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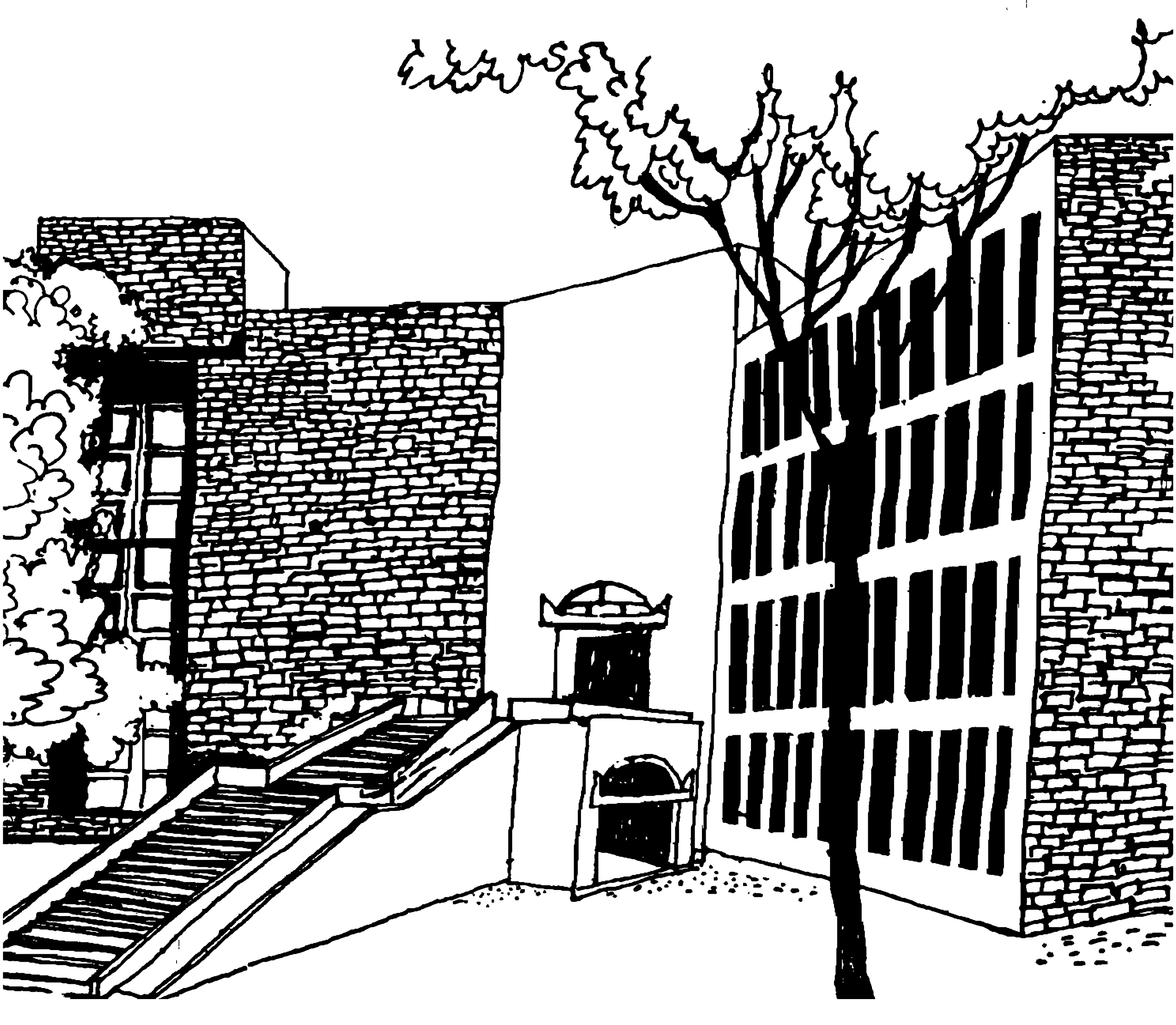

EXTENDED PARTIAL ORDERS: A NOTE

By

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Abstract

The purpose of this paper is to establish the equivalence of two axioms one of which appear in Nehring [1997] and the other in Nehring and Puppe [1999]. The one in Nehring and Puppe [1999] is due to Aizerman and Malishevski [1981]. We there by improve the existing characterisation of choice functions rationalized by extended partial orders. In an appendix to this paper we provide a proof of a related statement appearing in Nehring [1997]. This paper makes extensive use of the rather elegant device known as finite mathematical induction.

Extended Partial Orders: A Note

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1. **Introduction**:- The purpose of this paper is to establish the equivalence of two axioms one of which appear in Nehring [1997] and the other in Nehring and Puppe [1999]. The one in Nehring and Puppe [1999] is due to Aizerman and Malishevski [1981]. We there by improve the existing characterisation of choice functions rationalized by extended partial orders. In an appendix to this paper we provide a proof of a related statement appearing in Nehring [1997]. This paper makes extensive use of the rather elegant device known as finite mathematical induction.

2. **The Model**:- Let X be a finite non-empty set of alternatives and let [X] denote the set of all non-empty subsets of X. An extended relation (ER) is any non-empty subset of [X] x X. An extended relation P is said to be an extended preference relation (EPR) if it satisfies the following two properties:

(a) Irreflexivity (IRR): $\forall$ (A, x) $\in$ P, A \ {x} $\neq \phi$ and (A \ {x}, x) $\in$ P;

(b) Monotonicity(MON): If (A,x) $\in$ P and A $\subset$ B, then (B, x) $\in$ P.

An extended relation P is said to be acyclic if $\forall$ A $\in$ [X], there exists x $\in$ A: (A, x) $\notin$ P; it is said to be transitive if $\forall$ x, y $\in$ X and A $\in$ [X], (A$\cup$ {y}, x) $\in$ P & (B, y) $\in$P implies (A $\cup$ B, x) $\in$ P.

It has been observed in Nehring [1997] that if an EPR P is transitive then it is acyclic. As in Nehring [1997], we refer to a transitive EPR as an extended partial order .

A choice function is any function C: [X] $\rightarrow$ [X] such that, $\phi \neq$ C (A) $\subset$ A $\forall$ A $\in$ [X].

Given a choice function C, let P_c = { (A, x) $\in$ [X] x X / x $\notin$ C (A $\cup$ {x}) }.

Given an ER P, and A $\in$ [X], let L(A, P) = {x $\in$ A/(A, x) $\notin$ P}.

The following observation is immediate:

Observation 1:- Let C be a choice function. Then

(a) $C(A) = L(A, P_c)\ \forall A \in [X]$;

(b) P_c satisfies IRR and acyclicity;

(c) $(A, x) \in P_c$ implies $(A \cup \{x\}, x) \in P_c$ (: a property which we may refer to as Weak Monotonicity).

The following axioms on a choice function appear in the sequel:

A choice function C is said to satisfy:

(1) Chernoff's Axiom(CA) if $\forall A, B \in [X]$, $[A \subset B$ implies $C(B) \cap A \subset C(A)]$;

(2) New Quasi-Transitivity Axiom (NQTA) if $\forall A \in [X]$, $\forall x, y \in A \setminus C(A)$ implies $x \notin C(A \setminus \{y\})$;

(3) Generalized Axiom of Revealed Preference(GA) if $[y \in A \setminus C(A), C(A) \subset B]$ implies $[y \notin C(B)]$ $\forall A, B \in [X]$ and $y \in X$;

(4) Nehring's Axiom of Revealed Preference (NA) if $y \in A \setminus C(A)$ implies $y \notin C(C(A) \cup \{y\})$;

(5) Aizerman and Malishevski's Axiom (AMA) if $\forall A, B \in [X]$, $[C(A) \subset B \subset A] \rightarrow [C(B) \subset C(A)]$;

(6) Outcasting Axiom (OA) if $\forall A, B \in [X]$, $[C(B) \subset A \subset B]$ implies $C(A) = C(B)$.

CA is an assumption which now forms an integral part of rational choice theory; NQTA originally appears as axiom η in Nehring [1997], but is used under its present nomenclature to characterize quasi-transitive rational choice in Lahiri [1999]; GA and NA appear in Nehring [1997] with the latter under the name of ρ_4 ; AMA originates in the work of Aizerman and Malishevski [1981]. This axiom has been used in Nehring and Puppe [1999], and hence the main result reported here, has obvious implications in that paper as well. OA is an axiom which has been attributed to Nash [1950] by Suzumura [1983]. It appears under its present nomenclature in Aizerman and Aleskerov [1995].

3. The Main Results:-

Theorem 1:- AMA $\leftrightarrow$ NQTA

Proof:- The fact that AMA implies NQTA is obvious. Hence let us prove the converse and that too by induction. Thus suppose C is a choice function which satisfies NQTA. Let $A, B \in [X]$, $C(A) \subset B \subset A$, and x be an arbitrary element of $A \setminus C(A)$. We prove our result by backward induction on the cardinality of B.

Let B = A \ {y} for some y ∈ A \ C (A). By NQTA, x ∉ C (B). Since x is arbitrary, C (B) ⊂ C (A) whenever B = A \ {y} and y ∈ A \ C (A).

Now suppose for any $y_1, \ldots, y_r \in A \setminus C(A)$, if $B = A \setminus \{ y_1, \ldots, y_r \}$, then C (B) ⊂ C (A).

Let $y_{r+1} \in A \setminus C(A)$, $y_{r+1} \notin \{ y_1, \ldots, y_r\}$.

Let $B = A \setminus \{y_1, \ldots, y_r\}$ and thus $B = B \setminus \{y_{r+1}\}$

By NQTA, C (B) ⊂ C (*B*). However, by the induction hypothesis, C (*B*) ⊂ C(A). Hence, C (B) ⊂ C (A).

Since the result has been proved for r = 1 and has now been shown to be true for r+ 1 if it assume true for r, it is therefore true in general.

Q.E.D.

Theorem 2:- AMA & CA ↔ GA.

Proof:- Let C be a choice function which satisfies AMA and CA. Let A, B ∈ [X] and let y ∈ A \ C (A) with C (A) ⊂ B.

Consider A ∩ B. Clearly C (A) ⊂ A ∩ B ⊂ A. By AMA, C (A ∩ B) ⊂ C(A).

By CA, A ∩ B ⊂ B implies C (B) ∩ (A∩B) ⊂ C (A ∩ B).
Thus C(B) ∩ A ⊂ C(A ∩ B) ⊂ C (A).
Thus y ∉ C(B).
Thus C satisfies GA.

Conversely, let C satisfy GA. Then it obviously does satisfy AMA. To show that it satisfies CA, let A, B ∈ [X] with A ⊂ B.

Let x ∈ C(B) ∩ A. If x ∉ C(A), then since C(A) ⊂ B, by GA, x ∉ C(B) which is a contradiction. Thus, x ∈ C(A). Thus C(B) ∩ A ⊂ C(A). Thus C satisfies CA.

Q. E. D.

Example to show that AMA (↔ NQTA) does not necessarily imply GA: Let X = {x, y, z}, C(X) = {x, y}, C({x, y}) = {x}, C({y, z}) = {y}, C({x, z}) = {z}, C

$(\{a\}) = \{a\}\ \forall\ a \in X$. C satisfies AMA (and NQTA). However, $y \in \{x, y\} \setminus (\{x, y\})$, $C(X) \subset \{x, y\}$ and yet $y \in C(X)$. Thus C does not satisfy GA.

Example to show that CA does not necessarily imply GA: Let $X = \{x, y, z\}$, $C(X) = \{x\}$, $C(A) = A\ \forall\ A \in [X]$, $A \neq X$. C satisfies CA. However, $y \in X \setminus C(X)$, $C(X) \subset \{x, y\}$ and yet $y \in C(\{x, y\})$. Thus C does not satisfy GA.

Theorem 3: (a) CA & AMA implies OA; OA implies AMA;
(b) OA need not imply CA;
(c) AMA need not imply OA.

Proof: (a) is easy to establish.

(b) Let $X = \{x, y, z\}$, $C(X) = \{x, y\}$, $C(\{x,y\}) = \{x, y\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$. C satisfies OA. However, $\{x,z\} \subset X$, $x \in C(X) \cap \{x, z\}$ but $x \notin C(\{x,z\})$. Thus C does not satisfy CA.

(c) Let $X = \{x, y, z\}$, $C(X) = \{x, y\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$. C satisfies AMA. However, $C(X) = \{x, y\} \subset \{x, y\} \subset X$, but $C(\{x, y\}) \neq C(X)$. Thus C does not satisfy OA.

Q. E. D.

In view of Theorems 1, 2 and 3 above and Theorem 2 in Nehring [1997] we may now state the following:

Theorem 4: Let C be a choice function. Then the following are equivalent:
(a) P_c is an EPO
(b) C satisfies CA and NQTA
(c) C satisfies CA and AMA
(d) C satisfies CA and OA.

In Theorem 2 of Nehring [1997] it is also mentioned that P_c is an EPO if and only if C satisfies CA and NA. However, CA seems to play a more significant role in this result than in our Theorem 3 as the following result shows:

Theorem 5: AMA implies NA. However, the converse need not be true.

Proof:- Let C satisfy NA and let $A \in [X]$, $y \in X$ with $y \in A \setminus C(A)$. Thus $C(A) \subset C(A) \cup \{y\} \subset A$. By AMA, $C(C(A) \cup \{y\}) \subset C(A)$. Thus, $y \notin C(C(A) \cup \{y\})$. Thus C satisfies NA.

We show that the converse need not be true by means of an example. Let $X = \{x, y, z, w\}$. Let $C(X) = \{x\}$; $C(A) = A\ \forall\ A \in [X]$ with three elements; and for all $A \in [X]$ with one or two elements, $C(A) = \{x\}$ if $x \in A$ and $C(A) = A$ otherwise.

Now y, z $\in$ X \ C (X) and yet y $\in$ C (X \ {z}). Thus C does not satisfy NQTA, which has been shown in Theorem 1 above to be equivalent to AMA. Yet C satisfies NA.

Q. E. D.

However if the cardinality of X is three or less, then NQTA (: and hence AMA) is equivalent to NA. For cardinality of X equal to one or two, there is nothing to prove. If cardinality of X is three and C satisfies NA, then x, y $\in$ A \ C(A) for some A $\in$ [X] with x $\neq$ y, implies A = X. Thus if {x, y, z} = X, then C (X) = {z}. By NA, a $\notin$ C ({z, a}) where a $\in$ {x, y}. Thus b $\notin$ C (X \ {a}), where a, b $\in$ {x, y}. Thus C satisfies NQTA and hence AMA.

Example to show that NA does not necessarily imply GA:- Let X = {x, y, z}, C(X) = {x, y}, C({x, y}) = {x}, C({y, z}) = {y}, C({x, z}) = {z}, C ({a}) = {a} $\forall$ a $\in$ X. C satisfies NA. However, C ({x, y}) $\subset$ X, y $\in$ {x, y} \ C ({x, y}) and yet y $\in$ C(X). Thus C does not satisfy GA.

Appendix

Theorem:- Let C(A) = L (A, P) $\forall$ A $\in$ [X] where P is an EPO. Then $\forall$ A $\in$ [X], y $\in$ A \ C(A) implies (C(A), y) $\in$ P.

Proof:- We prove this theorem by induction on the cardinality of A \ C(A). Suppose A \ C (A) = {y}. Thus (A, y) $\in$ P and hence by Irreflexivity, (A \ {y}, y) $\in$ P. Thus (C(A), y) $\in$ P.

Suppose the theorem is true for A \ C(A) = {y_1 ,..., y_k} where k is any positive integer less than or equal to "r". Now suppose, A \ C(A) = {y_1 ,..., y_{r+1}}. Let x $\in$ C(A) and consider A \ {z} where z $\in$ {y_1 , ..., y_{r+1}}. Suppose towards a contradiction x $\notin$ C (A \ {z}). Thus (A \ {z}, x) $\in$ P. Thus by Monotonicity, (A, x) $\in$ P which contradicts x $\in$ C (A). Thus x $\in$ C (A \ {z}). Hence C(A) $\subset$ C (A \ {z}).

Suppose towards a contradiction w $\in$ C (A \ {z}) \ C(A).

$\therefore$ (A, w) $\in$ P. Thus ((A \ {z}) $\cup$ {z}, w) $\in$ P

Further (A, z) $\in$ P implies by Irreflexivity, (A \ {z}, z) $\in$ P. Since P satisfies transitivity, (A \ {z}, w) $\in$ P, contradicting w $\in$ C (A \ {z}).

$\therefore$ C(A \ {z}) = C(A)

Now $(A \setminus \{z\}) \setminus C(A \setminus \{z\})$ has cardinality "r". Hence by induction hypothesis, $(C(A \setminus \{z\}), y) \in P \forall y \in (A \setminus \{z\}) \setminus C(A \setminus \{z\})$

$\therefore (C(A), y) \in P \; \forall y \in (A \setminus \{z\}) \setminus C(A)$.

Choosing $z \neq y$, we establish the result for the case when the cardinality of $A \setminus C(A)$ is $r + 1$ having assumed it for the case when the cardinality of $A \setminus C(A)$ is r. The result has been shown to be true for the case when the cardinality of $A \setminus C(A)$ is one. Hence the result is true in general.

Q. E. D.

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