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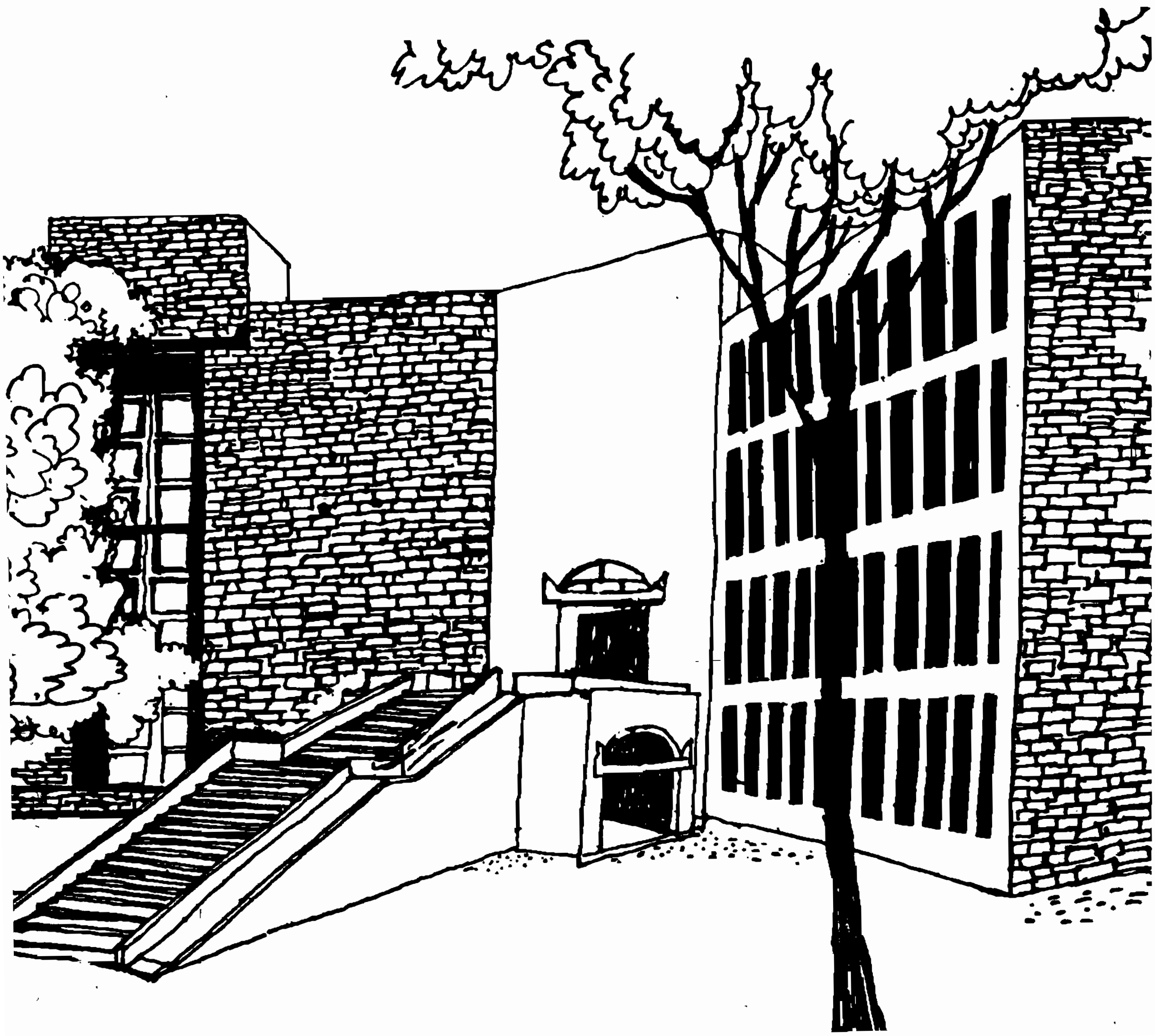

A COMMENT ON THE INDEPENDENCE OF IRRELEVANT EXPANSION ASSUMPTION FOR THE NON-SYMMETRIC NASH CHOICE FUNCTION

By

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W P No.1282
October 1995

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ABSTRACT

In this paper we establish three distinct results:

a) a unique characterization of the family of non-symmetric Nash choice functions using an Independence of Irrelevant Expansions Assumption;

b) a characterization of the entire family of choice functions (containing those determined by a weighted hierarchy and those which are 'mongrel' choice functions only) satisfying Pareto Optimality, Scale Translation Covariance and Weak Independence of Irrelevant Expansions;

c) as a by-product of (b) we obtain a characterization of the family of choice functions determined by a weighted hierarchy by slightly strengthening the Weak Independence of Irrelevant Expansions assumption.

1. INTRODUCTION:

The family of non-symmetric Nash choice functions, which was proposed for the first time in the seminal work of Harsanyi and Selten (1972), has been axiomatically characterized in almost the same way that Nash himself characterized its symmetric ancestor in his by now historic 1950 paper. A more recent and thorough investigation of the family of choice functions characterized by a weighted hierarchy (and containing the family of non-symmetric Nash choice functions) is the work of Peters (1986b). There an additional axiom called the consistency axiom is used, which however is not required for two dimensional choice problems. All the above mentioned characterizations of the non-symmetric family under discussion, rely heavily on an assumption which has often been questioned from various quarters: Nash's Independence of Irrelevant Alternatives Assumption (NIIA).

There has been several attempts to free the characterization of the Nash choice function from the grip of NIIA. Of interest in the present paper is a characterization for two dimensional choice problems presented in Thomson (1981), where instead of NIIA an assumption called Independence of Irrelevant Expansions (IEE) has been used. Interest in choice theory had since then shifted largely to the multidimensional cases and even more to choice problems with varying dimensions. A recent revival of interest in the two dimensional case (and solely that) is seen in the paper by Bossert (1994), where once again NIIA is used to characterize rational choice functions. Our theorem A in the present paper is an easy and valid extension of Thomson's original result to the non-symmetric cases.

In Peters (1986a) can be found a characterization of a family of choice functions determined by a weighted hierarchy for two dimensional choice problems using a slightly weakened version of Thomson's Independence of Irrelevant Expansions assumption. However, the domain chosen for the result deviates considerably from the conventional domain used by Thomson (1981) or Bossert (1994), in that it assumes that every choice problem admits infinite free disposability. Now, this is an assumption whose worth or meaningfulness depends on the context. If we assume that each choice problem represents a multisectoral investment planning problem for instance (i.e., dividing a dollar between several sectors, the returns being measured by concave, non-decreasing, non-constant and continuous revenue functions), then the kind of domain assumed in Peters (1986a) for the present purpose is not quite meaningful. That the set of investment planning problems is isomorphic to the domain of choice problems assumed in this paper, is however a result established in Lahiri (1994). So, the natural question that crops up is whether the result established by Peters is valid when the domain (as in the present paper) consists of non-empty, compact, convex, comprehensive subsets of two dimensional Euclidean spaces, each such set admitting a strictly positive vector. A cursory look at the proof of the result in Peters (1986a), shows that it is very dependent on his choice of domain. In fact, a couple of lemmas simply do not have any meaning in our framework. What is more noteworthy, is our Theorem B: the original result simply does not continue to hold. In addition to choice functions determined by weighted hierarchies, there is a family of 'mongrel' solutions determined by two different weighted hierarchies which now continue to satisfy the assumptions suggested by Peters. However that is all--there are no more solutions; and even the family of 'mongrel' solutions are restricted in a particular way.

These two theorems A and B comprise the main contribution of this paper. As a byproduct of theorem B, we obtain a theorem C uniquely characterizing the family of choice functions determined by a weighted hierarchy by slightly strengthening the Weak Independence of Irrelevant Expansions Assumption, which is used in theorem B.

2. THE MODEL

A (two dimensional) choice problem is a non-empty set S in R^2_+ (the non-negative orthant of two dimensional Enclidean space) satisfying the following properties:

i) S is compact, convex

ii) S is comprehensive i.e. $0 \leq y \leq x \in S \rightarrow y \in S$

iii) $\exists\, x \in S$ with $x = (x_1, x_2)$ and $x_i > 0$ for $i = 1, 2$.

Let Σ^2 denote the class of all two dimensional choice problems. A choice function (on Σ^2) is a function $F: \Sigma^2 \to R^2_+$ such that $F(S) \in S \;\forall\, S \in \Sigma^2$

The following Assumption on $F: \Sigma^2 \to R^2_+$ will be found useful in the sequel:

<u>Assumption 1:</u> F satisfies <u>Pareto Optimality</u> i.e. $\forall\, S \in \Sigma^2$, $y \in S$, $y \geq F(S) \to y = F(S)$

(For future reference, given $S \in \Sigma^2$, let $P(S) = \{x \in S / y \geq x,\ y \in S \to y = x\}$; $P(S)$ is called the Pareto Optimal set of S)

<u>Assumption 2:</u> F satisfies <u>Scale Transformation</u> <u>Covariance</u>: $\forall\, S \in \Sigma^2\ \forall\, a = (a_1, a_2) >> 0$, $F(aS) = a F(S)$, where for $x = (x_1, x_2) \in R^2_+$ $ax = (a_1 x_1, a_2 x_2)$ and $aS = \{ax: x \in S\}$ if $S \subseteq R^2_+$

<u>Assumption 3:</u> F satisfies <u>Independence</u> <u>of</u> <u>Irrelevant</u> <u>Expansions</u>: $\forall\, S \in \Sigma^2$ there exists a vector $p \in R^2_+$ with $p_1 + p_2 = 1$ such that (a) $p.x = p.F(S)$ is the equation of a supporting line of S at F(S), (b) $\forall\, T \in \Sigma^2$ with $S \subset T$ and $p.x \leq p.F(S)\ \forall\, x \in T$, we have $F(T) = F(S)$.

We are interested in a choice function defined thus:

<u>Definition:</u> A weight hierarchy H of {1,2} is an ordered (k + 1) - tuple of the form

$$H = \langle N^1, \ldots, N^k; w\rangle$$

where $(N^1, \ldots, N^k)$ is a partition of {1,2} and $w \in R^2_{++}$ with $\Sigma_{i \in N^l} w_i = 1$ for every $l = i, \ldots, k$. The set N^l is called the l^{th} class of H. By $H^{\{1,2\}}$ the family of all weighted hierarchies of {1,2} is denoted.

With each weighted hierarchy $H \in H^{\{1,2\}}$ a choice function F^H for Σ^2 will be associated thus;

If $H = \langle\{1\}, \{2\}; (1,1)\rangle$, then

$F^H(S) = (u_1(S), a_2(S)) \; \forall \; S \in \Sigma^2$

where $u_i(S) = \max\{x_i / (x_i, x_j) \in S, i \neq j\}$, $i=1,2$

and $a_j(S) = \max\{x_j / (x_j, u_i(S)) \in S, i \neq j\}$, $j=1,2$

If $H = \langle\{2\}, \{1\}; (1,1)\rangle$, then

$F^H(S) = (a_1(S), u_2(S))$

If $H = \langle\{1,2\}; w\rangle$ where $w_1 > 0$, $w_2 > 0$, $w_1 + w_2 = 1$,

then $F^H(S) = \text{argmax}\, \{x \in S / x_1^{w_1} \times x_2^{w_2} \geq y_1^{w_1} \, y_2^{w_2} \; \forall \; y = (y_1, y_2) \in S\}$

<u>Example:</u> Let $H = \langle\{1\}, \{2\}, (1,1)\rangle$.

Thus $F^H(S) = (u_1(S), a_2(S)) \; \forall \; S \in \Sigma^2$. But this F^H does not satisfy Independence of Irrelevant Expansions.

Take $S = \{x = (x_1, x_2) \in R^2 / x_1^2 + x_2^2 \leq 1\}$.

Clearly $F^H(S) = (1,0)$. At (1,0), the unique supporting hyperplane in the definition of Assumption 3 is given by

$p = (1,0)$. Now take $T = \{(x_1, x_2) \in R^2_+ / x_1 \leq 1, x_2 \leq 1\}$. Now T and S satisfy the conditions in Assumption 3, with $p = (1,0)$. But $F^H(T) = (1,1) \neq F^H(S)$.

This example excludes the weighted hierarchy $H = \langle\{1\}, \{2\}, (1,1)\rangle$ as well as the weighted hierarchy $H' = \langle\{2\}, \{1\}, (1,1)\rangle$ from the list of the possible candidates which could define a solution satisfying Assumption 3.

Hence the only possibilities are weighted hierarchies of the form $H = \langle\{1,2\}, w\rangle$. Let an F^H for such an H be called a non-symmetric Nash choice function.

3. A PRELIMINARY THEOREM

For the purpose of Section 3, the following convention is adopted. Let $F: \Sigma^2 \to R^2_+$ be a choice function satisfying Assumption 3. Given $S \in \Sigma^2$, let $p(F,S) = \{p \in R^2_+ \setminus \{0\} / p$ satisfies the conditions of Assumption 3 for S$\}$.

<u>Theorem A:</u> Let F be a choice function which satisfies Assumptions 1, 2 and 3. Then F is a non-symmetric Nash choice function. Conversely, every non-symmetric Nash choice function satisfies Assumption 1, 2 and 3.

<u>Proof:</u> It is easy and somewhat routine to verify that every non-symmetric Nash bargaining solution satisfies Assumptions 1, 2 and

3. Hence let us prove the converse. Hence assume that F is a choice function satisfying the desired assumptions.

Step 1: Let $\Delta = \{(x_1, x_2) \in R^2_+ / x_1 + x_2 \leq 1\}$. Then $F_i(\Delta) > 0$ for i = 1, 2.

Proof of Step 1: Let $S = \{ (x_1, x_2) \in R^2_+ / x^2_1 + x^2_2 \leq 1\}$. The example above show that (1,0) and (0,1) do not qualify as solutions for S which would satisfy assumption 3. Hence by Assumption 1 (Pareto optimality), there exists $(y_1, y_2) \gg 0$, with $y^2_1 + y^2_2 = 1$ such that $F(S) = (y_1, y_2)$.

Let $p.x = p.F(S)$ be the unique supporting line to S at F(S). Clearly $p_1 > 0$, $p_2 > 0$.

Let $T = \{(x_1, x_2) \in R^2_+ / p.x \leq p.F(S)\}$.
By Assumption 3, $F(T) = F(S) = y = (y_1, y_2) \gg 0$. By Assumption 2 (Scale Transformation Covariance), $F(\Delta) = w \gg 0$.

Step 2: Let $S \in \Sigma^2$, such that $(u_1(S), u_2(S)) \in S$. Then $F(S) = F^N(S) = (u_1(S), u_2(S))$

Proof: Obvious by Assumption 1.

Step 3: Let $S \in \Sigma^2$ and $(u_1(S), u_2(S)) \notin S$.
Then $(1,0), (0,1) \notin p(F,S)$

Proof: Suppose towards a contradiction $(1,0) \in p(F,S)$ (the proof for $(0,1)$ is similar). Clearly $F(S) = (u_1(S), a_2(S))$ with $a_2(S) < u_2(S)$. Let $T = \{(x_1, x_2) \in R^2_+ / x_i \leq u_i(S), i = 1, 2\}$. S and T satisfy the conditions of Assumption 3 with $p = (1,0)$. But $F(T) = (u_1(S), u_2(S))$ by Assumption 1; this contradicts assumption 3 and proves Step 3.

Step 4: Let $S \in \Sigma^2$. If $S = \{(x_1, x_2) \in R^2_+ / x_i \leq u_i(S), i = 1,2\}$, then Step 2 establishes that F(S) is a non-symmetric Nash Solution for S.

Hence assume $(u_1(S), u_2(S)) \notin S$. By Step 3 and Assumption 1, $(1,0)$, $(0,1) \notin p(F,S)$ irrespective of whether $F_1(S) = u_1(S)$ or $F_2(S) = u_2(S)$ or otherwise.

Let $p \in p(F,S)$ whereby Assumption 2 we may assume $F_i(S) = 1$, $i = 1, 2$.

Let $T = \{(x_1, x_2) \in R^2_+ / p_1 x_1 + p_2 x_2 \leq 1\}$.

Since $F(\Delta) = w$ by Assumption 2, $F(T) = F^H(T)$

where $H = \langle\{1,2\}, w\rangle$

But this T and S satisfy the condition of Assumption 3. Hence $F(S) = F(T) = F^H(T) = F^H(S)$.

Thus $F(S) = F^H(S)$.

Q.E.D.

Our next objective is to invoke the assumption of weak independence of irrelevant expansions defined in Peters (1986a) and establish a result similar to his.

Assumption 4: F satisfies Weak Independence of Irrelevant Expansions: $\forall$ $S \in \Sigma^2$ there exists a vector $p \in R^2_+$ with $p_1 + p_2 = 1$ such that:

a) $p.x = p.F(S)$ is the equation of a supporting line of S at $F(S)$;

b) $\forall$ $T \in \Sigma^2$ with $S \subset T$ and $p.x \leq p.F(S)$ $\forall$ $x \in T$, we have $F(S) \leq F(T)$.

Notice that Assumption 3 implies Assumption 4. Hence the non-symmetric Nash choice functions satisfy Assumption 4 as well.

4. THE MAIN THEOREM

For the purpose of Section 4, the following convention is adopted: Let $F: \Sigma^2 \to R^2_+$ be a choice function satisfying Assumption 4. Given $S \in \Sigma^2$, let $p(F,S) = \{p \in R^2_+ \setminus \{0\} / p$ satisfies the conditions of Assumption 4 for $S\}$.

Lemma 1: Let $V \in \Sigma^2$ with $u(V) \notin V$. Suppose $(0,1) \in p(F,V)$. If F satisfies Assumptions 1, 2 and 4, then $F(S) = (a_1(S), u_2(S))$ $\forall$ $S \in \Sigma^2$ such that $S \neq a\Delta$ for $a \in R^2_{++}$

Proof: $(0,1) \in p(F,V) \to F(V) = (a_1(V), u_2(V))$

Since $u(V) \notin V$, $\forall (p_1, p_2) \in R^2_{++}$, $\exists$ $u, v \in R^2_+$ with $u_2 = u_2(V)$, $v_1 = u_1(V)$, $a_1(V) \leq u_1 < v_1$, $a_2(V) \leq v_2 < u_2$ such that $V \subset$ comprehensive convex hull of $\{u,v\}$ and $\dfrac{u_2 - v_2}{u_1 - v_1} = -\dfrac{p_1}{p_2}$.

Let $T(u,v)$ denote comprehensive convex hull of $\{u,v\}$. By Assumption 1 and 4, $F(T(u,v)) = (a_1(T(u,v)), u_2(T(u,v))$ for u, v as defined above. By Assumption 2, $F(T(u,v)) = (a_1(T(u,v)), u_2(T(u,v))) \; \forall \; u, v \in R^2_{++}$, with $u_1 < v_1$ and $v_2 < u_2$. Now let $S \in \Sigma^2$. If $u(S) \in S$, then there is nothing to prove. Thus assume $u(S) \notin S$ and $F(S) \neq (a_1(S), u_2(S))$.

Case 1: $(1,0) \in p(F,S)$. Hence $F(S) = (u_1(S), a_2(S))$. Since $u(S) \notin S$, $\exists$ $u, v \in R^2_{++}$ such that $u_1 < v_1, v_2 < u_2$, $v_1 = u_1(S)$, $u_2 = u_2(S)$, such that $S \subset T(u,v)$. By Assumption 1 and 4 and since $(1,0) \in p(F,S)$, $F(T(u,v)) = (u_1(T(u,v)), a_2(T(u,v))) \neq (a_1(T(u,v)), u_2(T(u,v)))$ which is a contradiction. Hence Case 1 is not possible.

Case 2: $p = (p_1, p_2) \in p(F,S)$ with $p_1 + p_2 = 1$, $p_1 > 0$, $p_2 > 0$ and $F(S) = (u_1(S), a_2(S))$.

From what has been said before Case 1, it follows that $(0,1) \in p(F,T(u,v)) \; \forall \; u, v \in R^2_{++}$ with $u_1 < v_1$, $v_2 < u_2$. By applying Assumption 1 and 4, it thus follows that

$F(T(u,v)) = u \ \forall\ u \in R^2_{++}$, $V \in R^2_{+}$ with $u_1 < v_1$, $v_2 < u_2$.

Now let $V = F(S)$, $u_2 = u_2(S)$, $\dfrac{v_2 - u_2}{v_1 - u_2} = -\dfrac{p_1}{p_2}$

Since $S \subset T(u,v)$ and $p = (p_1, p_2)$ satisfies the conditions of Assumptions 4 for S and T (u,v), $F(T(u,v)) = F(S) = V \neq u$, which contradicts that $F(T(u,v)) = u$. Thus Case 2 is also not possible.

<u>Case 3:</u> $p = (p_1, p_2) \in p(F;S)$ with $p_1 + p_2 = 1$, $p_1 > 0$, $p_2 > 0$, $F(S) \neq (a_1(S), u_2(S))$, $F(S) \neq (u_1(S), a_2(S))$. Since $S \neq a\Delta$ with $a \in R^2_{++}$, $\exists$ $u, v \in R^2_{++}$ with $u_1 < v_1$, $v_2 < u_2$ such that

a) $S \subset T(u,v)$

b) $F(S)$ is a Pareto optimal point of $T(u,v)$, $F(S) \neq u, v$.

c) $\dfrac{v_2 - u_2}{v_1 - u_1} = -\dfrac{p_1}{p_2}$

By Assumptions 1 and 4, $F(T(u,v)) = F(S) \neq u$ which is a contradiction.

Hence, $F(S) = (a_1(S), u_2(S))$.

In the course of the above prof we have established the following Lemma:

<u>Lemma 2:</u> Let $V \in \Sigma^2$ with $u(V) \notin V$. Suppose $(1,0) \in p(F,V)$. If F satisfies Assumptions 1, 2 and 4, then:

a) $F(S) \neq (u_1(S), a_2(S))$ whenever $S \in \Sigma^2$

b) $F(S) = (a_1(S), u_2(S))$ whenever $S \in \Sigma^2 \setminus \{a\Delta \,/\, a \in R^2_{++}\}$

Proof:

a) follows from Cases 1 and 2 above

b) is Lemma 1.

By an identical reasoning, we can establish the following Lemma:

Lemma 3: Let $V \in \Sigma^2$ with $u(V) \notin V$. Suppose $(1,0) \in p(F,V)$. If F satisfies Assumptions 1, 2 and 4, then:

a) $F(S) \neq (a_1(S), u_2(S))$ whenever $S \in \Sigma^2$

b) $F(S) = (u_1(S), a_2(S))$ whenever $S \in \Sigma^2 \setminus \{a\Delta \,/\, a \in R^2_{++}\}$

We now have the following crucial Lemma:

Lemma 4: Suppose $(1,0), (0,1) \notin p(F,V)$ whenever $V \in \Sigma^2$. If F satisfies Assumptions 1, 2 and 4, then F is a non-symmetric Nash bargaining choice function.

Proof: Let $F(\Delta) = w \gg 0$ since $(1,0), (0,1) \notin p(F,V)$

Thus $F(a\Delta) = F^H(a\Delta) \;\forall\; a \in R^2_{++}$ where $H = \langle N, \dfrac{w_1}{w_1 + w_2}, \dfrac{w_2}{w_1 + w_2} \rangle$

Now let $S \in \Sigma^2$. Then $\exists\; p \in p(F,S),\; p \gg 0$

Let $T = \{(x_1, x_2) \in R^2 / p_1 x_1 + p_2 x_2 \leq p_1 F_1(S) + p_2 F_2(S)\}$

Clearly $F(T) = F^H(T)$ and $F(T) = F(S)$ the latter by Assumptions 1 and 4. Thus $F(S) = F^H_*(T)$. Since $F^H(T) = F^H(S)$, we have the desired result.

Q.E.D.

Note: By Assumption 2, if $F(\Delta) = F^H(\Delta)$ for some $H \in H^{\{1,2\}}$, then $F(a\Delta) = F^H(a\Delta)$ $\forall$ $a \in R^2_{++}$ and for the same H. Since $F(\Delta)$ is always equal to some $F^H(\Delta)$ with $H \in H^{\{1,2\}}$, $F(a\Delta)$ is always equal to $F^H(a\Delta)$ $\forall$ $a \in R^2_{++}$ for some fixed $H \in H^{\{1,2\}}$.

As a consequence of the above lemmas we have the following theorem.

Theorem B: Let F be a choice function on Σ^2 which satisfies Assumptions 1, 2 and 4. Then there are two weighted hierarchies H and H′ not necessarily distinct such that:

a) $F(a\Delta) = F^H(a\Delta)$ $\forall$ $a \in R^2_{++}$

b) $F(S) = F^{H'}(S)$ $\forall$ $S \in \Sigma^2 \setminus \{a\Delta / a \in R^2_{++}\}$

c) If H, H′ are both different from $\langle N, w \rangle$ whatever w, then H=H′

d) If H is either $\langle \{1\}, \{2\}, (1,1) \rangle$ or $\langle \{2\}, \{1\}, (1,1) \rangle$ then H = H′.

Conversely, any choice function F which satisfies (a), (b) and (c) also satisfies 1, 2 and 4.

Proof: The converse is easy and (a), and (b) have already been established. To prove (d), take $S=\{(x_1,x_2)\in R^2_+ / x^2_1 + x^2_2 \leq 1\}$ and assuming (c), let $H' = \langle N,w\rangle$ and H to be either $\langle\{1\}, \{2\}, (1,1)\rangle$ or $\langle\{2\}, \{1\}, (1,1)\rangle$ and obtain a contradiction if F satisfies Assumptions 1, 2 and 4 and is of the form determined by (a) and (b).

To prove (c), let $H = \langle\{1\},\{2\}; (1,1)\rangle$ and $H'=\langle\{2\},\{1\}; (1,1)\rangle$.
Thus $F(\Delta) = F^H(\Delta) = (1,0)$

Let $S = \{(x_1, x_2) \in \ / x_1 \leq ½\}$

$S\in\Sigma^2 \setminus \{a\Delta / a\in R^2_{++}\}$.

Thus $F(S) = F^{H'}(S) = (½, ½)$

But Assumptions 1 and 4 applied to and S imply $F(S) = F(\Delta)$ which is a contradiction. A similar contradiction arises if H and H′ are interchanged.

Q.E.D.

Note: 1) If $F(S) = F^H(S) \ \forall\ S\in\{a\Delta / a\in R^2_{++}\}$

$= F^{H'}(S) \ \forall\ S\in\Sigma^2 \setminus \{a\Delta / a\in R^2_{++}\}$

where $H = \langle N,w\rangle$ for some $w\in R^2_{++}$ and H′ is either $\langle\{1\}, \{2\}, (1,1)\rangle$ or $\langle\{2\}, \{1\}, (1,1)\rangle$ then we call F a "mongrel" choice function. It is easy to see, as asserted in Theorem B, that the entire family of "mongrel" choice functions satisfy Assumptions 1, 2 and 4. This leaves the characterization theorem B, somewhat open ended.

2) This open endedness of the characterization theorem B, may seem somewhat disconcerting. A simple way to remedy this is to assume that the choice function $F: \Sigma^2 \to R^2_+$ satisfies an assumption which may be called Modified Weak Independence of Irrelevant Expansion:

<u>Assumption 5:</u> F satisfies Modified Weak Independence of Irrelevant Expansion if F satisfies Assumption 4 and in addition the following: given S, $T \in \Sigma^2$, $S \subset T$, $F(T) \in S$ implies $p(F,T) \subset p(F,S)$.

The following theorem can now be established using the intermediate value theorem:

<u>Theorem C:</u> Let F be a choice function satisfying Assumptions 1, 2 and 5. Then there exists $H \in H^{\{1,2\}}$ such that $F = F^H$. Conversely, every F^H, $H \in H^{\{1,2\}}$ satisfies Assumptions 1, 2 and 5.

<u>Proof:</u> The proof consists in showing that:

$$F(S) = F^H(S) \ \forall \ S \in \{a\Delta / a \in R^2_{++}\} \text{ where } H = \langle N, w \rangle$$

$$\text{and } F(S) = F^{H'}(S) \ \forall \ S \in \Sigma^2 \setminus \{a\Delta / a \in R^2_{++}\} \text{ where}$$

H' is either <{1}, {2}, (1,1)> or <{2}, {1}, (1,1)> is untenable.

Clearly $F(\Delta) = w$ and by Assumption 2, $F(a\Delta) = aw$ $\forall \ a \in R^2_{++}$

Let $S = \{x = (x_1, x_2) \in R^2 / x_1^2 + x_2^2 \leq 1\}$

$\forall$ $x_1 \in (0,1)$, let $a(x_1) \in R^2_{++}$ be such that the Pareto-optimal points of $a(x_1)\Delta$, is contained in the straightline supporting S at

$(x_1, \sqrt{1-x_1^2})$. Define f: $(0,1) \to R$ as follows:

$$f(x_1) = x_1 - F_1(a(x_1)\Delta)$$

It is easy to show that f is continuous on (0,1) and

a) $f(x_1) \leq 0$ for values of x_1 close to 0

b) $f(x_1) \geq 0$ for value of x_1 close to 1.

Hence by the intermediate value theorem, there exists $\bar{x}_1 \in (0,1)$, such that $\bar{x}_1 = a(\bar{x}_1)$, i.e., $F_1(a(\bar{x}_1)\Delta) = \bar{x}_1$. Thus $F_2(a(\bar{x}_1)\Delta) = \sqrt{1-(\bar{x}_1^2)} = \bar{x}_2$

Let $T = a(\bar{x}_1)\Delta$

By Assumption 5, and the fact that p(F,S) is a singleton for the chosen S, (0,1) (1,0) $\notin$ p(F,S). Hence $F(S) \neq F^{H'}(S)$ if H' is either <{1}, {2}, (1,1)> or <{2}, {1}, (1,1)> which completes the proof.

Q.E.D.

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