

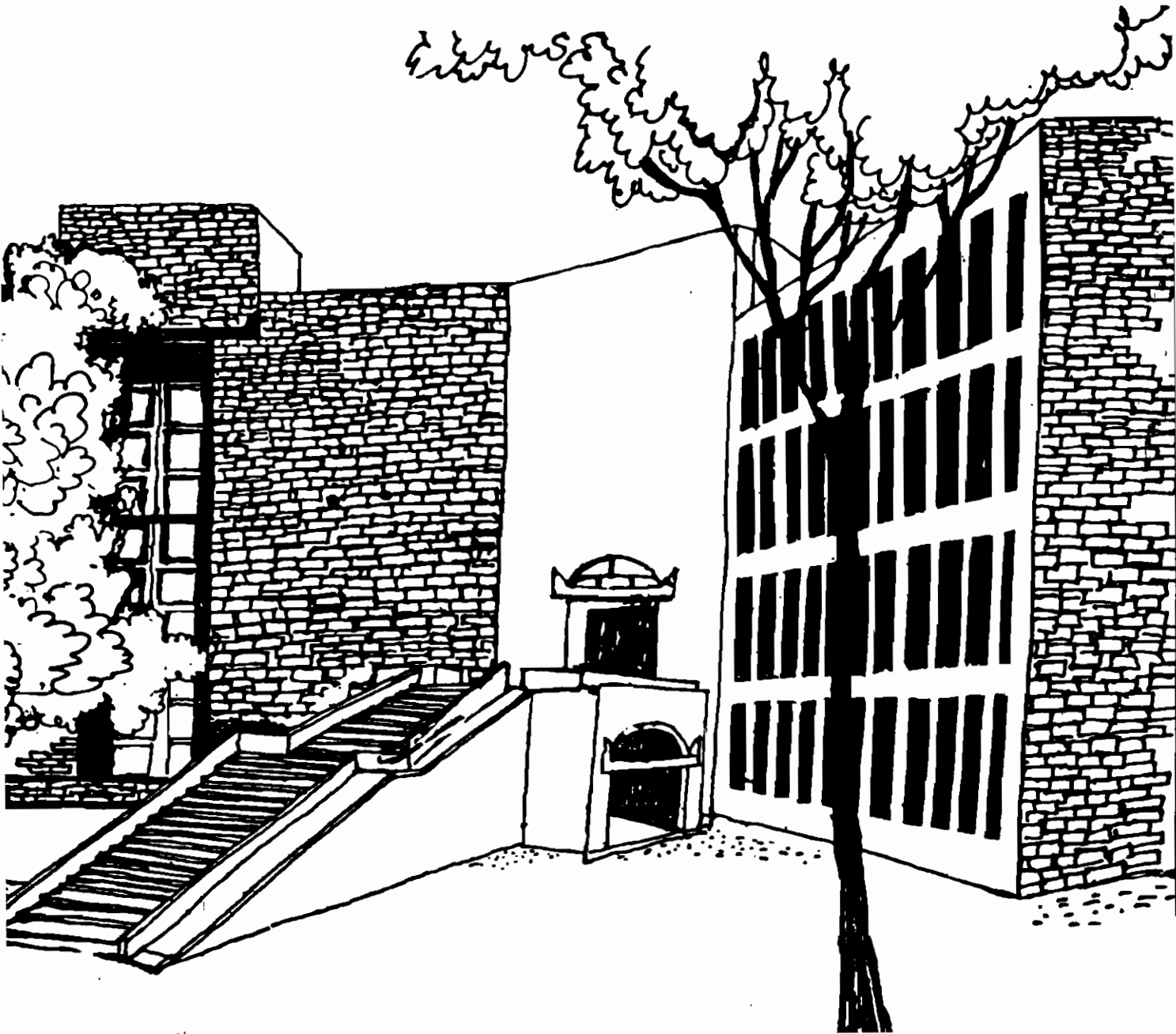


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Working Paper



**A NOTE ON COST MONOTONIC GROUP
DECISION MECHANISMS**

By

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Abstract

In this paper we study the problem of project selection as a group decision making problem and obtain a characterization of cost monotonic group decision mechanism. We furnish two examples of cost monotonic group decision mechanisms - the egalitarian mechanism and the egalitarian equivalent mechanism. The latter is shown to belong to the core of the group decision making problem. In the process of defining an egalitarian equivalent mechanisms we invoke the concept of a composite public good.

1. Introduction :- A large literature on the decentralized allocation of resources in the process of provision of a public project has grown up considering the public project to be a (one-dimensional) public good. A notable contribution in this line of activity is the paper by Moulin (1987), where he defines an egalitarian equivalent group decision mechanism and shows that it is the unique mechanism to satisfy cost monotonicity, Pareto efficiency and individual monotonicity.

In Mas-Colell (1980) can be found a model which studies decentralized resource allocation in the process of provision of a public project, where the public project is not necessarily a single public good. Take for instance the situation where a community decides to build a swimming pool for the sole consumption of the members of the community. Clearly the dimensions of the swimming pool - length, breadth and depth - influence the preferences of the agents. There is no way in which the swimming pool can be captured as a one dimensional public good in the process of consumption. It may be argued that after all what matters in calculating the cost of construction of a swimming pool is its surface area. However, the same surface area is compatible with several different dimensions e.g. one which is very deep but has less length and breadth and another which has comparatively greater length and breadth but is shallow. From the point of view of a swimmer, it is not the surface area, but the vector of dimensions which is important in deciding which swimming pool to choose. Thus we are compelled to consider the space of public projects as a general metric space as in Mas-Colell (1980). It may be worth noting that the concept of a ratio-equilibrium due to Kaneko (1977) is easily adapted to the general framework of Mas-Colell (1980).

It is precisely in this framework that we establish the equivalence of the two concepts of cost monotonicity (due to Moulin (1987)) and solidarity (due to Moulin (1991)) when the group decision mechanism is Pareto efficient. Cost monotonicity says that if one production process is uniformly cheaper than

another, then everyone should be at least as well off in the first production process as in the second. Solidarity says that given any two production processes, either everyone is at least as well off in the first as in the second or it is the other way round. The example of an egalitarian group decision mechanism (which satisfies the solidarity axiom) is furnished to show that Pareto efficient and cost monotonic group decision mechanisms are not scarce.

Next we consider public projects which appear like a composite public good to the consumer. This is typically the case when a public good has several dimensions (as for instance a swimming pool to be used by a community). In such a framework we define the concept of an egalitarian - equivalent state, originally due to Moulin (1987), and adopting the proof of a similar result in Moulin (1988), we show that for a large class of environments, an egalitarian - equivalent state belongs to the core of the group decision problem (as defined by Mas-Colell (1980)). Egalitarian-equivalent mechanisms are naturally cost-monotonic. This is the final result of our paper.

In our presentation, we follow closely the description of the model existing in Mas-Colell (1980).

2. The Model and Assumptions :- There is given a nonempty, metric space K of projects and a finite collectivity of agents $N = \{1, \dots, n\}$. Every agent $i \in N$ has preferences on tuples (x, m) of projects and amounts of a unique private good (to be called "money"), represented by a continuous-utility function $u_i : K \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that $\forall (x, m), (x, m') \in K \times \mathbb{R}_+, m > m' \Rightarrow u_i(x, m) > u_i(x, m')$. Further money is indispensable for each agent i.e. $\forall i \in N, \forall m > 0$ and $\forall x, x' \in K, u_i(x', m) > u_i(x, 0) \equiv 0$. We assume every $i \in N$ is endowed with a positive amount of money $w_i > 0$.

Let $c : K \rightarrow \mathbb{R}_+$ be a cost function for the provision of the project.

Definition 1 :- A state is a $(n+1)$ tuple $(x, m_1, \dots, m_n) \in K \times \mathbb{R}_+^n$. It is denoted by (x, \underline{m}) .

Definition 2 :- A state (x, \underline{m}) is feasible if

$$c(x) \leq \sum_{i \in N} w_i - \sum_{i \in N} m_i$$

Definition 3 :- A state (x, \underline{m}) is Pareto efficient if it is feasible and if there is no feasible state (x', \underline{m}') such that $u_i(x', m'_i) \geq u_i(x, m_i) \forall i \in N$ with strict inequality for at least one $i \in N$.

Definition 4 :- A state (x, \underline{m}) belongs to the core if it is feasible and there is no $S \subseteq N, S \neq \emptyset$ and a state (x', \underline{m}') with $c(x') \leq \sum_{i \in S} w_i - \sum_{i \in S} m'_i$ and $u_i(x', m'_i) \geq u_i(x, m_i) \forall i \in S$ with strict inequality for at least one $i \in S$.

Let $u_i, i \in N; w_i, i \in N$ be fixed and let c the cost function be a variable. Let \mathcal{Q} be a set of cost functions.

A group decision mechanism is a correspondence

$F: \mathcal{Q} \rightarrow K \times \mathbb{R}^n$, such that:

- (i) $(x, \underline{m}), (x', \underline{m}') \in F(c) \Rightarrow u_i(x, m_i) = u_i(x', m'_i) \forall i \in N$
- (ii) $(x, \underline{m}) \in F(c)$ and $u_i(x, m_i) = u_i(x', m'_i) \forall i \in N$
 $\Rightarrow (x', \underline{m}') \in F(c)$ where $\underline{m}' = (m'_i)_{i \in N}$.
- (iii) $F(c) \subseteq \{(x, \underline{m}) \in K \times \mathbb{R}^n / c(x) \leq \sum_{i \in N} w_i - \sum_{i \in N} m_i\}$.

Definition 5 :- A group decision mechanism F is said to be cost monotonic if $\forall c_1, c_2 \in \mathcal{Q}$ with $c_1(x) \leq c_2(x) \forall x \in K$ we have $u_i(x, m_i) \geq u_i(x', m'_i)$ whenever $(x, \underline{m}) \in F(c_1)$ and $(x, \underline{m}') \in F(c_2)$.

Definition 6 :- A group decision mechanism F is said to satisfy the axiom of solidarity if $\forall c_1, c_2 \in \mathcal{Q}$ and $(x, \underline{m}) \in F(c_1)$,

- $(x', \underline{m}') \in F(c_2)$ we have either (i) $u_i(x, m_i) \geq u_i(x', m'_i) \forall i \in N$
- or (ii) $u_i(x, m_i) \leq u_i(x', m'_i) \forall i \in N$.

3. Existence of Cost Monotonic Group Decision Mechanisms :-

Theorem 1 :- Let F be a Pareto efficient group decision mechanism i.e. $(x, \underline{m}) \in F(c) \Rightarrow (x, \underline{m})$ is Pareto efficient for $c \in \mathcal{Q}$. Then F is cost monotonic if and only if it satisfies the axiom of solidarity.

Proof :- That the axiom of solidarity implies cost monotonicity in the presence of Pareto optimality is obvious. Hence let us prove the converse.

Let F be cost monotonic and let $c_1, c_2 \in \mathcal{C}$. Let $c(x) = \min\{c_1(x), c_2(x)\} \forall x \in K$.

Let $(x^i, m^i) \in F(c)$. Thus $c(x^i)$ is either equal to $c_1(x^i)$ or $c_2(x^i)$. Without loss of generality assume, $c(x^i) = c_1(x^i)$. Since $c(x) \leq c_1(x) \forall x \in K$, by cost monotonicity,

$$u_i(x^i, m^i) = u_i[F(c)] \geq u_i[F(c_1)] \forall i \in N.$$

Now $c(x^i) = c_1(x^i)$ implies $c_1(x^i) \leq \sum_{i \in N} w_i - \sum_{i \in N} m^i$.

Hence by Pareto efficiency of F , $u_i[F(c)] = u_i[F(c_1)] \forall i \in N$.

But $c(x) \leq c_2(x) \forall x \in K$; by cost-monotonicity

$$u_i[F(c_1)] = u_i[F(c)] \geq u_i[F(c_2)] \forall i \in N.$$

This shows that F satisfies the axiom of solidarity.

Q.E.D.

Example :- The Egalitarian Solution :- Let $K = \mathbb{R}^k$, for some $k \in \mathbb{N}$ and $c: \mathbb{R}^k \rightarrow \mathbb{R}_+$ be continuous and strictly increasing with $c(0) = 0$. Thus the set $\{(x, m) \in \mathbb{R}^k \times \mathbb{R}^n, c(x) \leq \sum_{i \in N} w_i - \sum_{i \in N} m^i\}$ is compact and comprehensive.

Let $u_i: \mathbb{R}^k \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be strictly increasing $\forall i \in N$. It is easy to see as in Villar (1990), that there exists (x^i, m^i) which is Pareto efficient and such that $u_i(x^i, m^i) = u_j(x^i, m^j) \forall i, j \in N$.

Let \mathcal{Q} be the collection of all cost functions satisfying the above properties. Let $F: \mathcal{Q} \rightarrow \mathbb{R}^k \times \mathbb{R}^n$ be defined as follows:

$(x^i, m^i) \in F(c) \Leftrightarrow u_i(x^i, m^i) = u_j(x^i, m^j) \forall i, j \in N$ and (x^i, m^i) is Pareto efficient.

F satisfies the axiom of solidarity on \mathcal{Q} . Hence F is cost monotonic by Theorem 1.

4. A Composite Public Good :- Often times agents in a society treat a public project as a composite public good i.e. one which behaves like a single public good in the process of yielding satisfaction to the agents. The preferences of each agent $i \in N$ are then summarized by two functions $v_i: \mathbb{R}^2 \rightarrow \mathbb{R}_+$ and $t: k \rightarrow \mathbb{R}_+$ (the latter being independent of i) such that:

(a) v_i is continuous and strictly increasing

(b) t is continuous

(c) $v_i(t(x), m_i)$ measures the satisfaction to agent i from the consumption bundle $(x, m_i) \in K \times \mathbb{R}_+^n$.

(d) $v_i(t(x), 0) = 0 \forall x \in K$

Let $\bar{t} = \sup\{t / \exists (x, \underline{m}) \in K \times \mathbb{R}_+^n, \text{ with } c(x) \leq \sum_{i \in I} w_i - \sum_{i \in I} m_i \text{ and } v_i(t, w_i) \geq v_i(x, m_i) \forall i \in N\}$

A feasible state (x, \underline{m}) such that $v_i(t(x), m_i) = v_i(\bar{t}, w_i) \forall i \in N$ is called an egalitarian equivalent state (in the sense of Moulin (1987)).

Lemma 1 :- An egalitarian equivalent state is Pareto efficient.

Proof :- Suppose (x^*, \underline{m}^*) is an egalitarian equivalent state which is not Pareto efficient. Then there exists a feasible state (x', \underline{m}') such that

$$v_i(t(x'), m'_i) > v_i(t(x^*), m^*_i) \forall i \in N \text{ (by (a) and (d))}$$

Thus $v_i(t(x'), m'_i) > v_i(\bar{t}, w_i) \forall i \in N$ where $v_i(\bar{t}, w_i) = v_i(t(x^*), m^*_i) \forall i \in N$.

Hence $\exists \epsilon > 0$ such that $v_i(t(x'), m'_i) > v_i(\bar{t} + \epsilon, w_i) \forall i \in N$, contradicting the definition of \bar{t} and (x^*, \underline{m}^*) as an egalitarian equivalent state.

Lemma 2 :- If (x^*, \underline{m}^*) is an egalitarian equivalent state with $v_i(t(x^*), m^*_i) \geq v_i(\bar{t}, w_i) \forall i \in N$, then there exists a feasible state (x', \underline{m}') with $v_i(t(x'), m'_i) = v_i(\bar{t}, w_i) \forall i \in N$.

Proof :- Suppose $v_i(t(x^*), m^*_i) > v_i(\bar{t}, w_i)$ for some $i \in N$

$$\text{Clearly } m^*_i > 0 \text{ and } v_i(t(x^*), 0) = 0 \leq v_i(\bar{t}, w_i)$$

By the intermediate value theorem for continuous functions there exists $1 > \alpha > 0$ such that $v_i(t(x^*), \alpha m^*_i) = v_i(\bar{t}, w_i)$

$$\text{Clearly, } c(x^*) \leq \sum_{i \in N} w_i - \sum_{i \in N} m^*_i \leq \sum_{i \in N} w_i - \sum_{j \in N} m^*_j - \alpha m^*_i$$

$$j \neq i$$

This proves the lemma.

Q.E.D.

Theorem 2 :- Let (x^*, \underline{m}^*) be an egalitarian equivalent state. Then (x^*, \underline{m}^*) belongs to the core.

Proof :- Suppose towards a contradiction that there exists $\emptyset \neq S \subseteq N$, and a state (x', \underline{m}') with $c(x') \leq \sum_{i \in S} w_i - \sum_{i \in S} m'_i$ and v_i

$(t(x'), m'_1) \succ v_1 (t(x^*), m^*_1) \forall i \in S.$

Define $y' \in \mathbb{R}^n$, as follows:

$$y'_i = m^*_i \quad \forall i \in S$$

$$y'_i = w_i \quad \forall i \in N \setminus S.$$

By Lemma 1, $N \setminus S \neq \emptyset$

Let $v_1 (t(x^*), m^*_1) = v_1 (\bar{t}, w_1) \quad \forall i \in N.$

If $t(x') \leq \bar{t}$, then $v_1 (t(x'), m'_1) \leq v_1 (\bar{t}, w_1) \quad \forall i \in N,$

contradicting $v_1 (t(x'), m'_1) > v_1 (\bar{t}, w_1) \quad \forall i \in S.$

Thus $t(x') > \bar{t}$. Hence by strict monotonicity of v_1

Thus $v_1 (t(x'), w_1) > v_1 (\bar{t}, w_1) = v_1 (t(x^*), m^*_1) \quad \forall i \in N \setminus S$

Thus (x', y') which is a feasible state, Pareto dominates (x^*, m^*_1) , the latter being an egalitarian equivalent state. This contradicts the result in Lemma 1 and proves this theorem.

Q.E.D.

The significant thing to note about this theorem is that, although the public project is assumed to behave like a composite public good in the process of consumption, it is not assumed to function like a composite commodity in the process of it being produced. The cost of producing the public project may depend on one or more of the dimensions defining the project.

It should also be noted, the group decision mechanism $G:$

$\mathcal{Q} \rightarrow K \times \mathbb{R}^n$, such that $\forall c \in \mathcal{Q}$ $G(c)$ is the set of egalitarian equivalent states for c , defines a cost-monotonic group decision mechanism. However in view of our example in section 3 it is not the unique mechanism to yield cost-monotonic and Pareto efficient states.

Theorem 3 :- Let $G: \mathcal{Q} \rightarrow K \times \mathbb{R}^n$, be the egalitarian-equivalent mechanism i.e. $\forall c \in \mathcal{Q}$, $(x^*, m^*_1) \in G(c)$ if and only if (x^*, m^*_1) is an egalitarian equivalent state for c . Then G is cost monotonic.

Proof :- Let $c_1, c_2 \in \mathcal{Q}$ and suppose $c_1(x) \leq c_2(x) \quad \forall x \in K$. Let (x^*, m^*_1) be an egalitarian equivalent state for c_2 . Thus, there exists $\bar{t} \geq 0$, such that $v_1 (t(x^*), m^*_1) = v_1 (\bar{t}, w_1) \quad \forall i \in N$ (without loss of generality and by Lemma 2).

But (x^t, g^t) is a feasible state for c_1
Hence $\bar{t} \in (t^* / \exists (x, g)$ with $c_1(x) \leq \sum_{i \in N} w_i$
 $\cdot \sum_{i \in N} m_i$ and $v_i(t(x), m_i) \geq v_i(t^*, w_i) \forall i \in N) = A$
Thus $\sup_{t^* \in A} t^* \geq \bar{t}$

Let $t^t = \sup_{t^* \in A} t^*$

$\Delta v_i(t^t, w_i) \geq v_i(\bar{t}, w_i) \forall i \in N.$

since v_i is strictly increasing.

Thus $u_i[G(c_1)] = v_i(t^t, w_i) \geq v_i(\bar{t}, w_i) = u_i[G(c_2)]$
 $\forall i \in N$, proving the theorem.

Q.E.D.

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