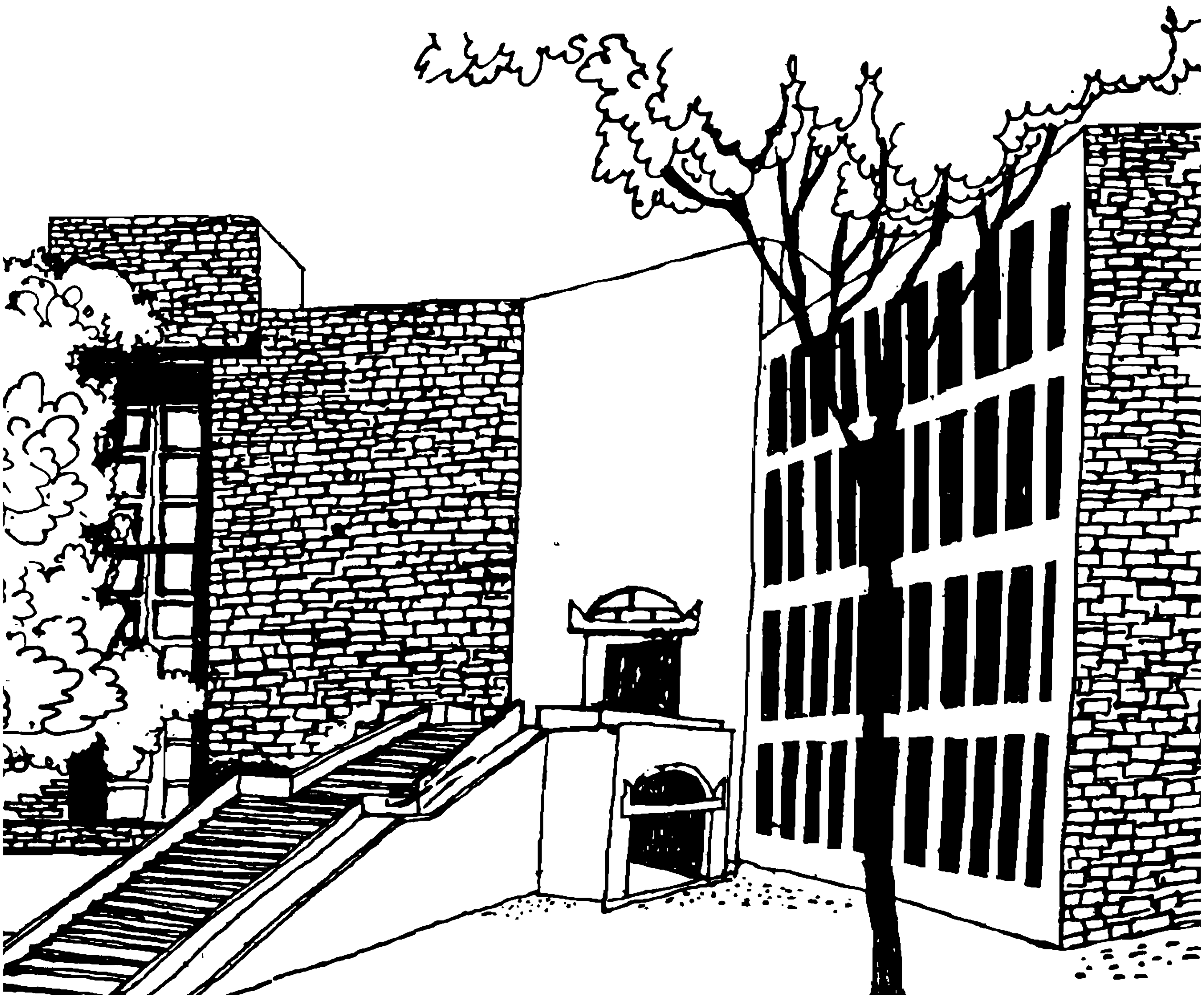
विद्याविनियोगाद्विकासः
IIM
AHMEDABAD

Working Paper



WEAK LOCALIZATION AND THE UTILITARIAN CHOICE FUNCTION: A NOTE

By

Somdeb Lahiri

W.P. No.1421
January 1998

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Abstract

In axiomatic bargaining (choice theory), a choice function of some importance is the utilitarian choice function. Basically, this choice function selects the vector of utilities whose sum is greatest, among all utility vectors. There have been several axiomatic characterizations of the utilitarian choice function. Notable among them are the ones due to Myerson [1981], and Moulin [1988]. A variant of the utilitarian choice function, called the additive choice function (: the latter being defined on a larger domain, than the domain permissible for the utilitarian choice function) has been axiomatically characterized in Lahiri [forthcoming].

In this paper, we present an axiomatic characterization of the utilitarian choice function, which is similar to the axiomatic characterization in Moulin [1988], except that we now replace Nash's Independence of Irrelevant Alternatives by an assumption called Weak Localization, essentially due to Peters [1992].

1. Introduction:- In axiomatic bargaining (choice theory), a choice function of some importance is the utilitarian choice function. Basically, this choice function selects the vector of utilities whose sum is greatest, among all utility vectors. There have been several axiomatic characterizations of the utilitarian choice function. Notable among them are the ones due to Myerson [1981], and Moulin [1988]. A variant of the utilitarian choice function, called the additive choice function (: the latter being defined on a larger domain, than the domain permissible for the utilitarian choice function) has been axiomatically characterized in Lahiri [forthcoming].

In this paper, we present an axiomatic characterization of the utilitarian choice function, which is similar to the axiomatic characterization in Moulin [1988], except that we now replace Nash's Independence of Irrelevant Alternatives by an assumption called Weak Localization, essentially due to Peters [1992].

2. The Model:- Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of non-negative reals and $\mathbb{R}_{++}$ the set of strictly positive reals. Given

$n \in \mathbb{N}$, $n \geq 2$, let $\mathbb{R}^n_+$ denote the non-negative orthant of n dimensional Euclidean space.

A bargaining problem (game) is a non-empty subset S of $\mathbb{R}^n_+$ satisfying the following properties:

(i) S is compact, convex, comprehensive (i.e. $0 \leq x \leq y \in S \to x \in S$)

(ii) there exists $x \in S$ with x >> 0 i.e. x is strictly greater than zero.

Let Σ denote the class of bargaining problems.

Given $\phi \neq T \subset \mathbb{R}^n_+$, let cch (T) denote the smallest comprehensive, convex set containing T. It is called the comprehensive, convex hull of T.

Given $\phi \neq T \subset \mathbb{R}^n_+$, let $P(T) = \{x \in T / y > x \rightarrow y \notin T\}$. P(T) is called the Pareto set of T.

(Note, $y > x \leftrightarrow y \geq x$ *and* $y \neq x$.)

Given $\phi \neq T \subset \mathbb{R}^n_+$, let $Z(T) = \{x \in T / y \gg x \rightarrow y \notin T\}$. Z(T) is called the Weak Pareto Set of T.

Given $T \epsilon \Sigma$, we say that T is semi-strictly convex if

and $\alpha \epsilon (0,1), \alpha x + (1 - \alpha) y \notin P(T)$.

Let $\Sigma^{\circ} = \{ S \epsilon \Sigma / S$ is semi-strictly convex $\}$

A domain is any non-empty subset of Σ.

Let V be a domain.

A choice function on V is a function $F : V \rightarrow \mathbb{R}^n_+$ such that $F(S) \in S \; \forall \; S \in V$.

Let $F : V \rightarrow \mathbb{R}^n_+$ be a choice function.

(1) F is said to be Pareto Optimal if $F(S) \in P(S) \forall S \in V$

(2) F is said to be Translation Covariant if $\forall \; S \in V, \; \forall \; a \in \mathbb{R}^n_+$ with

$cch(S+\{a\}) \in V, F(cch(S+\{a\})) = F(S) + a$.

(3) F is said to be symmetric if $\forall S \in V$ such that $S = \pi \; (S) \; \forall \; \pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ which are permutations, we have $F_i(S) = F_j(S) \; \forall \; i, j \in \{1, \ldots, n\}$.

(4) F is said to satisfy Weak Localization if $\forall \; S, T \in V$ such

that whenever there exists a neighbourhood W of F(S) in $\mathbb{R}^n_+$ *with* $W \cap S = W \cap T$, we have F(S) = F(T), provided $F(S) \in Z(S)$.

For some of the undefined expressions above (like that of a permutation) we refer the reader to Moulin (1988).

The utilitarian choice function $U : \Sigma^\circ \to \mathbb{R}^n_+$ is defined as follows:

$$U(S) = \{x \in S / \Sigma_{i=1}^n x_i \geq \Sigma_{i=1}^n y_i \ \forall\, y \in S\}$$

It is easily checked that u is well defined and satisfies all the above mentioned properties.

3. The Main Result:- We now establish the main result of the paper.

Theorem: The only choice function on Σ° which is Pareto Optimal, Translation Covariant, Symmetric and satisfies Weak Localization is U.

Proof: Having already asserted that U satisfies all the above properties, let F be any choice function which satisfies the mentioned properties and let $S \in \Sigma^\circ$. By Translation Covariance, Pareto Optimality and Symmetry we may assume U (S) = be where b > 0 and e is the vector with all coordinates equal to one. Towards a contradiction assume,

$z = F(S) \neq U(S)$. Thus $\sum_{i=1}^{n} z_i < nb$.

Let $K = \{ x \in \mathbb{R}^n_+ \ / \ \Sigma_{i=1}^{n} x_i \leq nb \}$.

Let T be any symmetric set in Σ° such that.

(i) $S \subset T \subset K$

(ii) $z \notin P(T)$

Clearly such a set T contains the smallest symmetric set in Σ° containing S. By Pareto Optimality and Symmetry, F (T) = U (T) = be.

Let $0 < \alpha < 1$ such that $z \in T' \setminus P(T')$ where $T' = \alpha T$. This is clearly possible for α close to 1.

Now $F(T') = U(T') = \alpha\, be$, since F and U are both Pareto Optimal and Symmetric.

Consider $S \cap T'$. By Weak Localization of F, and since $S \cap T'$ and S agree on a neighbourhood of z, $F(S \cap T') = F(S) = z$.

But $S \cap T'$. and T' agree on a neighbourhood of αbe. Thus, by Weak Localization, $F(S \cap T') = F(T) = \alpha be$.

Thus $z = \alpha\, be \notin P(S)$.

This contradicts Pareto Optimality of F and proves the theorem.

Q. E. D.

Example: Given $S \subset \mathbb{R}^n_+$, let $ch(S) = \{y \in \mathbb{R}^n_+ / y \leq x$ for some $x \in S\}$.

ch (S) is called the comprehensive hull of S.

Given $S \in \Sigma$, either

(i) there exists a $a(s) \in \mathbb{R}^n_{++}$ such that $S = ch(\{a(S)\})$ or

(ii) there exists $T(S) \in \Sigma$ and $a(S) \in \mathbb{R}^n_+$ such that

$$S = ch(T(S) + \{a(S)\}) \text{ and } P(T(S)) = Z(T(S)).$$

Define $F : \Sigma^\circ \to \mathbb{R}^n_+$ as follows:

$$F(S) = a(S) \text{ if } S = ch(\{a(S)\})$$

$$= a(S) + \bar{t}e \text{ if } S = ch(T(S) + \{a(S)\})$$

where $\bar{t} = \max\{t / a(S) + te \in S\}$.

F is Pareto Optimal, Symmetric and Translation Covariant, but F does not satisfy Weak Localization.

An example of a choice function which is Pareto Optimal, Symmetric, satisfies Weak Localization but is not Translation Covariant is the Nash choice function (see Peters [1992]).

Note:- There is no choice function F on Σ which is Pareto Optimal, Symmetric, Translation Covariant and satisfies Weak Localization. We prove this for the case n = 2. Towards a contradiction suppose H is a choice function satisfying the above properties. Let $S = cch\left\{(0,1),\left(\frac{1}{4},\frac{3}{4}\right)\right\}$

Case 1:- $H(S) \neq (0,1), H(S) \neq \left(\frac{1}{4},\frac{3}{4}\right)$.

Let K = cch $\{(0,1),\ (1,0)\}$.

By Pareto Optimality of H, in this case, there exists a neighbourhood W of H (S) such that $W \cap S = W \cap K$. Thus H(K)=H(S). However, by Pareto Optimality and Symmetry of

$H, H(K) = \left(\frac{1}{2}, \frac{1}{2}\right) \notin P(S)$. Hence Case 1 is not possible.

<u>Case 2</u>:- H(S) = (0, 1) or H(S) = $(\frac{1}{4}, \frac{3}{4})$.

Let $a = (\frac{3}{4}, 0)$.

Let T = cch (S + {a}) = cch $\left\{(\frac{3}{4}, 1), (1, \frac{3}{4})\right\}$.

By Pareto Optimality and Symmetry, H(T) = $\left(\frac{7}{8}, \frac{7}{8}\right)$.

By Translation Covariance (however), $H(T) = \left(\frac{3}{4}, 1\right)$ or $\left(1, \frac{3}{4}\right)$

Hence Case 2 is not possible either.

In view of this, it is not possible to extend the characterization theorem above, to non-convex problems such as those considered by Zhou [1996].

It remains to show that Weak Localization does not imply Nash's Independence of Irrelevant Alternatives (NIIA), the latter property having been used by Moulin [1988] in an earlier characterization of the utilitarian choice function.

A choice function $F : V \to \mathbb{R}^n_+$ is said to satisfy NIIA if

$\forall S, T \in V, S \subset T, F(T) \in S$ implies F(T) = F(S).

Let $V = \Sigma$ or Σ° and let $G : V \to \mathbb{R}^n_+$ be defined as follows:

$$G(S) = N(S) \text{ if } N(S) = e$$

$$= \frac{1}{2} N(S), \text{ otherwise;}$$

here $N : V \to \mathbb{R}^n_+$ is the Nash [1950] choice function.

G satisfies Weak Localization but does not satisfy NIIA. However, it is an open question whether along with Pareto Optimality, Weak Localization implies NIIA.

References:-

1. S. Lahiri [forthcoming]: "The Supporting Line Property And The Additive Choice Function For Two Dimensional Choice Problems," Pacific Economic Review.

2. H. Moulin [1988]: "Axioms of Cooperative Decision Making," Econometric Society Monograph, Cambridge University Press.

3. R. Myerson [1981]: "Utilitarianism, Egalitarianism And The Timing Effect In Social Choice Problems," Econometrica, 49(4), 883-97.

4. J. F. Nash [1950]: "The Bargaining Problem," Econometrica 18, 155-162.

5. H. Peters [1992]: "Axiomatic Bargaining Game Theory," Theory and Decision Library, Kluwer Academic Publishers.

6. L. Zhou [1996]: "The Nash Bargaining Theory With Non-Convex Problems," Econometrica, Vol.65, No.3, 681-685.