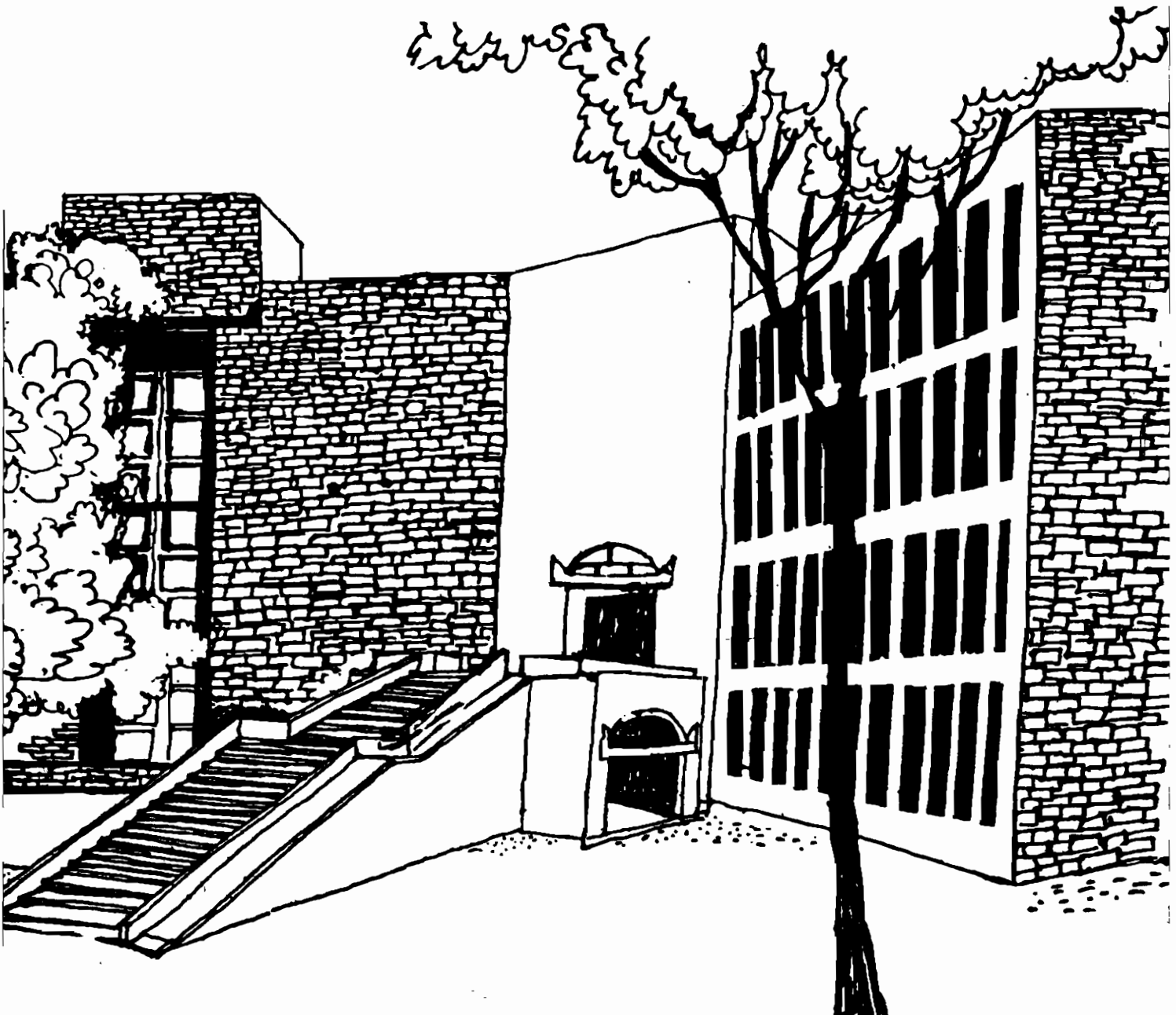




Working Paper

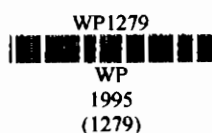


THE EQUAL LOSS CHOICE FUNCTION REVISITED

BY

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Abstract

Choice theory is a mathematical discipline which studies the problem of choosing a point from a set of points by studying the mathematical properties of maps assigning an outcome to each choice problem in some class of choice problems. A large literature has grown up concerning choice problems in Euclidean spaces. A typical choice problem is then a compact, convex, comprehensive subset of the non-negative orthant of a finite dimensional Euclidean space, containing a strictly positive vector.

For such choice problems, Yu (1973) and Freimer and Yu (1976) have introduced a class of solutions obtained by minimizing the distance of the "ideal point", measured by some norm. The equal loss solution is one such. However neither Yu (1973) nor Freimer and Yu (1976), succeeded in characterizing such solutions axiomatically. It was in Chun (1988) that we find a complete axiomatic characterization of the equal loss solution for the first time.

A brief glance at the proof of Chun's theorem, begs the questions, whether there is a simple alternative proof. The purpose of this paper is to provide such a proof, by modifying the technique suggested by Thomsom and Lensberg (1989) in their axiomatic characterization of the egalitarian solution.

In the later sections of the paper we consider choice problems with variable dimensions and obtain an axiomatic characterization of the equal-loss-choice function using a reduced choice problem property, first invoked in the relevant literature by Peters, Tijs and Zarzuelo (1994). We are thereby able to drop the assumption of Strong Monotonicity With Respect to the Ideal point, which is used in the original characterization.

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2. The Frame Work: Let $N = \{1, 2, \dots, n\}$ where $n \in \mathbb{N}$ be a set of indices. A choice problem in \mathbb{R}_+^N (the set of all functions from N to \mathbb{R}_+) is a nonempty set S in \mathbb{R}_+^N , satisfying the following conditions:

- i) S is compact and convex
- ii) S is comprehensive i.e. $x \in S, 0 \leq y \leq x \rightarrow y \in S$
- iii) there exists $x \in S$ with $x \gg 0$.

Let Σ^N be the class of all choice problems in \mathbb{R}_+^N . A choice function on Σ^N is a function $F: \Sigma^N \rightarrow \mathbb{R}_+^N$, such that $F(S) \in S \forall S \in \Sigma^N$.

The equal loss choice function on Σ^N is the function, $E^*: \Sigma^N \rightarrow \mathbb{R}_+^N$, defined as follows:

- (a) $E_i^*(S) - u_i(S) = E_j^*(S) - u_j(S) \forall i, j \in N$
- (b) $E_i^*(S) \in W(S) = \{x \in S / \nexists y \in S \text{ with } y \gg x\}$

Here $\forall i \in N$ and $S \in \Sigma^N$, $u_i(S) = \max \{x_i / x \in S\}$.

$u(S) = (u_1(S), \dots, u_n(S))$ is called the ideal point of S .

For all $S \in \Sigma^N$, $W(S)$ is called the Weakly Pareto Optimal set of S .

Let $F: \Sigma^N \rightarrow \mathbb{R}^N$, be a choice function. We say that

- (i) F satisfies Weak Pareto Optimality (WPO) if $\forall S \in \Sigma^N$, $F(S) \in W(S)$
- (ii) F satisfies Symmetry (SYM) if $\forall S \in \Sigma^N$ and if for all $p: N \rightarrow N$ which are 1-1, $p(S) = S$, then $F_i(S) = F_j(S) \forall i, j \in N$.

Here for $p: N \rightarrow N$ which is 1-1 and $x \in \mathbb{R}^N$, $p(x)$ is the vector in \mathbb{R}^N with $p(i)^{\text{th}}$ coordinate x_j ; $p(S) = \{p(x) / x \in S\}$.

(iii) F satisfies Translation Covariance (TC) if $\forall S \in \Sigma^N$ and all $x \in \mathbb{R}^N$, $T = \{y \in \mathbb{R}^N, / \exists z \in S \text{ with } y \leq x + z\}$ implies $F(T) = x + F(S)$.

(iv) F satisfies Strong Monotonicity With Respect to the Ideal Point (SMON) if $\forall S, T \in \Sigma^N$ with $u(S) = u(T)$ and $S \subset T$ implies $F(S) \leq F(T)$.

3. The Main Theorem

Theorem 1: The only choice function on Σ^N to satisfy WPO, SY, TC and SMON is E^* .

Proof: It is easy to check that E^* satisfies the desired properties. Hence let $F: \Sigma^N \rightarrow \mathbb{R}^N$, be a choice function satisfying the above properties and let $S \in \Sigma^N$.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, be such that
 $x_i = \max_{j \in N} (u_j(S)) - u_i(S)$.
 Let $T = \{y \in \mathbb{R}^n, \exists z \in S \text{ with } y \leq z + x\}$
 Thus $u(T) = u(S) + x$

Thus $E^s(T) = b e_N$ for some $b > 0$

Let $V =$ comprehensive convex hull $(E^s(T), (u_1(T), 0, \dots, 0), \dots, (0, 0, \dots, u_n(T)))$

By SYM and WPO, $F(V) = E^s(T)$

Now $u(V) = u(T)$ and $V \subset T$

Thus by (SMON), $F(T) \geq F(V) = E^s(T)$.

Case 1: $E^s(T) \in \{y \in T / x \in T, x \geq y \rightarrow x = y\} = P(T)$; then $F(T) = E^s(T)$

Case 2: $E^s(T) \notin P(T)$

Let $T_\epsilon =$ convex hull $(T \cup (\epsilon e_N + E^s(T)))$ for $\epsilon > 0$.

$E^s(T) \notin P(T) \rightarrow \exists y \in T$ such that $y \geq E^s(T)$, $y \neq E^s(T) = b e_N$

Let $y_i > b$ for some $i \in N$

Thus $u_i(T) > b$.

Since $u_j(T) = u_j(T) \forall j \in N$, $u_j(T) > b \forall j \in N$.

Hence for $\epsilon > 0$, ϵ small, $u(T_\epsilon) = u(T)$, $T \subset T_\epsilon$

But $E^s(T_\epsilon) = \epsilon e_N + E^s(T) \rightarrow$ (by case 1) $F(T_\epsilon) = \epsilon e_N + E^s(T)$

By SMON, $F(T) \leq F(T_\epsilon) = \epsilon e_N + E^s(T)$

Thus $E^s(T) \leq F(T) \leq \epsilon e_N + E^s(T) \forall \epsilon > 0$ sufficiently small. Letting ϵ go to zero, we get, $F(T) = E^s(T)$.

Note, $E^s(T) = E^s(S) + x$ and $F(T) = F(S) + x$, both by (TC).

Thus $F(S) = E^1(S)$.

Note: In the above e_N is the vector in \mathbb{R}^N , with all coordinates equal to 1.

4. **Variable Dimension Choice Problems:** This section is inspired by Peters, Tijs and Zarzuelo (1994).

Let $I \subset \mathbb{N}$ (the set of natural numbers) denote the set of potential dimensions, and G the class of all finite subsets of I . For $N \in G$, Σ^N is class of all choice problems in \mathbb{R}^N .

Let $X = \bigcup_{N \in G} \mathbb{R}^N$, and $\Sigma = \bigcup_{N \in G} \Sigma^N$

A choice function on Σ is a function $F : \Sigma \rightarrow X$ such that $F(S) \in S \forall S \in \Sigma$.

As before the equal loss choice function $E^1 : \Sigma \rightarrow X$ is defined by:

$$(i) \quad E^1_i(S) - u_i(S) = E^1_j - u_j(S) \quad \forall i, j \in N \in G$$

$$(ii) \quad E^1(S) \in W(S)$$

whenever $S \in \Sigma$.

Let L, M be non-empty elements of G , and let $S \in \Sigma^M$. If $L \subset M$, then $S_L = \{y \in \mathbb{R}^L : \text{there is } x \in S \text{ with } y = x_L\}$. Here given $x \in \mathbb{R}^M$, $x_L = (x_i)_{i \in L}$

Let $x \in S$, $x \geq 0$ with $x_L \neq 0$. Let $\lambda(S, x_L) = \min \{\lambda \in \mathbb{R}_+ : x_L \in \lambda S_L\}$.

The reduced choice problem of S with respect to L and x is the following choice problem for L :

$$S^x_L = \lambda(S_L, x_L) S_L$$

Because, $x_L \neq 0$, $\lambda(S_L, x_L) > 0$ and therefore $S_L^x \in \Sigma^L$. Note, $x_L \in W(S_L^x)$

Reduced Choice Problem Property (RCPP): For all nonempty subsets $L, M \in G$ and all $S \in \Sigma^M$ if $L \subset M$, $F(S) \succeq 0$ and $F_L(S) \neq 0$, then $F(S_{F(S)}^L) = F_L(S)$.

Let $F: \Sigma \rightarrow X$ be a choice function. We say that

- (1) F satisfies **Weak Pareto Optimality (WPO)** if $F(S) \in W(S) \forall S \in \Sigma$.
- (2) F satisfies **Anonymity (AN)** if $\forall L, M \in G$, and $p: L \rightarrow M$ one-to-one, if $S \in \Sigma^L$ and $T \in \Sigma^M$ with $T = p(S)$, then $F(T) = p(F(S))$. Here for $x \in \mathbb{R}^L$, $p(x) \in \mathbb{R}^M$, with $p(x) = (p(x)_{p(i)})_{i \in L}$ and $p(x)_{p(i)} = x_i \forall i \in L$; $x(S) = \{p(x) / x \in S\}$.

F satisfies **Translation Covariance (TC)** if $\forall S \in \Sigma^M, M \in G$ and all $x \in \mathbb{R}^M, T = \{y \in \mathbb{R}^M / \exists z \in S \text{ with } y \leq x + z\}$ implies $F(T) = x + F(S)$.

- (4) F satisfies **Homogeneity (HOM)** if $\forall S \in \Sigma^M, M \in G$ and $a > 0$, $F(aS) = a F(S)$.

(Here, for $x \in \mathbb{R}^M, ax \in \mathbb{R}^M$, with $(ax)_i = ax_i \forall i \in M$ and $aS = \{ax / x \in S\}$).

Theorem 2: Let $|I| \geq 3$ ($|I|$ means cardinality of I). A choice function $F: \Sigma \rightarrow X$ satisfies WPO, AN, TC, HOM, and RCPP if and only if $F = E^I$.

Proof: That E^k satisfies RCPP is clear. Let now F be a solution satisfying the four axioms. We first prove that if $|M| = 2$ and $S \in \Sigma^H$, then $F(S) = E^k(S)$.

Let $M = \{i, j\}$ and $S \in \Sigma^H$. By TC and the method in the proof of Theorem 1, we may assume $u_i(S) = u_j(S) = a > 0$.

Let $k \in I \setminus M$ and

$T_{(i,k)} = \text{convex hull of } \{S, (a e^k)\}$, where $a e^k \in \mathbb{R}^{(i,j,k)}$ with $e^{k_l} = 0$, if $l \neq k$, $e^{k_l} = 1$ if $l = k$.

Now by AN, $F(T_{(i,k)}) = (a/2, a/2)$ and $F(T_{(j,k)}) = (a/2, a/2)$

Now, by (RCPP) $F(T_{(i,k)}^{F(T)}) = F_{(i,k)}(T)$ and $F(T_{(j,k)}^{F(T)}) = F_{(j,k)}(T)$

$T_{(i,k)}^{F(T)} = \lambda (T_{(i,k)}, F_{(i,k)}(T)) T_{(i,k)}$ and $T_{(j,k)}^{F(T)} = \lambda (T_{(j,k)}, F_{(j,k)}(T)) T_{(j,k)}$

By HOM, $\lambda (T_{(i,k)}, F_{(i,k)}(T)) (a/2, a/2) = F_{(i,k)}(T)$

and $\lambda (T_{(j,k)}, F_{(j,k)}(T)) (a/2, a/2) = F_{(j,k)}(T)$

Thus $\lambda (T_{(i,k)}, F_{(i,k)}(T)) = \lambda (T_{(j,k)}, F_{(j,k)}(T)) = \bar{\lambda} > 0$

i. $F_i(T) = F_j(T)$

But $\lambda (T_{(i,j)}, F_{(i,j)}(T)) F(T_{(i,j)}) = F_{(i,j)}(T)$ by RCPP and HOM and $S = T_{(i,j)}$

Thus $F_i(S) = F_j(S)$

By WPO, $F(S) = E^k(S)$

Suppose now $|M| > 2$ and $S \in \Sigma^H$ with (without loss of generality by TC) $u_i(S) = a > 0 \forall i \in M$. Let $i, j \in M$

Then $F_i(S_{(i,j)}) = F_j(S_{(i,j)})$ by the above.

But $F_{(i,j)}(S) = \lambda (S_{(i,j)}, F_{(i,j)}(S)) F(S_{(i,j)})$ by RCPP and HOM. Thus

$F_i(S) = F_j(S)$

By WPO, $F(S) = E^k(S)$.

The following lemma has been established in Peters, Tijs and Zarzuelo (1994).

Lemma 1: Let F be a choice function on Σ satisfying RCPP, HOM and SIR where

F satisfies SIR (Strong Individual Rationality) if $\forall S \in \Sigma, F(S) \gg 0$.

Let $M \in G, M \neq 1$ and let $S \in \Sigma^M$. Then $F(S) \in W(S)$.

Using Lemma 1 and Theorem 2, we have the following Corollary.

Corollary 1: Let I be infinite. A solution F on Σ satisfies SIR, AN, TC, HOM and RCPP if and only if $F = E^1$.

Remark:

1. RCPP along with other axioms has been used in Lahiri (1995) to characterize uniquely the egalitarian choice function.
2. Our theorem 2 and Corollary 1, resembles the relevant characterization theorems for the Kalai-Smorodinsky choice function available in Peters, Tijs and Zarzuelo (1994). It is worth noting that the Kalai-Smorodinsky Choice function does not satisfy TC and the equal loss choice function does not satisfy Scale Invariance used in the other paper.
3. The difference between the result reported here and that reported in Lahiri (1995) is that the egalitarian choice function does not satisfy TC and the equal loss choice function does not satisfy the Nash's Independence of Irrelevant Alternatives assumption used there.

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