity (WM) property, then (I^3) along with (WM) uniquely characterizes the CEA solution.

Weak Monotonicity (WM):- Given $(d,S)\epsilon B$ if $d_i \leq d_j$ then $F_i(d,S) \leq F_j(d,S)$.

This property says that higher demanders do not get lesser amounts. It is easy to see that Weak Monotonicity implies Equal Treatment, though not conversely.

Theorem 1:- The only solution to satisfy WM and I^3 is CEA.

Proof:- It is easy to see that CEA satisfies these two properties. Hence suppose F is a solution which satisfies these two properties and towards a contradiction assume $F \neq CEA$. Thus there exists $(d,S) \in B^N$ such that $F(d,S) \neq CEA(d,S)$ Without loss of generality and in order to facilitate the proof assume $d_k \leq d_{k+1}$ $\forall k=1,\ldots,n-1$. Clearly there exists $i,j \in N, i<j$ such that $F_i(d,S)<d_i, F_j(d,S) \leq d_j$ and $F_i(d,S) \neq F_j(d,S)$. By $WM, F_i(d,S)<F_j(d,S)$. By WM once again we may assume, j = n and $i = \min\{k/F_k(d,S)<d_k\}$ By $WM, F_i(d,S)<F_n(d,S)$.

Define $d' \in \mathbf{R}^n_+$ as follows:

$$d'_k = d_k \ \forall k \neq i$$

$$d'_i = d_n$$

By $I^3, F_i(d',S) = F_i(d,S)$

By ET (which is implied by WM), $F_n(d',S) = F_i(d',S)$.

Thus $F_n(d',S) = F_i(d,S) < F_n(d,S)$.

Clearly there exists k such that i<k<n and $F_k(d',S) > F_k(d,S)$.

But k>i implies by $WM, F_k(d,S) \geq F_i(d,S) = F_n(d',S)$.

Thus $F_k(d',S) > F_n(d',S)$ which contradicts WM since k<n.

Q.E.D.

However for n = 2, (I^3) and (ET) uniquely characterizes the constrained equal award solution, as the following (which is a strengthening of the previous theorem) reveals.

Theorem 2: For n = 2, the only solution to satisfy (I^3) and ET is CEA.

Proof:- Suppose towards a contradiction, that there exists a rationing problem (d_1, d_2; S) and a solution F satisfying (I^3) and (ET) such that $F(d_1, d_2; S) \neq CEA(d_1, d_2; S)$. Let $(x_1, x_2) = F(d_1, d_2; S)$. Thus $x_1 \neq x_2$. There are two possible cases:

Case 1:- $x_1 < d_1$

Case 2:- $x_2 < x_1 = d_1$

Case 1:- If $x_1 < d_1$ where we have assumed without loss of generality $d_1 \leq d_2$, then by ET, we must have $d_1 < d_2$. Let $d_1' = d_2$.

By (I^3), $F_1(d_1', d_2; S) = x_1$.

By ET, $F_2(d_1', d_2; S) = x_1$.

$\therefore 2x_1 = S = x_1 + x_2$, *contradicting* $x_1 \neq x_2$.

Case 2:- $x_2 < x_1 = d_1$

$\therefore S = x_1 + x_2 < 2d_1$.

By ET, F $(d_1, d_1; S)$ = $(\frac{S}{2}, \frac{S}{2})$

By ET, once again $d_1 < d_2$.

By I', F_2 $(d_1, d_2; S)$ = $\frac{S}{2}$. Thus F_1 $(d_1\ d_2\ ; S)$ = $\frac{S}{2}$. Thus

$x_1 = x_2 = S$ contradicting $x_2 < x_1$.

This proves the theorem.

Q.E.D.

4. The Variable Population Model:-

There is a population of "potential agents", indexed by elements in a set I. Let P denote the set of all non-empty finite subsets of I. Given $M \epsilon P$, let $\mathbf{R}_+^M$ (*respectively* $\mathbf{R}_{++}^M$) denote the set of all functions from M *to* $\mathbf{R}_+$ (*respectively* $\mathbf{R}_{++}$) . Here $\mathbf{R}_+$ is the set of all non-negative real numbers and $\mathbf{R}_{++} \equiv \mathbf{R}_+ \backslash \{0\}$.

Given $M \epsilon P$, a rationing problem for M is an ordered pair $(d, S) \epsilon \mathbf{R}_{++}^M x \mathbf{R}_+$ *such that* $\sum_{i \epsilon M} d_i > S$. .

Let B^M denote the set of all rationing problems for M and $B \equiv \bigcup_{M \in P} B^M$. Let $X = \bigcup_{M \in P} \mathbf{R}_+^M$.

Given $(d,S) \in B^M, M \in P$, an allocation for (d,S) is a vector $x \in \mathbf{R}_+^M$ such that $\sum_{i \in M} x_i = S$ and $x_i \leq d_i \forall i \in M$.

A <u>solution</u> is a function $F: B \rightarrow X$ such that F(d,S) is an allocation for (d,S) whenever $(d,S) \in B$.

The <u>constrained equal awards solution</u> $CEA: B \rightarrow X$ is defined as follows: $\forall (d,S) \in B^M, M \in P, \forall i \in M$, $CEA_i(d,S) = \min\{\lambda, d_i\}$ with $\lambda \geq 0$ satisfying $\sum_{i \in M} \min\{\lambda, d_i\} = S$.

<u>No-envy, property</u>:- A solution F is said to satisfy the no-envy, property if $\forall M \in P, \forall (d,S) \in B^M \forall i,j \in M, d_i - F_i(d,S) \leq |d_i - F_j(d,S)|$

The no-envy property is quite simple: between any two agents there should not arise a situation where any one's unfulfilled demands exceed the deviation of the other's from the first agent's claim i.e. no one's excess demand should be greater than either the excess supply or excess demand of the other from the one's point of view. If the situation were otherwise, then there would be an agent who would want someone else's allotment, since that would lead to a lower loss for him/her, where loss is measured in terms of deviation from announced demands.

Individual Rationality from equal division:- A solution F is said to satisfy individual rationality from equal division if $\forall M\epsilon P, \forall (d,S)\epsilon B^{M}\ \forall i\epsilon M,\quad d_i - F_i(d,S) \leq |S/_{|M|} - d_i|$

Once again the meaning is clear: for every agent the excess demand should not exceed his deviation from equal division of resources.

The following theorem is immediate.

Theorem 3: (a) CEA satisfies the no-envy property

(b) CEA satisfies individual rationality from equal division.

Proof: Let $CEA(d,S) = x\epsilon\mathbf{R}_{+}^{M}$ *for some* $M\epsilon P$, $(d,S)\epsilon B^{M}$.

(a) Suppose towards a contradiction that there exists i, jϵM with $d_i - x_i > |d_i - x_j|$

Clearly $d_i \neq x_i$

$$\therefore 0 \le x_i = \lambda < d_i$$

where $\sum_{k \in M} \min \{ \lambda, d_k \} = S.$

Since $x_j \ne x_i$, we have $x_j \ne \lambda$.

Thus $x_j = d_j$

$$\therefore d_i - \lambda > |d_i - d_j| \text{ with } d_j < \lambda < d_i.$$

$$\therefore d_i - \lambda > d_i - d_j$$

$\therefore \lambda < d_j$ which is a contradiction.

This proves (a).

(b) Suppose towards a contradiction that there exists $i \in M$ with

$$d_i - x_i > |d_i - S/_{|M|}|.$$

Thus $x_i = \lambda$ where λ is as in (a) and λ

$$\therefore d_i - \lambda > | d_i - S/_{|M|} |$$

<u>Case 1</u>: $\lambda < S/_{|M|}$.

$$\therefore S = \sum_{k \in M} x_k = \sum_{\lambda \le d_k} \lambda + \sum_{d_k < \lambda} d_k < S$$

which is a contradiction. Thus Case 1 cannot occur and we have

Case 2:- $$\lambda \geq \frac{S}{|M|}$$

$$\therefore d_i > \lambda \geq \frac{S}{|M|}$$

$\therefore d_i - \lambda \leq d_i - \frac{S}{|M|} = | d_i - \frac{S}{|M|} |$ which is again a contradiction.

This proves (b).

Q.E.D.

We now invoke the following property:

Resource Monotonicity:- A solution F is said to satisfy resource monotonicity if $\forall M \in P$,

$(d,S) \in B^M, (d,S') \in B^M, S' \geq S$ implies $F(d,S') \geq F(d,S)$.

The meaning of resource monotonicity is simple and needs no further explanation.

5. Axiomatic Characterizations of the CEA Solution In Terms of Consistency:

Lemma 1:- Let $(d_i, d_j; S)$ be a two agent rationing problem. Suppose that solution F satisfies either no-envy or individual rationality from equal division. Suppose $d_i \le d_j$ and

$(x_i, x_j) = F(d_i, d_j; S) \neq CEA(d_i, d_j; S)$. Then

$x_i < d_i, x_i \neq x_j$.

Proof:- Suppose not. Then the only other possibility is $x_j < x_i = d_i \le d_j$.

Since $d_j - x_j > d_j - x_i = | d_j - x_i |$, F violates no-envy (:indeed j envies i).

Since $x_j < \frac{x_i + x_j}{2} < x_i = d_i \le d_j$,

$d_j - x_j > d_j - \frac{x_i + x_j}{2} = | d_j - \frac{x_i + x_j}{2} |$. Thus F violates individual rationality from equal divis

Q.E.D.

Lemma 2:- If a solution F satisfies no-envy and resource monotonicity, then it coincides with CEA solution for all two agent problems.

Proof:- Towards a contradiction assume that there exists $\{i,j\} \in P$ and $(d_i, d_j, S) \in B^{\{i,j\}}$ such that

$F(d_i, d_j, S) \neq CEA\ (d_i, d_j, S)$ where we have that F satisfies no-envy and resource monotonicity. Without loss of generality assume $d_i \leq d_j$. Then if $(x_i, x_j) = F(d_i, d_j; S)$ we must have $x_i < d_i$, and $x_j \neq x_i$. If $x_j < x_i$ then $|d_j - x_j| > |d_j - x_i|$

Contradicting no-envy. Thus $x_j > x_i$.

If $x_i < x_j < d_i$, then $|d_i - x_i| > |d_i - x_j|$ contradicting no-envy.

Thus $x_i < d_i < x_j$, In fact we must have $x_i < d_i < 2d_i - x_i \leq x_j$ so that no-envy is satisfied. Thus $2d_i \leq x_i + x_j$.

Hence if $S < 2d_i, F(d_i, d_j; S) = CEA(d_i, d_j; S)$. By resource monotonicity, $F(d_i, d_j; S) = CEA(d_i, d_j; S)$ if $S \leq 2d_i$.

Thus for $S = 2d_i, F(d_i, d_j; S) = (d_i, d_i)$.

For $S>2d_i$, by monotonicity, $F_i(d_i,d_j;S)=d_i$. This contradicts $x_i<d_i$.

Hence $F(d_i,d_j;S) = CEA\ (d_i,d_j;S)$.

<u>Q.E.D.</u>

<u>Lemma 3</u>:- If a solution F satisfies individual rationality from equal division and resource monotonicity, then it must coincide with the Constrained Equal Awards Solution for all two agent problems.

<u>Proof</u>:- As in Lemma 1, let us assume that $(d_i,\ d_j\ ;\ S)$ is a claims problem and F satisfies the properties listed in Lemma 2. Suppose $F(d_i,d_j;S) = (x_i,x_j) \neq CEA(d_i,d_j;S)$.

Assuming without loss of generality $d_i \leq d_j$, we must have $x_i<d_i$, $x_i \neq x_j$.

Suppose $x_j < x_i < d_i \leq d_j$.

Then $d_j-\frac{x_i+x_j}{2}<d_j-x_j$ contradicting individual rationality from equal division. Thus $x_i<x_j$

If $x_i < x_j \le d_i \le d_j$, *then* $d_i - \frac{x_i + x_j}{2} < d_i - x_i$, once again contradicting individual rationality from equal division. Thus $d_i < x_j$

Suppose $\frac{x_i + x_j}{2} < d_i$.

Then $d_i - x_i \le d_i - \frac{x_i + x_j}{2}$

implies $x_i \ge \frac{x_i + x_j}{2}$ contradicting $x_j > x_i$.

Thus $x_i + x_j \ge 2d_i$

Hence for $S < 2d_i, F(d_i, d_j; S) = CEA(d_i, d_j; S)$

By resource monotonicity, $S > 2d_i$. implies $F_i(d_i, d_j; S) = d_i$ which contradicts $x_i < d_i$. *Thus* $F(d_i, d_j; S) = CEA(d_i, d_j; S)$.

<u>Q.E.D.</u>

<u>Consistency</u>: A solution F is said to satisfy <u>consistency</u> if $\forall M \in P, (d, S) \in B^M, x = F(d, S), \phi \ne N \subset M, \left(d_N, \sum_{i \in N} x_i\right) \in B^N$, implies

$$x_N = F\left(d_N, \sum_{i \in N} x_i\right)$$

Here $d_N = (d_i)_{i \in N}$ and $x_N = (x_i)_{i \in N}$.

Bilateral Consistency is simply the same property as above requiring in addition that N should be a set consisting of exactly two members.

Converse-Consistency: A solution F is said to satisfy converse-consistency if $\forall M \in P, (d,S) \in B^M, x$ is an allocation for (d,S) and $\forall \phi \neq N \subset M, N$ has exactly two members, $x_N = F\left(d_N, \sum_{i \in N} x_i\right)$, then $x = F(d,S)$.

The following lemma is easy to prove:

Lemma 4:- CEA satisfies consistency and converse-consistency.

We need one more lemma, before we can state the results that we promised in the introduction.

Lemma 5:- If F is a solution which satisfies bilateral consistency and agrees with CEA for all two agent problems, then F = CEA.

Proof:- Essentially the proof of Lemma 4 in Dagan (1996b).

We now have the following two major characterization theorems, by using the results obtained so far.

Theorem 3:- The unique solution on B to satisfy bilateral consistency, no-envy and resource monotonicity is CEA.

Theorem 4:- The unique solution on B to satisfy bilateral consistency, individual rationality from equal division and resource monotonicity is CEA.

6. Axiomatic Characterizations of the CEA Solution In Terms of Population Monotonicity:-

Let $\mathbb{N}$ be the set of natural numbers and let $I = \mathbb{N}$.

Resource Continuity: F is said to satisfy resource continuity if given $M \in P$, $(d,S) \in B^M$ and $\epsilon > 0$, there exists $\delta > 0$

such that $|S'-S|<\delta$, $(d,S') \in B^M \rightarrow ||F(d,S)-F(d,S')||<\epsilon$ where the norm is simply the Euclidean norm.

Resource Continuity is really a mind regularity assumption.

Population Monotonicity: F is said to satisfy population monotonicity if $\forall Q \in P$ and

$k \in \mathbb{N}-Q$, $(d,S) \in B^Q$, $(d',S) \in B^{Q\cup\{k\}}$, if $d'_i = d_i \ \forall i \in Q$, then

$F_i(d', S) \leq F_i(d, S) \; \forall i \in Q.$

Population monotonicity says that the arrival of a new agent, should not increase the wards for existing agents. This assumption seems quite reasonable.

Replication-Invariance: F is said to satisfy replication invariance if $\forall \, Q \in P$ and $k \in \mathbf{N}$, if $Q' \in P$ with $|Q'| = k \, |Q|$

and for

$i \in Q$ (i, 1), . . . , (i, k) $\in$ Q' then for $(d, S) \in B^{Q}$ and

$(d', kS) \in B^{Q'}$, with $d_{(i,j)} = d_i, j = 1, \ldots, k, i \in Q,$ x = F (d, S)

implies $y_{(i,j)} = x_i' \; \forall \; i \in Q, \; j = 1, \ldots, k,$ where y = F (d', kS)

$\in \mathbf{R}_{+}^{Q'}$.

The meaning of replication invariance is quite simple: if a rationing problem is replicated k times (i.e.) the available supply is multiplied k times and corresponding to each original agent there are now k agents with the same demand) then each replica in the replicated problem gets what the original agent in the original problem got. This assumption seems harmless.

We now prove the main theorem of this section, which states that the only solution to satisfy no-envy, population monotonicity, resource continuity and replication invariance is the CEA solution.

Theorem 5:

The only solution to satisfy no-envy, population monotonicity, replication invariance and resource continuity is CEA.

Proof:

That CEA satisfies the above properties has been discussed earlier. Hence, let us establish the converse. Thus, suppose F is a solution which satisfies the desired properties and towards a contradiction assume that there exists $L \in P$, $(d, S) \in B^L$ such that $F(d,S) \neq CEA\,(d,S)$. Thus there exists $i, j \in L$ such that

$x_i < d_i,\ x_i \neq x_j$

where x = F (d, S).

By no-envy, we must have

$x_i < d_i \leq 2d_i - x_i \leq x_j \leq d_j$

If we keep the available supply fixed at S, and simply replicate each agent 'k' times, then by no-envy, each agent of the same type gets the same amount. By population monotonicity and no-envy, we must have

either $x_i^k \leq x_i < d_i \leq 2d_i - x_i \leq 2d_i - x_i^k \leq x_i^k \leq x_j \leq d_j$

or $x_j^k = x_j^k$

where x_i^k is the common amount that a type i agent gets in the replicated problem (where the supply remains) fixed.

If (1) holds $\forall$k, then

$$kx_j^k \geq k\ (2d_i - x_i) > S$$

for $k \in$ **N** sufficiently large.

Hence for a sufficiently large replication, (2) holds.

Since i and $j \in L$ were arbitrarily chosen, we get that there exists $k^* \in$ **N**, such that if each agent is replicated k^* times and the supply is held fixed at S, then F(d', S) = CEA (d', S') were d' is as defined in the statement of the replication invariance property.

However, by replication invariance,

$F_{(i,l)}\ (d', k^*S) = F_i\ (d, S)\ \forall\ i \in L,\ l = 1, \ldots k^*$ where (i,l) is the l^{th} agent of type i (i.e. the l^{th} replica of agent i in the original problem).

Thus, there exists $i, j \in L$ such that

$$x_i < d_i \leq 2d_i - x_i \leq x_j \leq d_j$$

and $x_i^{k^*} = x_j^{k^*} < d_i$

As the total resources are increased form S to k^*S, the individual awards of type i and type j agents change from $x_i^{k^*}$ to x_i and $x_j^{k^*}$ to x_j respectively. By resource continuity, there exists $S' > S$, $S' < k^*S$ such that if y_i is what a type i agent gets at S' and y_j, is what a type j agent gets as S', then $y_i < y_j < d_i$

Thus no-envy is easily seen to be violated; infact, i envies j.

This contradiction establishes the theorem.

Q.E.D.

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