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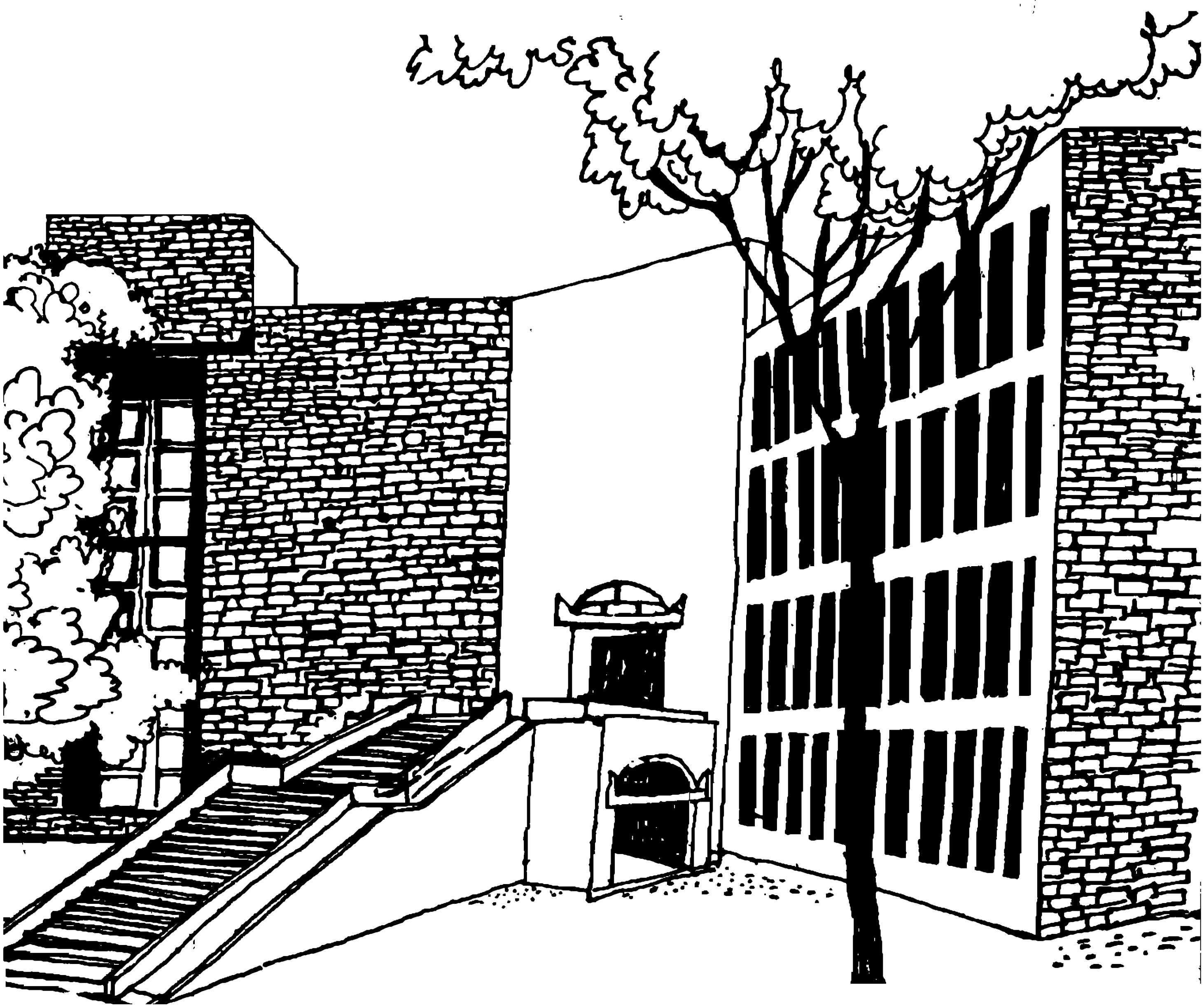

A THEOREM IN UTILITY THEORY REVISITED

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

A THEOREM IN UTILITY THEORY REVISITED

Somdeb Lahiri
Indian Institute Of Management,
Ahmedabad-380015,
India.

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Abstract:-We provide in this paper a simple proof of the proposition that comparability of first differences of utility implies cardinality when the range of the utility function is a connected subset of the reals.

1. Introduction:- In an interesting paper,Basu (1982) addresses and answers the problem arising out of comparability of the first differences of utility levels.In particular he resolves the issue concerning cardinality of utility functions.His theorem though correct relies on a proof whose argument is not that transparent.In this paper we propose a simpler proof of the same result.In a final section of this paper we provide two general characterization theorems of a functional equation which is implied in the proof of the above result.

2. The Framework:- As in Basu (1982) let X be the set of alternatives.A utility function u,is a real-valued function on X,i.e. $u : X \to \Re$, where $\Re$ denotes the set of real numbers. A **tranformation** is a mapping from
$\Re$ to $\Re$.An individual is characterized by a reference utility function, u, and a set of permitted transformations, Ω.
The following two axioms were investigated by Basu (1982):

Axiom 1: (Lange) $\forall\, u_1, u_2, u_3, u_4 \in \Re, \forall f \in \Omega$, $[u_1 - u_2 \geq u_3 - u_4 \leftrightarrow f(u_1) - f(u_2) \geq f(u_3) - f(u_4)]$.

Axiom 1: (Basu) $\forall\, u_1, u_2, u_3, u_4 \in u(X), \forall f \in \Omega$, $[u_1 - u_2 \geq u_3 - u_4 \leftrightarrow f(u_1) - f(u_2) \geq f(u_3) - f(u_4)]$.

Basu (1982) attributes the first axiom to Oscar Lange.It is well known that under Axiom 1, each f must be a strictly monotically increasing affine transformation from the reals to the reals.Under Axiom 2, the same conclusion does not hold and Basu (1982) establishes that utility is cardinal if u(X) is a connected subset of $\Re$.We propose an alternative simpler proof in this paper.In what follows let N denote the set of positive integers.

3. The Main Result:-
Let [a, b] be a non-degenerate interval in $\Re$.

Lemma 1:- Suppose f: $[a, b] \to \Re$ satisfies the following property:
$\forall\, u_1, u_2, u_3, u_4, \in [a,b]$, $u_1 - u_2 \geq u_3, - u_4 \leftrightarrow f(u_1) - f(u_2) \geq f(u_3) - f(u_4)$ (*)
Then there exists $c \in \Re$ and $d > 0$ such that $f(x) = c + dx\ \forall x \in [a, b]$.

Proof:- Taking $u_2 = u_4$, we get from (*):

$u_1 \geq u_3 \leftrightarrow f(u_1) \geq f(u_3)\ \forall u_1, u_3 \in [a, b]$. (1)
Taking, $u_1 - u_2 = u_3 - u_4$ we get from (*), $u_1 - u_2 = u_3 - u_4 \leftrightarrow f(u_1) - f(u_2) = f(u_3) - f(u_4)$.
Taking $u_2 = u_3 = (u_1 + u_4)/2$, we get, $f(u_1) + f(u_4) = 2f([u_1 + u_4]/2) \forall u_1, u_4 \in [a, b]$.
i.e. $f(u) + f(v) = 2f([u + v]/2) \forall u, v \in [a, b]$. Now $f(a) + f(b) = 2f([a + b]/2)$, implies that in $\Re^2$, ([a + b]/2, f ([a + b]/2)) lies on the straight line joining (a, f (a) and (b, f(b)).
Let, $I^{(k)} = \{(m/2^k)a + (1-(m/2^k)b : m = 0, 1, \ldots, 2^k\}$, for $k \in N$.

Let $I = \cup \{ I^{(k)} : k \in N\}$.By a simple induction argument on k (elaborated in the proof of a lemma 2),it follows that (r,f(r)) lies on the straight line joining (a,f(a)) and (b,f(b)) $\forall$ r $\in$I;i.e.
$f(r)=f(a)+([r-a]/[b-a])[f(b)-f(a)]$ $\forall$ r $\in$I.
Let u $\in$ [a, b].Then given $\varepsilon >0$,there exists r_1 , r_2 $\in$I, such that $r_1 \geq u \geq r_2$ and $\varepsilon/2 \geq \max.\{ r_1 - u, u- r_2\}$.
Now, $f(r_1)-f(a)-([u-a]/[b-a])[f(b)-f(a)] \geq f(u)-f(a)-([u-a]/[b-a])[f(b)-f(a)] \geq f(r_2)-f(a)-([r_2-a]/[b-a])[f(b)-f(a)]$.
Further $f(r_2)-f(a)-([u-a]/[b-a])[f(b)-f(a)] \geq f(r_2)-f(a)-([r_2-a]/[b-a])[f(b)-f(a)] -(\varepsilon/2[b-a]) [f(b)-f(a)]$,
since $r_2+\varepsilon/2 \geq u$;and
$f(r_1)-f(a)-([u-a]/[b-a])[f(b)-f(a)] \geq f(r_1)-f(a)-([r_1-a]/[b-a])[f(b)-f(a)] +(\varepsilon/2[b-a]) [f(b)-f(a)]$,
since $u \geq r_1+\varepsilon/2$.
Therefore, $-(\varepsilon/2[b-a]) [f(b)-f(a)] \leq f(u)-f(a)-([u-a]/[b-a])[f(b)-f(a)] \leq +(\varepsilon/2[b-a]) [f(b)-f(a)]$.
Since ε is an arbitrary positive real number we get: $f(u)=f(a)+([u-a]/[b-a])[f(b)-f(a)]$ $\forall$ u $\in$[a,b].
By (1) once again ,f(b)>f(a).Let d= [f(b)-f(a)]/[b-a].Clearly,d > 0.Let c=f(a)-a([f(b)-f(a)]/[b-a]).Thus, f(u)=c+du $\forall$ u $\in$[a,b],as desired.

Q.E.D.

Note:-A study of the above proof reveals that we have established the following result:
Let f:[a,b] $\rightarrow \Re$,be a function satisfying (i) [f(x)+f(y)]/2=f([x+y]/2) $\forall$ x,y $\in$[a,b];and (ii) x,y$\in$[a,b], x $\geq$y, implies f(x) $\geq$ f(y).Then,there exists real numbers c and d, with d $\geq$ 0,such that f(u)=c+du $\forall$ u $\in$[a,b]. Further c=f(a)-a([f(b)-f(a)]/[b-a]) and d= [f(b)-f(a)]/[b-a].
Unlike the previous lemma,we are unable to assert that d is strictly positive,since we are allowing for the possibility that f may assume the same value at two different points.

As a consequence of the above lemma we have the following theorem:

Theorem 1 (Basu):-If u(X) is a connected subset of $\Re$,and if (u,Ω) satisfies Axiom 2,then for each f $\in\Omega$, there exists real numbers c and d (depending on f) with d > 0, such that f(u)=c+du $\forall$ u $\in$[a,b].

Proof :-If u(X) is a degenerate connected subset of $\Re$,then there is nothing to prove.If u(X) is non-degenerate, by Lemma 1, f is an affine function with positive slope on each closed and bounded subinterval of u(X).Hence, f is an affine function with positive slope on all of u(X);i.e. there exists real numbers c and d (depending on f) with d > 0, such that f(u)=c+du $\forall$ u $\in$u(X).

Q.E.D.

Note :- Nowhere in the proof above have we appealed to continuity or differentiability of the transformations.

In the above manner we are able to resolve the problem concerning whether an individual's preferences are represented by a cardinal utility function or not.Our proof is easy to comprehend and relies on a simple induction argument.

4.General Characterization Theorems:-In this section we discuss a functional equation, a special instance of which was invoked in the previous section.In particular we consider a problem of the following sort:

Let X be a convex subset of $\Re^n$ and f : X$\rightarrow\Re$ be a function satisfying [f(x)+f(y)]/2=f([x+y]/2) $\forall$ x,y $\in$X. What is f ?
In what follows let N(k)={1,...,k} where k is any positive integer and let $P(k)=N(2^k)$.
We begin with the following lemma:

Lemma 2 :-Suppose f : X$\rightarrow\Re$ be a function satisfies [f(x)+f(y)]/2=f([x+y]/2) $\forall$ x,y $\in$X.Then, f(tx+(1-t)y) =tf(x)+(1-t)f(y) $\forall$ t $\in$ $\{m/2^k : m=0,1,..., 2^k;k\in N\}$ whenever, x,y $\in$X.

Proof:-Fix x,y in X.Let $I_k = \{m/2^k : m=0,1,..., 2^k\}$, k$\in$N.For k=1, by the condition of the lemma (tx+(1-t)y,tf(x)+(1-t)f(y)) lies on the straight line joining (x,f(x)) and (y,f(y)) in $\Re^{n+1}$.
Assume that the lemma is true for k=1,...,q-1.Pick t $\in I_q$.Then tx+(1-t)y=(1/2)(sx+(1-s)y,s' x+(1- s')y) for some s, s' $\in I_{q-1}$.By the hypothesis of the lemma,f(tx+(1-t)y)=(1/2)f(sx+(1-s)y)+(1/2)f(s' x+(1- s')y), and

hence $(tx+(1-t)y, f(tx+(1-t)y))$ lies on the straight line joining $(sx+(1-s)y, f(sx+(1-s)y))$ and $(s'x+(1-s')y, f(s'x+(1-s')y))$ in $\Re^{n+1}$ .By the induction hypothesis , $(sx+(1-s)y, f(sx+(1-s)y))$ and $(s'x+(1-s')y, f(s'x+(1-s')y))$ both lie on the straight line joining $(x,f(x))$ and $(y,f(y))$ in $\Re^{n+1}$.Thus, $(tx+(1-t)y, f(tx+(1-t)y))$ lie on the straight line joining $(x,f(x))$ and $(y,f(y))$ in $\Re^{n+1}$ as well.As a consequence of a standard induction argument,the validity of the lemma follows for the chosen x and y.Since x and y were arbitrarily chosen points of X, the validity of the lemma holds for all pairs of points in X.

Q.E.D.

A first characterization result is the following:

Theorem 2 :-Suppose $f:X \to \Re$ is a continuous function satisfying $[f(x)+f(y)]/2=f([x+y]/2)\ \forall\ x,y \in X$. Then there exists real numbers $a_1,\ldots,a_n$ and b such that $f(x)= \Sigma_{i\in N(n)} a_i x_i +b\ \forall\ x \in X$.

Proof:-By lemma 2 and continuity ,$f(tx+(1-t)y)=tf(x)+(1-t)f(y)\ \forall\ t \in[0,1]$,whenever $x,y \in X$.Thus f is both convex and concave.Thus there exists real numbers $a_1,\ldots,a_n$ and b such that $f(x)= \Sigma_{i\in N(n)} a_i x_i +b\ \forall\ x \in X$.

Q.E.D.

Remark:- In view of Lemma 2, $[f(x)+f(y)]/2 =f([x+y]/2)\ \forall\ x,y\in X$,implies $f(x/2^k) =f([x/2^k]+[1-(1/2^k)]0) =(1/2^k)f(x)+ [1-(1/2^k)]f(0),\ \forall\ x\in X$ and $\forall\ k\in N$. Hence, $[f(x)+f(y)]/2 =f([x+y]/2)\ \forall\ x,y\in X$,implies $f(x)-f(0)= 2^k [f(x/2^k)-f(0)]$, $\forall\ x\in X$ and $\forall\ k\in N$.

Lemma3:- Let $f:X\to\Re$ where X is a convex subset of $\Re^n$ satisfy $[f(x)+f(y)]/2 =f([x+y]/2)\ \forall\ x,y\in X$. Then, $\forall\ k\in N$ and $x^i \in X$, $i=1,\ldots, 2^k$, $f([1/2^k]\Sigma_{i\in P(k)} x^i) =[1/2^k]\ \Sigma_{i\in P(k)} f(x^i)$.

Proof:- For k=1,the lemma follows from the hypothesis. Assume that the lemma holds for k=1,...,r.Let k=r+1. Thus, $f([1/2^{r+1}]\Sigma_{i\in P(r+1)} x^i)= f((1/2)[1/2^r]\Sigma_{i\in P(r)} x^i + (1/2)[1/2^r]\Sigma_{i\in P(r+1)\setminus P(r)} x^i)$

$= ((1/2)f([1/2^r]\Sigma_{i\in P(r)} x^i) + (1/2)f([1/2^r]\Sigma_{i\in P(r+1)\setminus P(r)} x^i)$, by the hypothesis of the lemma.

Now, $f([1/2^r]\Sigma_{i\in P(r)} x^i) =[1/2^r]\ \Sigma_{i\in P(r)}f(x^i)$, by the induction hypothesis.Since, $2^{r+1} -2^r =2^r$, $f([1/2^r]\Sigma_{i\in P(r+1)\setminus P(r)} x^i)= [1/2^r]\ \Sigma_{i\in P(r+1)\setminus P(r)}f(x^i)$. Thus, $f([1/2^{r+1}]\Sigma_{i\in P(r+1)} x^i) =[1/2^{r+1}]\ \Sigma_{i\in P(r+1)} f(x^i)$.The by a standard induction argument it follows that lemma holds $\forall\ k\in N$.

Q.E.D.

Corollary to Lemma 3:- Let $f:X\to\Re$ where X is a convex subset of $\Re^n$ containing 0,satisfy $[f(x)+f(y)]/2 =f([x+y]/2)\ \forall\ x,y\in X$. Then, $\forall\ m\in N$ and $x^i \in X$, $i=1,\ldots, m$, $f(\Sigma_{i\in N(m)} x^i)-f(0) = \Sigma_{i\in N(m)} [f(x^i)-f(0)]$.

Proof:-Let $m\in N$ and suppose $k\in N$ with $2^k\geq m$.If $m=2^k$ for some $k\in N$,then the corollary follows directly from Lemma 3.Hence assume $2^k> m$.Define, $x^i=0$,for $i=m+1,\ldots, 2^k$.Then, $f([1/2^k]\Sigma_{i\in P(k)} x^i) = [1/2^k]\ \Sigma_{i\in P(k)} f(x^i)$.This follows from Lemma 3.However,by the remark following Lemma 2, $f(\Sigma_{i\in P(k)} x^i)-f(0) = 2^k [f([1/2^k]\Sigma_{i\in P(k)}x^i)-f(0)]$. Thus, $\Sigma_{i\in P(k)} f(x^i)= f(\Sigma_{i\in P(k)} x^i)-f(0)+ 2^k f(0)$. Hence, $\Sigma_{i\in N(m)} f(x^i)+ (2^k-m)f(0) = f(\Sigma_{i\in N(m)} x^i)-f(0)+ 2^k f(0)$.Thus, $f(\Sigma_{i\in N(m)} x^i)-f(0) = \Sigma_{i\in N(m)} [f(x^i)-f(0)]$ as was required to be proved.

Q.E.D.

A function $f:X \to\Re$ is said to be **weakly monotone increasing** ,if $\forall\ x,y \in X$, $x \geq y$ implies $f(x) \geq f(y)$. For $i \in N(n)$, let c_i , d_i be real numbers with $c_i < d_i$ .Let $X= \Pi_{i\in N(n)}[c_i , d_i]$.Let $g: \Pi_{i\in N(n)}[0, d_i - c_i] \to\Re$,be defined by $g(x)=f(c+x)$ where $c=(c_1 ,\ldots,c_n)$.Then,

(1)$f(x)+f(y)=2f([x+y]/2)\ \forall\ x,y \in X$ if and only if $g(x)+g(y)=2g([x+y]/2)\ \forall\ x,y \in \Pi_{i\in N(n)}[0, d_i - c_i]$;

(2) f is weakly monotone increasing if and only if g is weakly monotone increasing;

(3) there exists real numbers $a_1,\ldots,a_n$ and b such that $f(x)= \Sigma_{i\in N(n)} a_i x_i +b\ \forall\ x \in X$ if and only if there exists real numbers $r_1,\ldots,r_n$ and s such that $g(x)= \Sigma_{i\in N(n)} r_i x_i +s\ \forall\ x \in \Pi_{i\in N(n)}[0, d_i - c_i]$.

In what follows without loss of generality assume $X=\Pi_{i\in N(n)}[0, c_i]$ for some positive real numbers $c_1,\ldots,c_n$.

Theorem 3:-Let $f:X \to \Re$ be weakly monotone increasing and satisfy $f(x)+f(y)=2f([x+y]/2)\ \forall\ x,y \in X$. Then, there exists real numbers $a_1,\ldots,a_n$ such that $f(x)-f(0)= \Sigma_{i\in N(n)}\, a_i\, x_i\ \ \forall\ x \in X$.

Proof:-Let e_i denote the ith unit coordinate vector in $\Re^n$,and for $i \in N(n)$,let $g_i :[0, c_i] \to \Re$ be the function defined by $g_i(\alpha)=f(\alpha e_i)-f(0)$.

Now,for all $\alpha,\beta \in [0, c_i]$,(i) $g_i(\alpha)+g_i(\beta)=2\, g_i([\alpha+\beta]/2)$;and (ii) $\alpha \geq \beta$ implies $g_i(\alpha)\geq g_i(\beta)$.Hence by the note following lemma 1,there exists a non-negative real number a_i such that $g_i(\alpha)=a_i\,\alpha+ g_i(0)$,for all $\alpha \in [0, c_i]$.Let $x \in X$ and let x_i be its ith co-ordinate.Thus $x=\Sigma_{i\in N(n)}\, x_i\, e_i$ .Hence, by the corollary to lemma 3, $\Sigma_{i\in N(n)}\, g_i(x_i)= \Sigma_{i\in N(n)}\, [f(x_i\, e_i)-f(0)]= f(x)-f(0)$.Since, $g_i(0)=0\ \forall\ i \in N(n)$,the theorem easily follows.

Q.E.D.

5.Conclusion:-We have thus solved a functional equation uniquely.The general solution is a generalization of Cauchy's functional equation (see Aczel (1992)).If we interpret the function as a utility function, then the defining property as assumed in the paper is sufficient to guarantee that the utility function is affine.

6.References:-

1. J.Aczel (1992): "A Short Course on Functional Equations",Theory and Decision Library,Kluwer Academic Publishers.

2. K.Basu (1982): "Determinateness of the Utility Function:Revisiting a Controversy of the Thirties", Review of Economic Studies XLIX,307-311.