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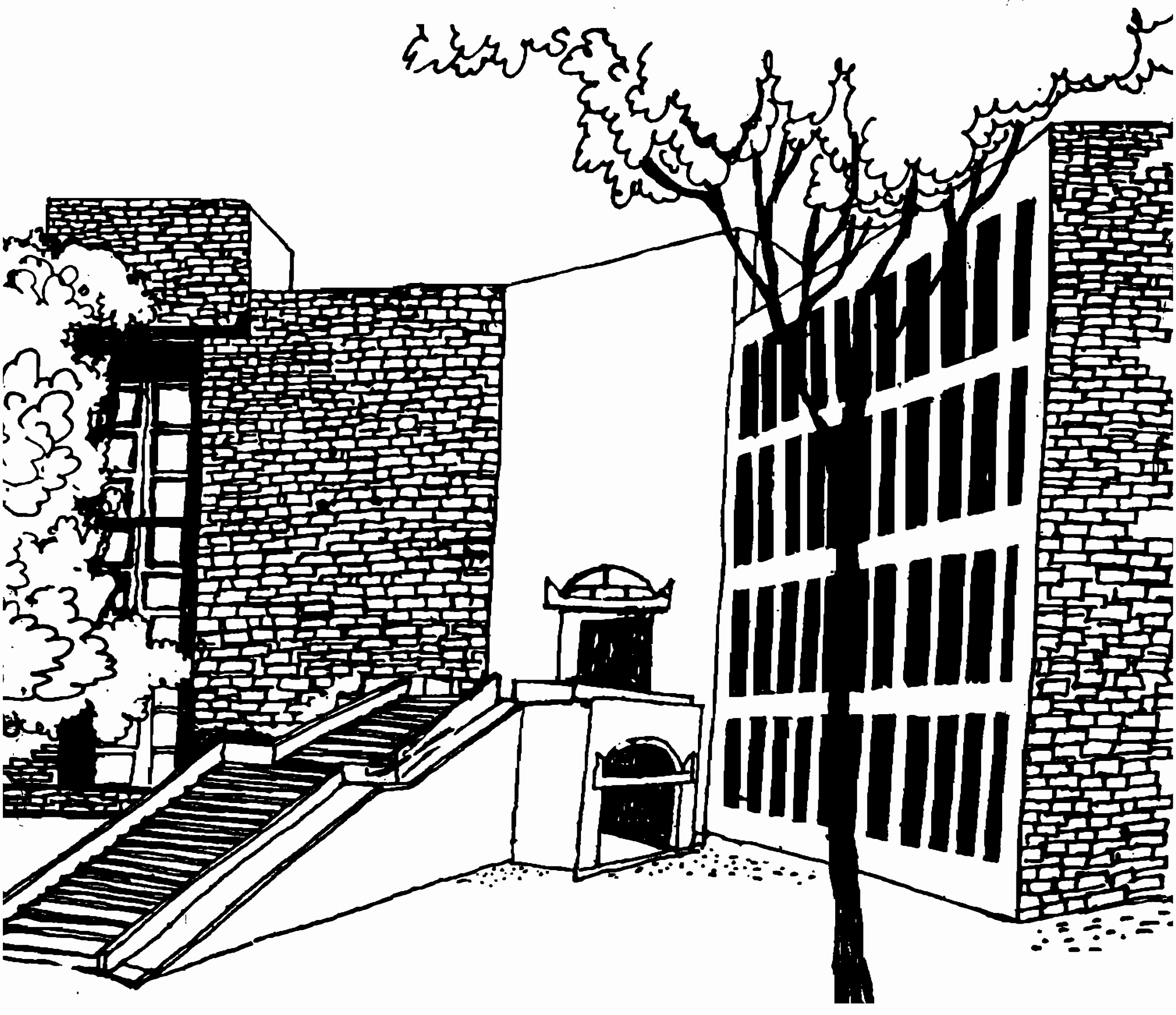

A NOTE ON NUMERICAL REPRESENTATIONS OF QUASI-TRANSITIVE BINARY RELATIONS

By

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A Note On Numerical Representations Of Quasi-Transitive Binary Relations

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1. Introduction:- A binary relation is said to be quasi-transitive if its asymmetric part is transitive. A binary relation is said to be a weak-order if it is reflexive, complete and transitive. One may refer to French [1986] for a synopsis of these and similar definitions. An easy consequence of the result in Donaldson and Weymark [1998], which states that, any binary relation which is reflexive,complete and quasi-transitive can be expressed as an intersection of weak orders, is the result (which for a finite domain may be traced back to Aizerman and Malishevsky [1981],[see Aizerman and Aleskerov [1995] as well]) that the asymmetric part of a quasi-transitive binary relation can be expressed as the intersection of the asymmetric parts of weak-orders.In this note we provide a new and an independent proof of this result (which is what we refer to as Theorem 1 in this note)considering its abiding importance in decision theory. We also,use our Theorem 1 to prove the well known result due to Dushnik and Miller [1941],which states that any asymmetric and transitive binary relation is the intersection of binary relations which are asymmetric , transitive and complete. Finally,we provide a new proof of the result due to Donaldson and Weymark[1998],mentioned above.Our proof appears to be simpler than the one provided by them ,and is established with the help of the theorem due to Dushnik and Miller [1941].

2. The Framework:-Let X be a non-empty universal set of alternatives. A binary relation Q on X is any non-empty subset of X x X. Given a binary relation Q on X its asymmetric part denoted $P(Q) = \{(x, y) \in Q / (y, x) \notin Q\}$ and its symmetric part denoted $I(Q) = \{(x, y) \in Q/(y, x) \in Q\}$. A binary relation Q on X is said to be

(i) reflexive if $(x, x) \in Q\ \forall\ x \in X$;
(ii) complete if $x, y \in X$, $x \neq y$ implies $(x, y) \in Q$ or $(y, x) \in Q$;
(iii) transitive if $\forall\ x, y, z \in X$, $(x, y) \in Q$ and $(y, z) \in Q$ implies $(x, z) \in Q$;
(iv) quasi-transitive if $\forall\ x, y, z \in X$, $(x, y) \in P(Q)$ and $(y, z) \in P(Q)$ implies $(x, z) \in P(Q)$;
(v) asymmetric if $(x,y) \in Q$ implies $(y,x) \notin Q$;
(vi) a weak order if it is reflexive, complete and transitive.

A binary relation R on X is said to be extend the binary relation Q on X if

(i) $Q \subset R$ and
(ii) $P(Q) \subset P(R)$.

We are concerned here with the following theorem:

Theorem 1:- If Q is a quasi-transitive binary relation then $P(Q) = \cap \{P(R)/ R \in A\}$, where $\phi \neq A \subset \{R \subset X \times X / R$ is a weak order$\}$.

This theorem for X finite, is really a consequence of two theorems in Aizerman and Malishevsky [1981] and these two theorems have been reproduced in Aizerman and Aleskerov [1995]. This theorem is also an easy consequence of the theorem in Donaldson and Weymark [1998]. It is important enough from the stand point of multi-objective decision making to merit an independent proof.

3. Proof of Theorem 1:- Let $\Delta = \{(x, x)/x \in X\}$ and let $R_1 = \Delta \cup P(Q)$. R_1 is reflexive and transitive. Hence by Szpilrajn's Extension Theorem (see Fishburn [1970]), there exists a weak order R^1 on X such that $R_1 \subset R^1$ and $P(R_1) \subset P(R^1)$. Let $A = \{R/R$ is a weak order on X with $R_1 \subset R$ and $P(R_1) \subset P(R)\}$.
Now $P(R_1) = P(Q)$.
Let $(x, y) \in P(Q)$. Thus $(x, y) \in P(R)$ whenever $R \in A$.
Now suppose $(x, y) \in \cap \{P(R)/ R \in A\}$. Towards a contradiction suppose $(x, y) \notin P(Q)$. Since $(y, x) \in P(Q) \subset \cap\{P(R)/ R \in A\}$ contradicts [$(x, y) \in P(R)$ whenever $R \in A$], clearly $(y, x) \notin P(Q)$ and also $(y, x) \notin P(R)$ whenever $R \in A$.
Let $\overline{P} = T(P(Q) \cup \{(y, x)\})$, where if S is any binary relation on X then $(a, b) \in T(S)$ if and only if there exists a positive integer m and elements $z_1, \ldots, z_m$ in X with $a = z_1$, $b = z_m$ satisfying $(z_i, z_{i+1}) \in S$ whenever $i \in \{1, .., m-1\}$. Let $\overline{R} = \Delta \cup \overline{P}$. Suppose towards a contradiction that (z,w) and (w,z) both belong to $\overline{P}$. Thus there exists a positive integer m and elements $z_1, \ldots, z_m$ in X with $z = z_1 = z_m$, $w = z_n$ (with n less than m, and $(z_i, z_{i+1}) \in P(Q) \cup \{(y, x)\}$ $\forall i \in \{1, \ldots, m-1\}$. If $(z_i, z_{i+1}) \in P(Q)$ $\forall i \in \{1, \ldots, m-1\}$, then we get by transitivity of P(Q), that $(z_1, z_m) \in P(Q)$ i.e. $(z, z) \in P(Q)$, contradicting asymmetry of P(Q). Hence $(z_i, z_{i+1}) = (y, x)$ for some $i \in \{1, \ldots, m-1\}$.

Observe that 'm' is greater than two, for if $m \leq 2$, then (z, w) and (w, z) belong to $P(Q) \cup \{(y, x)\}$ which is not possible since by hypothesis $x \neq y$.

Case 1:-Cardinality of $\{i \in \{1, \ldots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$ is one.

If $(z_1, z_2) = (y, x)$, then $z_m = y$ implies by transitivity of P(Q) that $(x, y) \in P(Q)$ which is a contradiction.

If i>1, then $(z_1, y) \in P(Q)$ and $(x, z_1) \in P(Q)$ by transitivity of P(Q), so that $(x, y) \in P(Q)$ by transitivity of P(Q) which is a contradiction.

Case 2:-Cardinality of $\{i \in \{1, ..., m-1\}/(z_i, z_{i+1}) = (y, x)$ is greater than one. Let $j = \min \{i \in \{1, ..., m-1\}/(z_i, z_{i+1}) = (y, x)\}$ and $k = \min \{i \in \{j+1, ..., m-1\}/(z_i, z_{i+1}) = (y, x)\}$. Thus $z_{j+1} = x$, $z_k = y$ and by transitivity of P(Q), $(x, y) \in P(Q)$ which is a contradiction.

Thus $\overline{P}$ is asymmetric and further $\overline{R}$ is reflexive and transitive. By Szpilrajn's Extension theorem, there exists a weak order R° such that (i) $\overline{R} \subset R°$ and (ii) $\overline{P} \subset P(R°)$. Thus $P(Q) \subset P(R°)$ and $R_1 \subset R°$. Thus $R° \in A$. However, $(y, x) \in \overline{P}$ implies $(y, x) \in P(R°)$. This contradicts $(x, y) \in \cap \{P(R)/R \in A\}$. Thus $(x, y) \in P(Q)$. Hence the proof is complete.

Q.E.D.

Corollary (due to Aizerman and Malishevsky [1981]): Suppose X is finite and Q is a quasi-transitive binary relation on X. Then there exists a positive integer n and function $f_i: X \rightarrow \Re$, $i \in \{1, ..., n\}$ such that $P(Q) = \{(x, y) \in X \times X / f_i(x) > f_i(y) \ \forall \ i \in \{1, ..., n\}\}$.

Proof:- Since X is finite, the set A above is finite. Let $A = \{R_1, ..., R_n\}$. Since each R_i is a weak-order on the finite set X, for $i \in \{1, .., n\}$, there exists $f_i : X \rightarrow \Re$ such that $(x, y) \in R_i$ if and only if $f_i(x) \geq f_i(y)$ (see Feldman [1980]). Thus $(x, y) \in P(R_i)$ if and only if $f_i(x) > f_i(y)$. Hence by the theorem above, we get the corollary.

Q.E.D.

4. The Dushnik and Miller Theorem:-Now we use our Theorem 1 to prove the following well known theorem due to Dushnik and Miller [1941]:

Theorem 2:-Let P be any asymmetric and transitive binary relation on X.Then $P = \cap \{Q / Q \in B\}$,where, $\phi \neq B \subset \{Q \in XxX / Q$ is asymmetric,transitive and complete $\}$.

Remark1:- A binary relation which is asymmetric,transitive and complete is called a strict linear order.

Remark2:-It is easy to see that Theorem 2 can be used to prove Theorem 1.

Proof of Theorem2:-If P is complete there is nothing to prove.Hence assume,that P is not complete.Let $\Delta = \{(x,x) / x \in X\}$, and let $N = \{(x,y) \in (XxX) / (x,y) \notin P \cup \Delta$

and $(y,x) \notin P \cup \Delta$ }. By Theorem 1, there exists a nonempty subset A of { R/ R is a weak order on X }, such that $P = \cap \{ P(R) / R \in A \}$. Let $R \in A$. If P(R) is complete then there is nothing more to be done with P(R). Hence suppose that P(R) is not complete.

An [$N \cup P(R)$]-cycle is any non-empty finite subset $\{ (z_i, z_{i+1}) / i \in \{1,\ldots,m-1\} \}$ of $N \cup P(R)$, where m is some positive integer and $z_1 = z_m$. An acyclic subset of $N \cup P(R)$ is any nonempty subset of $N \cup P(R)$, which does not contain any [$N \cup P(R)$]-cycle.

Let D (R) = { S/ S is an acyclic subsets of $N \cup P(R)$} and let $F(R) = \{T(S)/S \in D(R)\}$, ordered by set inclusion. F(R) is nonempty since P(R) is not complete. By Zorn's lemma, F (R) has a maximal element. Let E (R) be the set of all maximal elements of F(R). Let T(S) belong to E(R) and let $(x,y) \in N$. Suppose neither (x,y) nor (y,x) belong to U(S). Then $T(S \cup \{(x,y)\})$ belongs to F(R) and strictly contains T(S) thereby contradicting the maximality of T(S). Thus either (x,y) or (y,x) belongs to T (S). Further if $(x,y) \in N$ then there exists some S in E (R) (possibly a singleton)such that (x,y) belongs to T (S).

Given R in A and S in E (R), let Q (R,S) = $T(P(R) \cup T(S))$. Clearly Q (R,S) is transitive and due to the observation above we can conclude that Q (R,S) is complete. Further since S is an acyclic subset of $N \cup P(R)$, Q(R,S) is asymmetric.

Let (x,y) belong to P. Then by Theorem 1, (x,y) belongs to P(R) for all R in A. Therefore (x,y) belongs to Q(R,S) whenever R belongs to A and S belongs to E (R). Now suppose (x,y) belongs to $\cap \{Q(R,S) / R \in A \text{ and } S \in E(R)\}$. Suppose towards a contradiction (x,y) does not belong to P. If (y,x) belongs to P then by Theorem 1, (y,x) belongs to P(R) whenever R belongs to A. Hence (y,x) belongs to Q (R,S) whenever R belongs to A and S belongs to E(R), which is not possible since Q (R,S) is asymmetric. Thus (x,y) belongs to N. Thus (y,x) also belongs to N. Hence by the observation made above there exists $\bar{S}$ in E(R) such that $(y,x) \in T(\bar{S}) \subset Q(R, \bar{S})$. By asymmetry of $Q(R, \bar{S})$, (x,y) does not belong to $Q(R, \bar{S})$. Let $B = \{Q(R,S) / R \in A \text{ and } S \in E(R)\}$. This proves the Theorem.

<u>Q.E.D.</u>

We can now use Theorem 2,to establish the following theorem due to Donaldson and Weymark [1998] :

<u>Theorem 3</u>:- Let Q be a reflexive and transitive binary relation on X.
Then, $Q = \cap \{R / R \in A\}$, where $\phi \neq A \subset \{ R \subset X \times X / R \text{ is a weak order}\}$.

<u>Proof of Theorem 3</u>:- Let $I(x) = \{y \in X / (x,y) \in I(Q)\}$, and let $\Xi = \{I(x)/x \in X\}$. Clearly given x,y in X either I(x)=I(y) or $I(x) \cap I(y) = \phi$. By reflexivity of Q, $x \in I(x)$ whenever $x \in X$. Let, $\Pi = \{ (I(x),I(y)) \in \Xi \times \Xi / (x,y) \in P(Q)\}$. Since Q is transitive, Π is well defined and transitive. By Theorem 2,there exists $\phi \neq C \subset \{\Sigma / \Sigma \text{ is a strict linear order on } \Xi\}$,such that $\Pi = \cap \{\Sigma / \Sigma \in A\}$. Let $\Lambda(\Sigma) = \{(x,y) / (I(x),I(y)) \in \Sigma\} \cup \{(I(x),I(x)) / x \in X\}$. It is easy to verify that $\Lambda(\Sigma)$ is a weak order on Ξ, whenever $\Sigma \in C$. Let $R(\Sigma) = \{(x,y) / (I(x),I(y)) \in \Lambda(\Sigma)$. $R(\Sigma)$ is a weak order on X whenever $\Sigma \in C$ and $Q = \cap \{ R(\Sigma) / \Sigma \in C \}$. Let A= $\{ R(\Sigma) / \Sigma \in C \}$. This completes the proof.

<u>Q.E.D.</u>

Our proof above does not require the use of the theorem due to Suzumura [1976], and thus appears to be a little simpler than the one due to Donaldson and Weymark [1998].

Appendix

In this appendix,we give a new independent proof of the theorem due to Donaldson and Weymark [1998] ,which has been referred to as Theorem 3,in our paper.This proof is similar to the one we provided for our Theorem 1. Note that, neither does this proof make use of the theorem due to Suzumura [1976].

Alternative Proof of Theorem 3:-By Szpilrajn's theorem ,$A = \{ R / R$ is a weak order that extends $Q\} \neq \phi$, and it is easy to see therefore $Q \subset \cap \{R / R \in A\}$ with $P(Q) \subset \cap \{P(R) / R \in A\}$.Now suppose that $(x,y) \in \cap \{R / R \in A\}$.Towards a contradiction suppose,that $(x,y) \notin Q$.Thus $x \neq y$ since Q is reflexive.Further $(y,x) \notin Q$ since otherwise $(y,x) \in P(Q) \subset \cap\{P(R) / R \in A\}$ contradicting $(x,y) \in \cap \{R / R \in A\}$. Let $R_1 = T(\{(y,x)\} \cup Q)$. R_1 is reflexive and transitive. Hence by Szpilrajn's Extension Theorem there exists a weak order R° on X such that $R_1 \subset R^\circ$ and $P(R_1) \subset P(R^\circ)$. Let $(z,w) \in P(Q)$. Thus $(z,w) \in R_1$ and $z \neq w$ by asymmetry of P(Q).Towards a contradiction suppose $(w,z) \in R_1$. Thus there exists a positive integer m and elements $z_1, \ldots, z_m$ in X with $z = z_1 = z_m$,$w = z_n$ (with n less than m), and $(z_i, z_{i+1}) \in Q \cup\{(y, x)\}$ $\forall\, i \in \{1, \ldots, m-1\}$.
If $(z_i, z_{i+1}) \in Q \forall\, i \in \{1, \ldots, m-1\}$, then we get by transitivity of Q, that $(w,z) \in Q$, contradicting $(z,w) \in P(Q)$.Hence $(z_i, z_{i+1}) = (y, x)$ for some $i \in \{1, \ldots, m-1\}$.
Observe that `m' is greater than two, for (a) if m=2, then (z, w) and (w, z) belong to $Q \cup \{(y, x)\}$ which is not possible since by hypothesis $x \neq y$; and (b)if m =1 then z=w contradicting $(z,w) \in P(Q)$.
Case 1:-Cardinality of $\{i \in \{1, \ldots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$ is one.
If $(z_1, z_2) = (y, x)$, then $z_m = y$ implies by transitivity of Q that $(x, y) \in Q$ which is a contradiction.
If i>1, then $(z_1, y) \in Q$ and $(x, z_m) \in Q$ by transitivity of Q, so that $(x, y) \in Q$ by transitivity of Q and the fact that $z_1 = z_m$, which is a contradiction.
Case 2:-Cardinality of $\{i \in \{1, \ldots, m-1\}/(z_i, z_{i+1}) = (y, x)$ is greater than one.
Let $j = \min \{i \in \{1, \ldots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$ and $k = \min \{i \in \{j+1, \ldots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$. Thus $z_{j+1} = x$, $z_k = y$ and by transitivity of Q, $(x, y) \in Q$ which is a contradiction.
Thus $P(Q) \subset P(R_1) \subset P(R^\circ)$. However , $R^\circ \in A$ and $(y, x) \in P(R^\circ)$. This contradicts $(x, y) \in \cap \{R / R \in A\}$. Thus $(x, y) \in Q$. This completes the proof.

Q.E.D.

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