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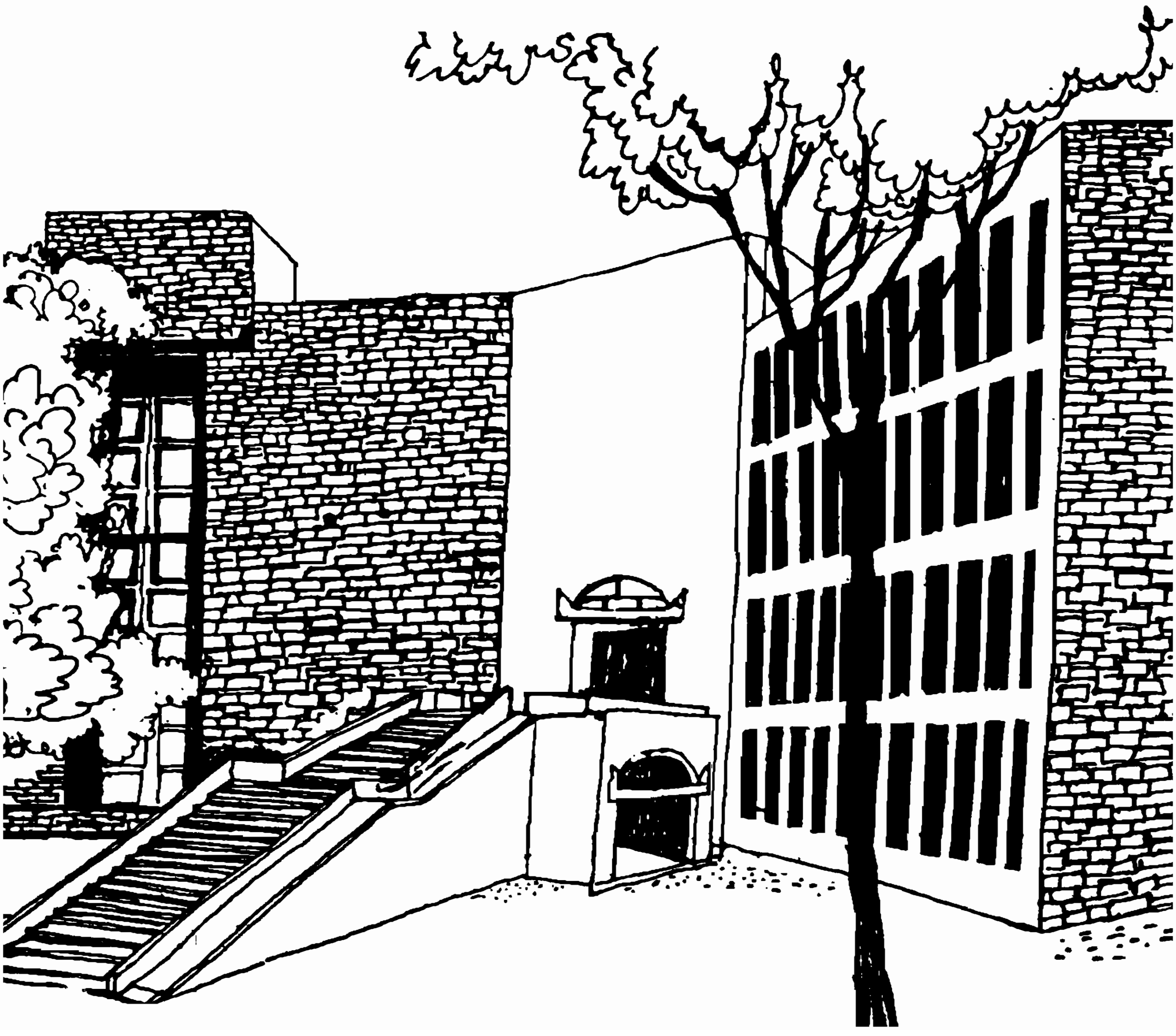

ZORN'S LEMMA IN RATIONAL CHOICE THEORY: A NOTE

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Abstract

The first significant use of Zorn's Lemma in rational choice theory is in the paper by Szpilrajn[1930], where it is established that every strict partial order can be embedded within a strict linear order. Subsequently, Dushnik and Miller [1941] showed that every strict partial order is the intersection of all the strict linear orders in which it can be embedded. Richter ([1966], [1971]), Hansson [1968], Suzumura ([1976], [1983]), Deb [1983] and Lahiri [1999a], all use Szpilrajn's theorem, to establish conditions for various shades of rationalizability of choice functions. In recent times Donaldson and Weymark [1998], Duggan [1999] and Lahiri [1999b] use Szpilrajn's theorem to establish results similar to those available in Dushnik and Miller [1941]. In Lahiri [1999b] an independent proof of the theorem due to Dushnik and Miller is given which uses Zorn's Lemma explicitly.

Szpilrajn's theorem is a deep theorem in its own right and the fact that it uses Zorn's Lemma, often makes it inaccessible to someone who has had no formal training in advanced set theory. This is because, Zorn's Lemma is proved using the axiom of choice and an intermediate theorem is the well-ordering theorem. Much of this is beyond the scope of an individual who has not studied advanced set theory. It would be considerably easier to grasp those aspects of rational choice theory where Zorn's Lemma is applied, if there was a simpler way to obtain the celebrated lemma. This is what we do in this note by replacing the axiom of choice by what we call 'chain axiom'. The proof of Zorn's lemma which now does not require the axiom of choice or the well-ordering theorem, can be established quite easily using elementary set theory.

Rational choice theory comprises a body of results which are sufficiently challenging in their own right. Szpilrajn's theorem is a major building block of rational choice theory. It does the subject or its students no good by making it unnecessarily inaccessible. By making the journey to a crucial result more arduous than it need be, we shift the focus of rational choice theory from the analysis of decision making to an important result in set theory. Our chain axiom may require a giant leap of faith for a set theorist. For us, however, it is a major step towards simplification.

Zorn's Lemma In Rational Choice Theory : A Note

Somdeb Lahiri
Indian Institute of Management
Ahmedabad 380 015
India

1. The first significant use of Zorn's Lemma in rational choice theory is in the paper by Szpilrajn[1930], where it is established that every strict partial order can be embedded within a strict linear order. Subsequently, Dushnik and Miller [1941] showed that every strict partial order is the intersection of all the strict linear orders in which it can be embedded. Richter ([1966], [1971]), Hansson [1968], Suzumura ([1976], [1983]), Deb [1983] and Lahiri [1999a], all use Szpilrajn's theorem, to establish conditions for various shades of rationalizability of choice functions. In recent times Donaldson and Weymark [1998], Duggan [1999] and Lahiri [1999b] use Szpilrajn's theorem to establish results similar to those available in Dushnik and Miller [1941]. In Lahiri [1999b] an independent proof of the theorem due to Dushnik and Miller is given which uses Zorn's Lemma explicitly.

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2. Let F be a non-empty collection of sets. For A, B $\in$ F we will write '$A \subset B$' to denote 'A is a subset of B' and '$A \subset\subset B$' to denote 'A is a proper subset of B'. We will say that F is closed under arbitrary unions if whenever $\{A_\alpha\}_{\alpha \in \Gamma}$ is a non-empty collection of sets in F, then $\bigcup_{\alpha \in \Gamma} A_\alpha \in F$. Let M (F) = { A $\in$ F/ there does not exist B $\in$ F with $A \subset\subset B$}. A non-empty subset G of F is said to be a chain in F if $\forall$ A, B $\in$ G either $A \subset B$ or $B \subset A$. If G

is a chain in F we let B(G) denote the set $\cup\{B/ B \in G\}$. We say that B(G) exists if B(G) $\in$F.We motivate the main result of this paper by means of a lemma and an example.

Lemma: Suppose F is closed under arbitrary unions. Then $M(F) \neq \phi$.

Proof:- Towards a contradiction suppose that F is closed with respect to arbitrary unions and $M(F) = \phi$. Let $A \in F$ and let $H(A) = \{ B \in F/A \subset B\}$.

Since $A \notin M(F)$, there exists $B \in H(A)$ such that $A \subset\subset B$.

Let $D = \cup \{B/B \in H(A)\}$.
Clearly $D \in F$ and $A \subset\subset D$. However, $D \notin M(F)$. Hence there exists $C \in F$ such that $D \subset\subset C$. However, $A \subset\subset C$ implies $C \subset D$, leading to a contradiction. Thus $M(F) \neq \phi$.

Q.E.D.

The requirement that F is closed under arbitrary unions is important as the following example shows:

Example:- For $n \in \mathbf{N}$ (: the set of natural numbers), let $I_n = \{ x \in \Re/-2 + (1/n)< x <2-(1/n)\}$ where $\Re$ is the set of real numbers. Let $F = \{I_n/n \in \mathbf{N}\}$. Now $\cup\{I_n/n \in \mathbf{N}\} = (-2,2) \notin F$. Hence F is not closed under arbitrary unions. Further $M(F) = \phi$.

However, $F= \{I_n / n\in \mathbf{N}\}$ satisfies the following property:

Chain Property:- A non-empty family F of sets is said to satisfy the chain property if there exists a chain H in F such that if there exists $B \in F \setminus M(F)$,and there exists $A \in H$,with $A\subset B$,then there exists $C \in H$ with $B \subset\subset C$.

Suppose F satisfies the Chain Property.

Zorn's Lemma :- Suppose that whenever G is a chain in F, B (G) exists. Then $M(F) \neq \phi$.

Proof :- Towards a contradiction suppose $M(F) = \phi$, even though whenever G is a chain in F, B(G) exists . By the Chain Property there exists a chain H in F such that if there exists $B \in F \setminus M(F)$,and there exists $A \in H$ with $A\subset B$,then there exists $C \in H$ with $B \subset\subset C$.Now $B(H) = \cup \{ B/B\in H\}$ and hence $B(H) \in F$, since H is a chain in F. Since $M(F) = \phi$, $B(H) \notin M(F)$. Let $B \in H$. Clearly, $B \subset B(H)$. Thus ,by Chain Property, there exists $C \in H$ with $B(H) \subset\subset C$. Thus $B(H) \subset\subset \cup \{ B/B\in H\}$ which is the required contradiction.

Q.E.D.

Note:-In the Chain Property above, the requirement "there exists A $\in$H,with A$\subset$ B" is important.For if F =L$\cup$ M,where L={J_n /n $\in$ **N**},M={K_n /n $\in$ **N**},and for n $\in$ **N**, J_n ={ x $\in$ $\Re$/-2 + (1/n)< x < 0} and K_n={ x $\in$ $\Re$/ 0< x <2- (1/n)},then a chain H in F is either subset of L or of M.If H is a subset of L then a B in M would suffice to violate the modified property;a similar violation would occur if H was a subset of M,whence B should be chosen from L.
However, F satisfies the following axiom :

Chain Axiom :- If F is a non-empty family of sets such that B(G) exists whenever G is a chain in F, then F satisfies the chain property.

Main Theorem :-Assume F satisfies the Chain Axiom and suppose B(G) exists whenever G is a chain in F.Then M(F) $\neq\phi$.

Proof :- Follows immediately from above.

It is instructive to note that Zorn's Lemma easily implies the Chain Axiom.

Note :-In the chain axiom above ,the requirement "B(G) exists for all chains in F", is important.For otherwise, if F={{x_1 ,...,x_n}/n$\in$**N** & $|x_i|$ = i ,for i=1,...,n}, there exists a chain G such that B(G) does not exist,namely G={{x_1 ,...,x_n}/n$\in$**N** & x_i = i ,for i=1,...,n}. Further,given any m $\in$**N**,m>1,both {x_1 ,...,x_{m-1} ,m} and{x_1 ,...,x_{m-1}, -m} belong to F and contains {x_1 ,...,x_{m-1} }.However,there is no set in F which contains both m and -m at the same time.Thus,this F invalidates the modified axiom.

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