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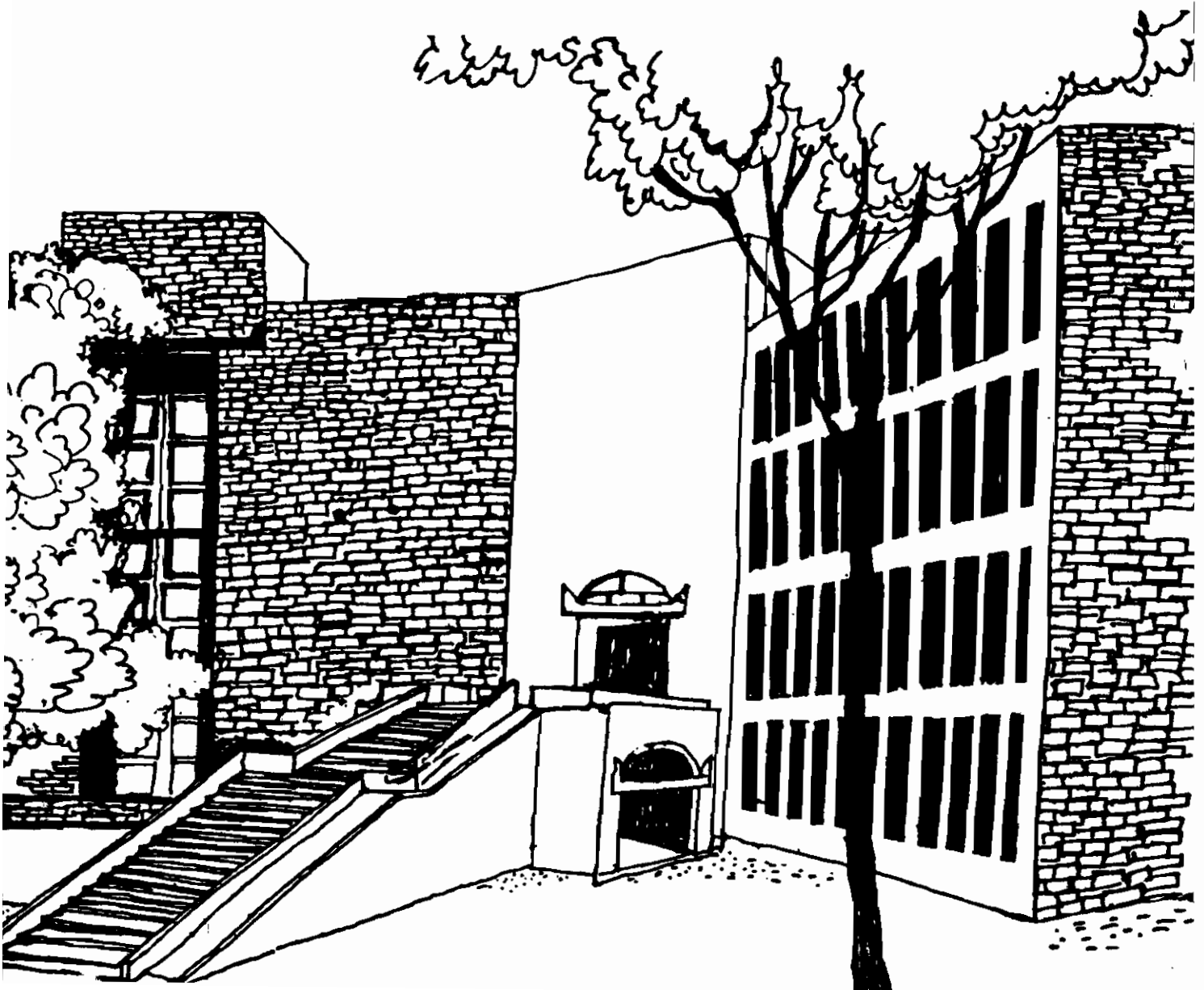
EXTENSION FUNCTIONS ON POWER SETS

By

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Abstract

In Kannai and Peleg(1984) the following problem was posed:Given a positive integer 'n',is it possible to define a positive integer valued function on all non-empty subsets of the first n positive integers,so that singletons preserve their original ranking and further the function satisfies two apparently reasonable properties?The same paper shows that for n greater than five,such a function cannot be defined.A large literature spawned out of this work,where modifications of the properties desired by Kannai and Peleg lead to possibility results.Notable among them are the following: Barbera,Barrett and Pattanaik (1984), Barbera and Pattanaik(1984), Fishburn(1984),Heiner and Packard(1984), Holzman (1984), Nitzan and Pattanaik(1984), Pattanaik and Peleg(1984), Bossert (1989).Our own efforts in this direction culminated in Lahiri(1999),where several of the above contributions have been discussed and studied.

The above mentioned result lead to the search for a possibility result for n equal to five, resulting in the paper by Bandopadhyay (1988).In this paper we provide another different possibility result for n equal to five.Our method of proof suggests an alternative (: and perhaps simpler) approach to the result established in Bandopadhyay(1988) as well.

Extension Functions On Power Sets

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1.In Kannai and Peleg(1984) the following problem was posed:Given a positive integer 'n',is it possible to define a positive integer valued function on all non-empty subsets of the first n positive integers,so that singletons preserve their original ranking and further the function satisfies two apparently reasonable properties?The same paper shows that for n greater than five,such a function cannot be defined.A large literature spawned out of this work,where modifications of the properties desired by Kannai and Peleg lead to possibility results.Notable among them are the following: Barbera,Barrett and Pattanaik (1984), Barbera and Pattanaik(1984), Fishburn(1984),Heiner and Packard(1984), Holzman (1984), Nitzan and Pattanaik(1984), Pattanaik and Peleg(1984), Bossert (1989).Our own efforts in this direction culminated in Lahiri(1999),where several of the above contributions have been discussed and studied.

The above mentioned result lead to the search for a possibility result for n equal to five, resulting in the paper by Bandopadhyay (1988).In this paper we provide another different possibility result for n equal to five.Our method of proof suggests an alternative (: and perhaps simpler) approach to the result established in Bandopadhyay(1988) as well.

2.Let N denote the set of positive integers and for $n \in N$, let $N_n = \{i \in N / i \leq n\}$i.e. the set of first n positive integers.Given, $n \in N$,let $[N_n]$ denote the set of all non-empty subsets of N_n. Given, $n \in N$ and $A \in [N_n]$, let $M(A)$ be the unique element of A such that $M(A)$ is greater than or equal to every element in A,and let $m(A)$ be the unique element of A such that $m(A)$ is less than or equal to every element in A.Further let $\#(A)$ denote the cardinality of A.

Given, $n \in N$ and $p \in N$ a p-dimensional extension function is a function $F: [N_n] \to N^p$ such that for all $i,j \in N_n$ with $i \neq j$, $F(\{i\}) >> F(\{j\})$ if and only if $i > j$, where given $a,b \in N^p$ (i)$a \geq b$ means $a_k \geq b_k$ for all $k \in N_n$;$a > b$ b means $a \geq b$ and $a \neq b$; $a >> b$ means $a_k > b_k$ for all $k \in N_n$.

Let $n \in N$ and let $F: [N_n] \to N^p$ be a p-dimensional extension function.

The following two axioms were used by Kannai and Peleg:

Gardenfors Principle (GP):For all $A \in [N_n]$ and $y \in N_n \setminus A$:(i)$m(A) > y$ implies $F(A) >> F(A \cup \{y\})$;(ii)$y > M(A)$ implies $F(A \cup \{y\}) >> F(A)$.

Weak Independence (WI): For all $A,B \in [N_n]$ and $y \in N_n \setminus (A \cup B)$:$[F(A) >> F(B)$ implies $F(A \cup \{y\}) \geq F(B \cup \{y\})$.

$\therefore 10j + i > 33$.

$\therefore$ either $j = 3$ or $j = 4$.

Suppose $j = 4$. Then $5 > i \geq j$ implies $i = 4$.

$\therefore \{i, j\} = \{4\}$. But then $i \geq k$, $j \geq r$, contradicting (i) and (ii).

Suppose $j = 3$. Then $5 > i \geq j$ implies $i = 4$ or 3. If $i = 4$, then $i \geq k$, $j \geq r$ contradicting (i) and (ii). Thus $i = 3$. Thus $\{i, j\} = [3\}$.

$\therefore G(\{i, j\}) = 33 = G(\{k, r\})$, contradicting $G(\{i, j\}) > G(\{k, r\})$.

Hence Case 3 is ruled out.

Case 4 :- $k < 5$, $\{k, r\} \neq \{4, 2\}$, $\{i, j\} = \{4, 2\}$:

$\therefore G(\{k, r\}) = 10r + k$,

and $G(\{i, j\}) = 33$.

$\therefore r = 3, 2$ or 1.

If $r = 3$, then $k \geq r$ implies $G(\{k, r\}) \geq 33 = G(\{i, j\})$, contradicting $G(\{i, j\}) > G(\{k, r\})$.

Thus $r \neq 3$. Thus $r < j$. Hence not (ii). Hence by (i), $r < j = 2$, Thus $r = 1$.

Further $k > i = 4$ implies $k = 5$, contradicting $k < 5$. Hence Case 4 is ruled out.

Case 5 : - $k < 5$, $\{k, r\} \neq \{4, 2\}$, $\{i, j\} \neq \{4, 2\}$, $i < 5$:

Thus $G(\{i, j\}) = 10j + i$

and $G(\{k, r\}) = 10r + k$.

$G(\{i, j\}) > G(\{k, r\})$ implies $j \geq r$. Hence not (ii). Hence by (i) $r < j$ and $i < k$.

Let $y = 5$, Then $G(\{i, y, j\}) = 50 + j > 50 + r = G(\{k, y, r\})$.

Let $y = 4$. Thus $y \geq k > i$. Then $G(\{i, y, j\}) = G(\{y, j\})$ and $G(\{k, y, r\}) = G(\{y, r\})$

Further $j > r$ implies $G(\{y, j\}) > G(\{y, r\})$.

$\therefore G(\{i, y, j\}) > G(\{k, y, r\})$.

Let $y = 3$. Thus $i < k < 5$ implies $\{i, k, j, r\} \subset \{4, 2, 1\}$ since $3 \notin \{i, k, j, r\}$.

Further $k > i \geq j > r$ implies $k = 4$, $i=j=2$, $r=1$.

$\therefore G(\{i, y, j\}) = 23 > 14 = G(\{k, y, r\})$

Let $y = 2$. Thus $i < k < 5$ implies $\{i, k, j, r\} \subset \{4, 3, 1\}$ since $2 \notin \{i, k, j, r\}$.

Further $k > i \geq j > r$ implies $k = 4$, $i = j = 3$, $r = 1$.

$\therefore G(\{i, y, j\}) = 23 > 14 = G(\{k, y, r\})$.

Let $y = 1$. Thus $1 < k < 5$ implies

$\{i, k, j, r\} \subset \{4, 3, 2\}$ since $1 \notin \{i, k, j, r\}$.

Further $k > i \geq j > r$ implies $k = 4$, $i = j = 3$, $r = 2$, contradicting $\{k, r\} \neq \{4, 2\}$.

Hence, we may conclude that if $G(\{i, j\}) > G(\{k, r\})$ with $i \geq j$ and $k \geq r$ and if $y \notin \{i, j, k, r\}$ then $G(\{i, y, j\}) \geq G(\{k, y, r\})$. Thus by Theorem 1, G satisfies GP and WI.   Q.E.D.

Remark 1 :- Given $n \in N$, let $q \in N$ such that $10^q > n$. Define $F : [N_n] \to N$ as follows :
$F_1(A) = 10^q M(A) + m(A) \ \forall A \in [N_n]$. Define $F_2(A) = 10^q m(A) + M(A) \ \forall \ A \in [N_n]$.
Both F_1 and F_2 are 1-dimensional extensions satisfying GP, as is easily verified. However, neither F_1 nor F_2 satisfies WI. For let $n \geq 5$. Then, $F_1(\{2, 4\}) > F_1(\{3\})$ but, $F_1(\{5, 3\}) > F_1(\{2, 4, 5\})$. Similarly, $F_2(\{3\}) > F_2(\{2, 4\})$ but, $F_2(\{3, 1\}) > F_2(\{2, 4, 1\})$.

The following result can be found in Bossert(1989).The simple proof is being provided for completeness.

Theorem 1: Let $F: [N_n] \to N^p$ be a p-dimensional extension function satisfying GP and WI.Then for all $A \in [N_n]$, $F(A)=F(\{M(A),m(A)\})$.

Proof: For #(A) equal to one or two the theorem is self evident.Hence assume #(A)>2. Let $A=\{j_1,...,j_k\} \in [N_n]$,with k>2 and $j_i < j_{i+1}$,for all $i \in \{1,...,k-1\}$.Hence $m(a)= j_1$ and $M(A)= j_k$.By successive applications of GP,$F(\{j_k\}) >> F(\{j_2,...,j_k\})$ and by WI, $F(\{j_1,j_k\}) \geq F(A)$.Similarly,by successive applications of GP, $F(\{j_1,...,j_{k-1}\}) >> F(\{j_1\})$ and by WI, $F(A) \geq F(\{j_1,j_k\})$.Hence the theorem.   Q.E.D.

Example due to Kannai and Peleg (1984): Let $F: [N_n] \to N^2$ be defined by $F(A)=(m(A),M(A))$>Then F satisfies GP and WI.

Corollary 1 of Theorem 1: Let $F: [N_n] \to N^p$ be a p-dimensional extension function satisfying GP and WI.Then for all $A \in [N_n]$ with #(A) ≥ 2, $F(A)>>F(A\setminus\{y\})$ implies y=M(A) and $F(A\setminus\{y\})>>F(A)$ implies y=m(A).

Proof:By Theorem 1, $F(A)=F(\{M(A),m(A)\})$,so that if $y \notin \{M(A),m(A)\}$,then $F(A)=F(A\setminus\{y\})$.On the other hand as a consequence of GP,y=M(A) implies $F(A)>>F(A\setminus\{y\})$ and y=m(A) implies $F(A\setminus\{y\})>>F(A)$.Hence the corollary.   Q.E.D.

Proposition 1: Let $F: [N_n] \to N^p$ be a p-dimensional extension function satisfying GP.Then for all $i,j,k,r \in N_n$, $i \geq j \geq r, i \geq k \geq r$ implies $F(\{i,j\}) \geq F(\{k,r\})$.Further if either i>k or j>r,then $F(\{i,j\}) >> F(\{k,r\})$.

Proof:If i=k and j=r,there is nothing to prove.Hence assume that either i>k or j>r.Suppose i>k.Hence $M(\{i,j,k\})=i$ and $m(\{i,j,k\}) \leq j$. By GP, $F(\{i,j\}) >> F(\{k,j\})$.Now $j \geq r$ implies $F(\{k,j\})= F(\{k,r\})$ if j=r,and $F(\{k,j\}) >> F(\{k,r\})$ if j>r,where the latter follows from GP. Combining the inequalities,we get the desired result for the case i>k.A similar conclusion obtains for the case j>r.   Q.E.D.

We now prove a partial converse of Theorem1.

Theorem 2: Let $F: [N_n] \to N^p$ be a p-dimensional extension function such that for all $A \in [N_n]$, $F(A)=F(\{M(A),m(A)\})$.Suppose:

(a)for all $i,j,k \in N_n, [i \geq j > k$ implies $F(\{i,j\}) \geq F(\{i,k\})]$;

(b)for all $i,j,k \in N_n, [k > i \geq j$ implies $F(\{k,j\}) \geq F(\{i,j\})]$;

(c)for all $i,j,k,r,y \in N_n$, with $i \geq j$, $k \geq r$ and $y \notin \{i,j,k,r\}, [F(\{i,j\}) >> F(\{k,r\})$ implies $F(\{i,j,y\}) \geq F(\{k,r,y\})]$.

Then F satisfies GP and WI.

Proof: Follows easily from the following: for $y \in N_n$:
(i)   if $A \in [N_n]$ and y < m(A), then $M(A\cup\{y\}) = M(A)$ and $m(A\cup\{y\}) = y$;
(ii) if $A \in [N_n]$ and y > M(A), then $M(A\cup\{y\}) = y$ and $m(A\cup\{y\}) = m(A)$;
(iii) if $A \in [N_n]$, then $M(A\cup\{y\}) = M(\{M(A),y\})$ and $m(A\cup\{y\}) = m(\{m(A),y\})$. Q.E.D.
Kannai and Peleg [1984] proved the following :

Theorem 3 : Let $n \geq 6$. Then there does not exist any 1-dimensional extension satisfying GP and WI.

question that naturally arose out of this theorem is :For n =5, does there exist any 1-ensional extension satisfying GP and WI? The answer to this implied by Theorem 1 in dopadhyay [1988] is the following :

orem 4: Let $F : [N_5] \rightarrow N$ be defined as follows :

$\quad$ A) = 10 + M(A) if $1 \in M(A)$

$\qquad$ = 33 if A = {2, 4}

$\qquad$ = 10 M(A) + m(A), otherwise.

ıen F is a 1-dimensional extension function satisfying GP and WI.

fact the conclusion of Theorem 1 in Bandopadhyay [1988] is equivalent to the onclusion of Theorem 4 above, and as we shall see subsequently,by providing an ıalogous but different result,that the proof in Bandopadhyay [1988] would have been ıuch simpler, had the theorem there been Theorem 4 of this paper.

)ur proposal is the following : Let $G : [N_5] \rightarrow N$ be defined by

$\quad$ G(A) = 50 + m(A) if $5 \in A$

$\qquad$ =33 if M(A), m(A)) = (4,2)

$\qquad$ =10 m(A) + M(A), otherwise.

It is easy to see that G is indeed an extension. Further, $G(A) = G(\{M(A), m(A)\})$ for all A $\in [N_5]$.

<u>Lemma 1</u> : Let i, j, k, r $\in N_n$ with $i \geq j \geq r$, $i \geq k \geq r$, $G(\{i,j\}) \geq G(\{k, r\})$. Further if either i>k or j>r,then $G(\{i,j\}) >> G(\{k,r\})$.

Proof :- Easily verified.

<u>Note</u> :- $G(\{5,1\}) = 51 > 33 = G(\{2,4\})$. However, if F is as defined in Theorem 4, then $F(\{1,5\}) = 15 < 33$. Hence the rankings of the non-empty subsets of N_5 given by F and G are indeed different.

<u>Theorem 5</u> :- G is a 1-dimensional extension satisfying GP and WI.

Proof :- That G is a 1-dimensional extension has already been observed. Similarly, (a) and (b) of Theorem 2 are easily verified (∴hence G satisfies GP). Thus it remains to show that (c) of Theorem 2 holds as well. Let i, j, k, r, y $\in N_5$ with $i \geq j$, $k \geq r$ and $y \notin \{i, j, k, r\}$. Suppose $G(\{i, j\}) > G(\{k, r\})$. Suppose $i \geq k$ and $j \geq r$. Then $M(\{i, y\}) \geq M(\{k, y\}) \geq m(\{r, y\})$ and $M(\{i, y\}) \geq m(\{j, y\}) \geq m(\{r, y\})$. Hence $G(\{i, j, y\}) = G(\{M(\{i, y\}), m(\{j, y\})\}) \geq G(\{M(\{k, y\}), m(\{r, y\})\})$ (∴ as a consequence of Lemma 1). However, $G(\{M(\{k, y\}), m(\{r, y\})\}) = G(\{k, y, r\})$. Hence $G(\{i, y, j\}) \geq G(\{k, y, r\})$.

Hence assume either (i) i < k and j > r or (ii) i > k and j < r. (The remaining case is excluded by Lemma 1 and the requirement that $G(\{i, j\}) > G(\{k, r\})$).

<u>Case 1</u> :- k = 5: Then $G(\{k, r\}) = 50 + r$. Now $G(\{i, j\}) > G(\{k, r\})$ implies i = 5. Hence, $G(\{i, j\}) = 50 + j$. Thus j > r and i ≥ k contradicting both (i) and (ii).Hence Case 1 is ruled out.

<u>Case 2</u> :- k < 5, i = 5: Thus y < 5.Thus $F(\{i, y, j\}) \geq 50 > F(\{k, y, r\})$, since $m(\{r,y\}) \leq M(\{k, y\}) < 5$.

<u>Case 3</u> :- i < 5, {k, r} = {4,2}: ∴ $G(\{k, r\}) = 33$ Now $G(\{i, j\}) > G(\{k, r\})$ implies {i, j} ≠ {4, 2}, {i, j} ≠ {3} ∴ $G(\{i, j\}) = 10j + i$.

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