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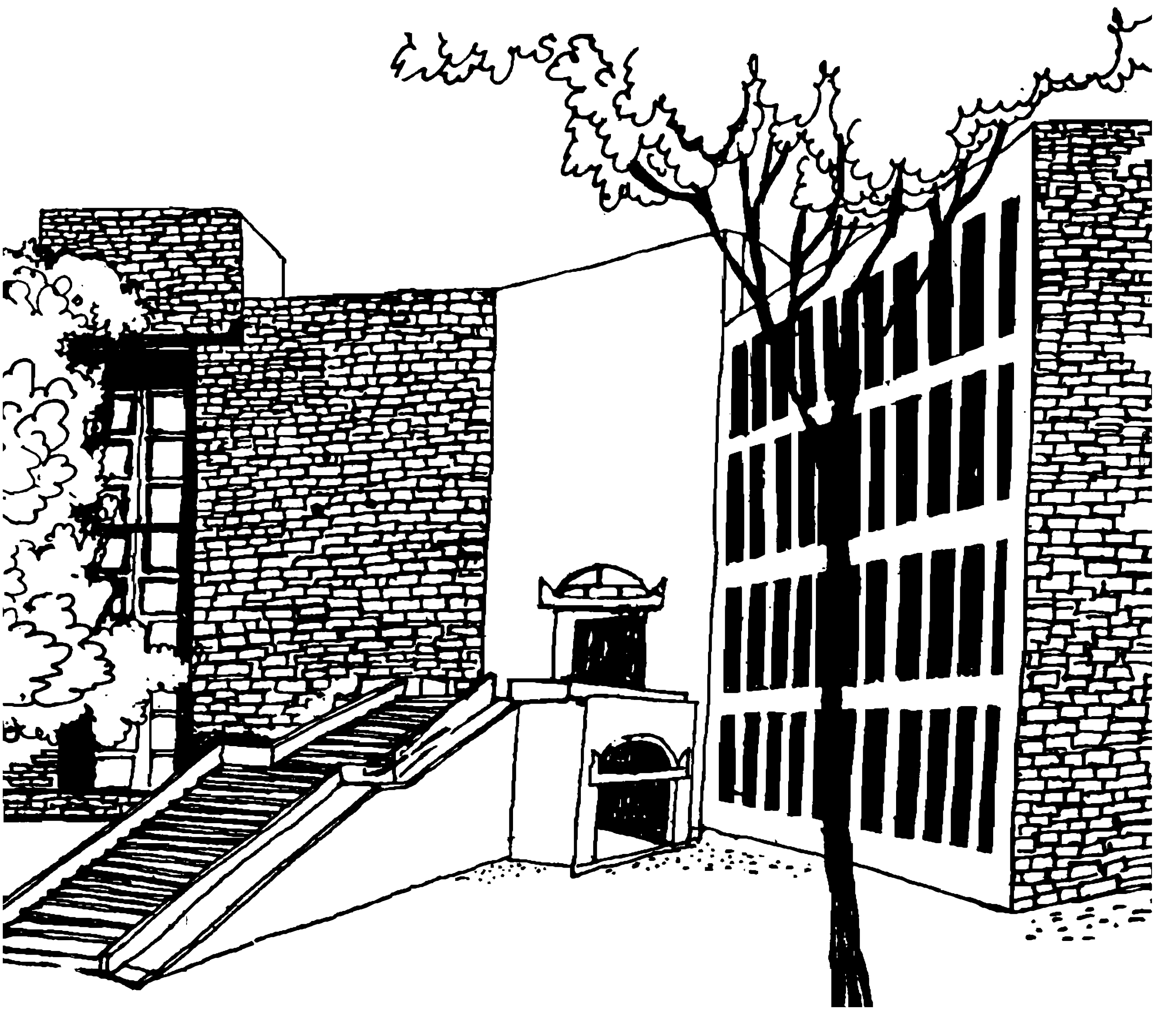

# A NON-HAUSDORFF TOPOLOGICAL SPACE IN WHICH EVERY CONVERGENT SEQUENCE CONVERGES TO A UNIQUE LIMIT

By

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# A Non-Hausdorff Topological Space In Which Every Convergent Sequence Converges To A Unique Limit

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**September 1999.**

**Abstract: Does there exist a non Hausdorff topological space, such that every convergent sequence in it converges to a unique limit? Considering the obvious scarcity of such spaces we were compelled to construct one by ourselves.**

1. Let (X,T) be a topological space with X the underlying set on which the topology T is defined. Let $\emptyset$ denote the empty set. Given x in X, let $\wp(x) = \{U \in T: x \in U\}$. (X,T) is said to be Hausdorff, if given x , y $\in$X, with x $\neq$ y, [there exists U $\in \wp(x)$, V$\in \wp(y)$ such that $U \cap V = \emptyset$].Otherwise it is said to be non-Hausdorff.

A sequence in X is a function S:N$\to$X where N denotes the set of natural numbers.Let $\Re$ denote the set of real numbers and Q the set of rational numbers.Thus $\Re \backslash Q$ denotes the set of irrational numbers.Given a,b in $\Re$ with a<b, let $(a,b)=\{x \in \Re: a<x<b\}$. A sequence S is said to converge to x in X if for every U in $\wp(x)$ there exists M in N (:possibly depending on U and x) such that [n$\in$N, n $\geq$ M implies S(n) $\in$ U].Let L(S)={x$\in$X:S converges to x}.An element of L(S) is called a limit of S.A sequence S is said to be convergent if L(S) $\neq \emptyset$.

Example 1:Let X={1,2} and T={1},{2},{1,2},$\emptyset$}.Let S:N$\to$X be defined as: S(n)=1 if n is odd;S(n)=2 if n is even.Then L(S)= $\emptyset$.

Example 2: Let X={1,2} and T={{1,2},$\emptyset$}.Let S:N$\to$X be any sequence. Then, L(S)=X.

The following result is well known:

Theorem 1: If (X,T) is a Hausdorff topological space, then given any sequence S, either $L(S)= \varnothing$ or L(S) is a singleton.

The result is by now very standard and a proof can be found in Lipschutz (1965) or Munkres (1975).

Let (X,T) be a topological space and let $x \in X$. It is said to be first countable at x, if there exists a countable subset $\Omega$ (x) of $\wp(x)$ such that for every U in $\wp(x)$ there exists V in $\Omega(x)$ with $V \subset U$. The topological space (X,T) is said to be first countable if it is first countable at every point in X.

The following result is well known and a proof can be found in Lipschutz (1965).

Theorem 2: Let (X,T) be a first coutable topological space. If for every sequence S, either $L(S)= \varnothing$ or L(S) is a singleton then (X,T) is Hausdorff.

The question that follows is: Does there exist a non-Hausdorff topological space, such that every convergent sequence in it converges to a unique limit? Clearly, in view of Theorem 2, such a space cannot be first countable. Considering the obvious scarcity of such spaces we were compelled to construct one by ourselves.

2. Let $X=\Re \cup \{*\}$. For x in $\Re$ and $\delta >0$, let $I(x, \delta) = (x-\delta, x+\delta)$ and let $I^o(x, \delta) = (x-\delta, x+\delta) \setminus \{x\}$. Let $A= \{1/n : n \in N\}$. Let $B(0, \delta) = I(0, \delta) \setminus A$, for all $\delta >0$; $M(*, \delta) = (I^o(0, \delta) \cup \{*\}) \cap Q$, if $\delta >0$ and $\delta$ is irrational; $B(*, \delta) = [(I^o(0, \delta) \cup \{*\}) \cap (\Re \setminus Q)] \cup (A \cap I^o(0, \delta))$, if $\delta >0$ and $\delta$ is rational; $B(x, \delta) = I(x, \delta)$, for all $\delta >0$, whenever x belongs to $\Re \setminus \{0\}$. Let T be the smallest topology containing $\{ B(x, \delta): x \in X \text{ and } \delta >0\}$. Clearly, T is non-Hausdorff, since given any U in $\wp(0)$ and V in $\wp(*)$, $U \cap V \neq \varnothing$. Further, every convergent sequence in X converges to a unique limit. Indeed , if S is a convergent sequence in X, and belongs to $X \setminus A$ infinitely often , then S converges to a unique real number. If S is eventually constant, then it converges to the unique element which occurs after a finite number of terms. If

S is not eventually constant, but remains in A after a finite number of terms , then it converges to *, and to no other element of X.

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