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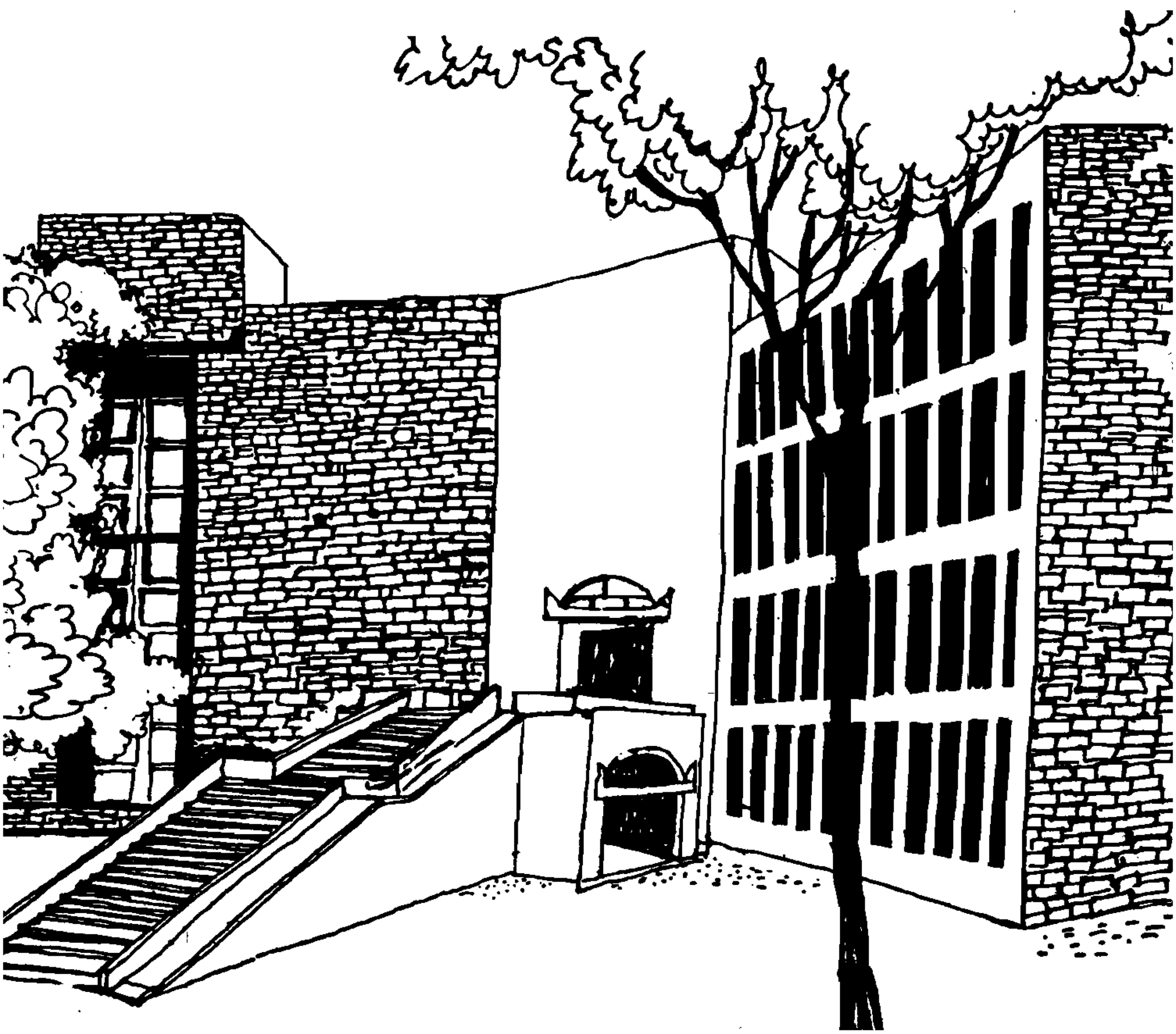

AXIOMATIC CHARACTERIZATIONS OF VOTING OPERATORS

By

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Axiomatic Characterizations of Voting Operators

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Abstract. In this paper we provide a model for the analysis of the electoral process. We prove some theorems in this paper, which characterize some voting operators. Apart from other well-known voting operators existing in the literature, we also provide an axiomatic characterization of the first past the post voting rule. In a final section, we take up the problem of a rationalizable voting operators. It is observed that except in the trivial case where every feasible alternative that is voted for by somebody, is chosen, no other voting operator is rationalizable. However, we are able to offer a necessary and sufficient condition for voting operators, to always select the best elements from the feasible set, according to a reflexive, complete and transitive binary relations.

1 Introduction

We consider a problem, in which a group of persons participate in collective choice of options.

Historically, this problem has its origin in the seminal work of Arrow, followed by a stream of publications : Black (1958), Black (1969), Bouysson (1992), Brown (1975), Hardin (1962), Kelly (1978), Kim and Roush (1980), Kirman and Sonderman (1972), Moulin (1988) [which contains significant original contributions], Pattanaik (1976), Sen (1977), Vincke (1982), Wilson (1972). In this approach, agents or voters are assumed to announce their preferences over options, and the voting rule aggregates them into a social preference. Alternatively, on the basis of the announced preferences, society elects collective outcomes. A good introduction to the literature can be found in French (1986).

A significant modification to this approach was brought about by Aizerman and Aleskerov (1986), who instead assumed that individual voters announce their own decision rules on the basis of which they select outcomes from a given set of outcomes. Although they make significant use of the individual decision rules in their approach, their essential philosophy is that the chosen alternatives of the individuals (i.e., profiles of individual ballots) are converted into social choice of outcomes by a voting operator. Further results along this approach can be found in Ilyunin, Popov and Elkin [1998] and Stefanescu [1997].

In this paper we push this approach further. We assume that what each individual announces is merely his set of chosen options, from a prespecified set of given options. The announced set is called a ballot for the individual. On the basis of the ballot profile, society now elects its outcomes.

Apart from providing a model for the analysis, we prove some theorems in this paper, which characterize some voting operators. Our voting operators are analogous to the voting operators discussed in Aizerman and Aleskerov [1986, 1995]. We also characterize axiomatically, the first past the post voting rule. This voting rule, is the basis of Westminster style parliamentary democracies as practised for instance in India and U.K. Oligarchy - a system where a subset of the voters are in a position to assert their common choices - is axiomatically characterized in a subsequent section.

In a final section, we take up the problem of rationalizable voting operators. It is observed that except in the trivial case, where every feasible alternative that is voted for by somebody, is chosen, no other voting operator is rationalizable. However, by modifying a property due to Deb [1983], we are able to offer a necessary and sufficient condition for voting operators, to always select the best elements from the feasible set, according to a reflexive, complete and transitive binary relation.

2 The Model

Let n be a natural number greater than or equal to two. Let $N = \{1,\ldots,n\}$ be the set of agents or voters. Let X be a non-empty universal set of alternatives and let Σ be the set of all finite subsets of X (including the empty set ϕ). An element of Σ is called a <u>ballot</u>.

Let Σ^N denote the set of all functions from N to Σ. Any element $(S_1,\ldots,S_n) \in \Sigma^N$, is called a <u>ballot profile</u>.

A voting operator is a correspondence $C : \Sigma^N \rightarrow\rightarrow X$ such that for all $(S_1,\ldots S_n) \in \Sigma^N$: (1) $C(S_1,\ldots,S_n) \subset \bigcup_{i\in N} S_i$; (2) if $\bigcap_{i\in N} S_i \neq \phi$, then $C(S_1,\ldots,S_n) \cap (\bigcap_{i\in N} S_i) \neq \phi$. In order to distinguish a correspondence (i.e., a set valued mapping) from a point valued function we use double arrows.

Thus an element which appears on the ballot of every individual is invariably chosen, and an element which appears on no ballot is never chosen. As a consequence of our definition it easily follows that given any $x \in X$, there exists $(S_1,\ldots,S_n) \in \Sigma^N$ such that $\{x\} = C(S_1,\ldots,S_n)$: simply take $S_i = \{x\}$ $\forall$ $i \in N$.

In the sequel we will be considering the following properties of voting operators :

Monotonicity : Let $x \in C(S_1,\ldots,S_n)$ and let $(S_1,\ldots,S_n)$ and $(T_1,\ldots,T_n) \in \Sigma^N$ with $\{i \in N / x \in S_i\} \subset \{i \in N / x \in T_i\}$. Then $x \in C(T_1,\ldots, T_n)$.

Neutrality with regard to options : Let $x, y \in X$ and $(S_1,\ldots,S_n)$, $(T_1,\ldots,T_n) \in \Sigma^N$. Suppose $\{i \in N/x \in S_i\} = \{i \in N/y \in T_i\}$.Then $x \in C(S_1,\ldots,S_n) \leftrightarrow y \in C(T_1,\ldots,T_n)$.

Context independence : Let $x \in X$ and $(S_1,\ldots,S_n)$, $(T_1,\ldots,T_n)$ Σ^N. Suppose $\{i \in N/x \in S_i\} = \{i \in N/x \in T_i\}$. Then, $x \in C(S_1,\ldots,S_n) \leftrightarrow x \in C(T_1,\ldots,T_n)$.

Option independence : Let $x, y \in X$ and $(S_1,\ldots,S_n) \in \Sigma^N$. If $\{i \in N/x \in S_i\} = \{i \in N/y \in S_i\}$, then $x \in C(S_1,\ldots,S_n) \leftrightarrow y \in C(S_1,\ldots,S_n)$.

Note :

a) Neutrality with regard to options implies context independence;
b) Neutrality with regard to options implies option independence.

Anonymity : Let $\eta : N \to N$ be a onto function and suppose $(S_1,...,S_n) \in \Sigma^N$, $(T_1,...,T_n) \in \Sigma^N$ with $T_{\eta(i)} = S_i \; \forall \; i \in N$. Then $C(S_1,...,S_n) = C(T_1,...,T_n)$.

Let $C : \Sigma^N \twoheadrightarrow X$ be a voting operator.

Definition : C is said to be a federation operator if there exists $\Omega = \{w_1,...,w_q\}$, a collection of nonempty subsets of N, such that

$$C(S_1,...,S_n) = \bigcup_{j=1}^{q} \bigcap_{i \in w_j} S_i \; \forall \; (S_1,...,S_n) \in \Sigma^N .$$

Definition : C is said to be a representative operator if there exists $E = \{e_1,...,e_p\}$, a collection of nonempty subsets of N such that

$$C(S_1,...,S_n) = \bigcap_{j=1}^{q} \bigcup_{i \in e_j} S_i \; \forall \; (S_1,...,S_n) \in \Sigma^N .$$

Note : Since a finite union of a family comprising a finite intersection of sets is always representable as a finite intersection of a family comprising a finite union of sets, and vice-versa, a federation operator is always a representation operator and a representation operator is always a federation operator.

Observe if $\Omega = \{w_1\}$, then we can take $E = \{\{i\} \,/\, i \in w_1\}$. If $E = \{e_1\}$, then we can take $\Omega = \{\{ i \} \,/\, i \in e_1\}$.

c) C is said to be an oligarchy if C is a federation operator with $\Omega = \{ w_1 \}$.
d) C is said to be minimal if C is a representative operator with $E = \{ e_1\}$.
e) C is said to be a k-votes operator (: where 'k' is a positive integer with $k \leq n$) if C is a federation operator with $\Omega = \{ w \subset N \,/\, w$ has exactly k elements$\}$.
   A k-votes operator selects only those elements which appear on at least k-ballots.
f) C is said to be dictatorial if there exists $i \in N$ (: called a dictator) such that $C(S_1,...,S_n) = S_i \; \forall \; (S_1,...,S_n) \in \Sigma^N$.

3 Characterization of Federation and Representation Operators

The first two theorems are minor modifications of those in Aizerman and Aleskerov [1986] and are being provided along with their proofs for the sake of clarity and ease in later reference.

Theorem 1 : a. A voting operator satisfies monotonicity and neutrality with regard to options if and only if it is a federation operator.
b. A voting operator satisfies monotonicity and neutrality with regard to options if and only if it is a representation operator.

Proof : In view of the note following the definition of a representation operator, it is enough to prove (a).

It is easy to verify that a federation operator satisfies monotonicity and neutrality with regard to options. Hence assume C is a voting operator satisfying monotonicity and neutrality with regard to options. We will show that it is a federation operator.

Let $x \in X$ and let w be any subset of N such that $T_i = \{ x \} \ \forall\ i \in w$ and $T_i = \phi$ if $i \notin w$ implies $C(T_1,\ldots,T_n) = \{ x \}$. Such a subset will be called a decisive set.

A minimal decisive set is any decisive set such that it does not contain any proper subset which is also a decisive set. Let $\Omega = \{ w_1,\ldots,w_q)$ be the collection of minimal decisive sets. By neutrality with regard to options, Ω is independent of x. By monotonicity,

$$\bigcup_{j=1}^{q}\left(\bigcap_{i\in w_j} T_i\right)\subset C(T_1,\ldots,T_n)\ \forall\,(T_1,\ldots,T_n)\in\Sigma^N$$

(: Let $x \in \bigcup_{j=1}^{q}\left(\bigcap_{i\in w_j} T_i\right)$; thus there exists w_m such that $x\in \bigcap_{i\in w_m} T_i$; thus $x \in C(S_1',\ldots,S_n')$ where $S_i' = \{ x \}$ if $i \in w_m$

$= \phi$ otherwise.

By monotonicity $x \in C(S_1,\ldots,S_n)$ where $S_i = \{ x \}$ if $i \in w_m$

$= T_i$ otherwise.

By monotonicity $x \in C(T_1,\ldots,T_n)$.)

Let $(T_1,\ldots,T_n) \in \Sigma^N$ and suppose $x \in C(T_1,\ldots,T_n) \setminus (\bigcup_{j=1}^{q}\bigcap_{i\in w_j} T_i)$

Let $w = \{ i \in N/x \in T_i\}$.

Clearly, there does not exist $w_j \in \Omega$ such that $w_j \subset w$.

Let $S_i = \{ x \}$ if $i \in w$

$= \phi$ otherwise.

Clearly, $x \notin C(S_1,\ldots,S_n)$. Hence, by monotonicity, $x \notin C(T_1,\ldots,T_n)$, which is a contradiction.

Thus, $C(T_1,\ldots,T_n) \subset \bigcup_{j=1}^{q}\bigcap_{i\in w_j} T_i$. Thus C is a federation operator.

Q.E.D.

As a corollary to Theorem 1, we have the following theorem :

Theorem 2 : A voting operator satisfies monotonicity, neutrality with regard to options and anonymity if and only if it is a k-votes operator.

Proof : It is easy to see that a k-votes operator satisfies the desired properties. Hence assume C is a voting operator satisfying monotonicity, neutrality with regard to options and anonymity. By Theorem 1, there exists $\Omega = \{ w_1,\ldots,w_q \}$, $\phi \neq w_j \subset N\ \forall\ j = 1,\ldots,q$ such that

$C(S_1,...,S_n) = \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} S_i) \; \forall \, (S_1,...,S_n) \in \Sigma^N$. Let, $\phi \neq w \subset N$ such that cardinality of w is equal to the cardinality of w_1.

Let $(T_1,...,T_n) \in \Sigma^N$ such that

$$T_i = S_i \; \forall \, i \notin w \cup w_1$$
$$T_{\rho(i)} = S_i \; \forall \, i \in w \cup w_1$$

where $\rho : N \to N$ is onto such that

$\rho(i) = i \; \forall \, i \notin w \cup w_1$

$\rho(w) = w_1$

$\rho(w_1) = w$.

By anonymity, $C(T_1,...,T_n) = C(S_1,...,S_n)$.

Thus, $w \in \Omega$.

Thus, C is a k-votes operator.

Q.E.D.

The following axiom is essentially due to Ilyunin, Popov and Elkin [1998].

Weak Neutrality with regard to options: Let $\sigma : X \to X$ be a bijection and let $(S_1,...,S_n), (T_1,...,T_n) \in \Sigma^N$ with $T_i = \{\sigma(x) / x \in S_i\} \; \forall i \in \{1,...,n\}$.
Then

$$C(T_1,...,T_n) = \{\sigma(x) / x \in C(S_1,...,S_n)\}.$$

Theorem 3 : A voting operator satisfies Monotonicity. Anonymity and Weak Neutrality with regard to options if and only if it is a k-votes operator.

Proof : That a k-votes operator satisfies the above properties is easily verified. Hence let C be a voting operator satisfying the above properties. Let $x \in X$. Let, $k = \min\{r \in N / S_i = \{x\} \; \forall \, i = 1,...,r;\; S_i = \phi$, otherwise implies $C(S_1,...,S_n) = [x]\}$.

By weak neutrality with regard to options, k is independent of x. By anonymity, $k = \min \{r \in N / \left|\{ i \in N / S_i = \{x\} \right| = r ,\; S_i \subset \{x\} \; \forall \, i$ implies $C(S_1,...,S_n) = \{x\}\}$

By monotonicity $y \in C(S_1,...,S_n)$ whenever $\left|\{i \in N / y \in S_i \}\right| \geq k$.

Now suppose $y \in C(S_1,...,S_n)$ and towards a contradiction suppose $\left|\{i \in N / y \in S_i \}\right| < k$. Let $T_i = \{y\}$ if $y \in S_i$.

$T_i = \phi$ otherwise.

By monotonicity, $y \in C(T_1,...,T_n)$ and yet $\left|\{i \in N/y \in T_i \}\right| < k$. This contradicts the minimality of k. Hence the theorem.

Q.E.D.

Suppose X is finite. Given $S \in \Sigma$, let $1_s : X \to X$ be the function defined as follows :

$$1_s(x) = 1 \text{ of } x \in S$$
$$= 0 \text{ otherwise.}$$

Given $S,T \in \Sigma$, let $d(S, T) = \sum_{x \in X} \left| 1_s(x) - 1_T(x) \right|$

d(S, T) is called the distance between S and T.

The median voting operator (see Ilyunin, Popov and Elkin [1988] $C_M : \Sigma^N \to \Sigma$ is defined as follows : $\forall\ (S_1,\dots,S_n) \in \Sigma^n$,

$$\sum_{i\in N} d(C_M(S_1,\dots, S_n),S_i) \le \sum_{i\in N} d(T, S_i)\ \forall T\in \Sigma$$

Let $x \in X$ and suppose $(S_1,\dots,S_n) \in \Sigma^N$ with $|\{i \in N / x \in S_i\}| < N/2$.

Suppose $x \in T$. Then

$$\sum_{i\in N} d(T,S_i) = \sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|+|1_T(x)-1_{S_i}(x)|\right]$$
$$= \sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|+|1-1_{S_i}(x)|\right]$$
$$= \sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|\right]+\sum_{i\in N/x\notin S_i} 1$$
$$> \frac{N}{2}+\sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|\right]$$

On the other hand

$$\sum_{i\in N} d(T\setminus\{x\},S_i) = \sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|+|0-1_{S_i}(x)|\right]$$
$$> \frac{N}{2}+\sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|\right]$$

Thus $x \notin C_M(S_1,\dots,S_n)$. Hence, $C_M(S_1,\dots,S_n) \subset \{x \in X / |\{ i \in N / x \in S_i\}| \ge N/2\}$.

Now suppose $x \in X$ and suppose $(S_1,\dots,S_n) \in \Sigma^N$ with $|\{i \in N / x \in S_i\}| > N/2$.

Suppose $x \notin T$. Then,

$$\sum_{i\in N} d(T,S_i) = \sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|+|0-1_{S_i}(x)|\right]$$
$$> \frac{N}{2}+\sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|\right]$$

$$\sum_{i\in N} d(T\cup\{x\},S_i) > \frac{N}{2}+\sum_{i\in N}\left[\sum_{y\neq x}|1_T(y)-1_{S_i}(y)|\right]$$

Thus, $x\in C_M(S_1,\dots,S_n)$. Hence $\left\{x \in X / |\{i \in N / x \in S_i\}| > \frac{N}{2}\right\} \subset C_M(S_1,\dots,S_n)$.

4. Characterization of First Past the Post Rule

Let $(T_1,...,T_n) \in \Sigma^N$ and let $x \in \bigcup_{i \in N} T_i$. Let $r(x; T_1,...,T_n)$ be the cardinality of the set $\{i \in N / x \in T_i\}$. A voting rule $C : \Sigma^N \to\to x$ is said to be the first past the post rule if

$C(T_1,...,T_n) = \{ x \in \bigcup_{i \in N} T_i / r(x;T_1,...,T_n) \geq r(y;T_1,...,T_n) \; \forall \; y \in \bigcup_{i \in N} Ti\}$

We need three other properties to characterize this rule :

Non-emptiness : Let $(S_1,...,S_n) \in \Sigma^N$. Unless $S_i = \phi \; \forall \; i \in N$, $C(S_1,...,S_n) \neq \phi$.

Minimal-responsiveness : Let $(S_1,...,S_n) \in \Sigma^N$ and suppose $x, y \in C(S_1,...,S_n)$ with $x \neq y$. Suppose $x \in S_1$. Let $T_1 = S_1 \setminus \{x\}$ and $T_i = S_i \; \forall \; i \neq 1$.
Then $C(T_1,...,T_n) = C(S_1,...,S_n) \setminus \{x\}$.

Option Monotonicity : Let $x \in C(S_1,...,S_n)$ with $(S_1,...,S_n) \in \Sigma^N$. If $y \in X$ and $\{i \in N / x \in S_i\} \subset \{i \in N/y \in S_i\}$, then $y \in C(S_1,...,S_n)$.

We are now in a position to characterize the first past the post rule.

Theorem 4 : A voting rule satisfies anonymity, non-emptiness, minimal responsiveness and option montonicity if and only if it is the first past the post rule.

Proof : It is easy to verify that the first past the post rule satisfies the above mentioned properties. Hence assume C is a voting rule which satisfies the above properties. Let $(S_1,...,S_n) \in \Sigma^N$ and suppose $C(S_1,...,S_n) = \{x_1,...,x_q\}$.

Let $w_j = \{i \in N / x_j \in S_i\}$.

Thus $C(S_1,...,S_n) \subset \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} S_i)$. Note $x_j \in \bigcap_{i \in w_j} S_i$.

Let $x \in \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} S_i)$. Thus $x \in (\bigcap_{i \in w_j} S_i)$ for some j. Thus, $w_j \subset \{i \in N / x \in S_i\}$.

By option monotonicity, $x \in C(S_1,...,S_n)$.

Thus, $C(S_1,...,S_n) = \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} S_i)$.

By anonymity and an argument similar to that used in the proof of Theorem 2, there exists a positive integer 'k' such that $\Omega = \{w \subset N$ / w has exactly 'k' elements$\}$.

Since, C satisfies non-emptiness,

$\{x \in \bigcup_{i \in N} S_i / r(x;S_1,...,S_n) \geq r(y;S_1,...,S_n) \; \forall \; y \in \bigcup_{i \in N} S_i \} \subset C(S_1,...,S_n)$.

Let $x, y \in C(S_1,...,S_n)$ with $x \neq y$ ($\because$ if $C(S_1,...,S_n)$ is a singleton then by the above it selects the first past the post winner). Suppose $r(y; S_1,...,S_n) < r(x;S_1,...,S_n)$. Without loss of generality and by anonymity assume, $x \in S_1$. Let $T_1 = S_1 \setminus \{x\}$ and $T_i = S_i \; \forall \; i \neq 1$.It is easy to check that owing to minimal responsiveness (i.e., $C(T_1,...,T_n) = C(S_1,...,S_n) \setminus \{x\}$),

$$C(T_1,...,T_n) = \bigcup_{w \in \Omega} (\bigcap_{i \in w} T_i),$$

where $\overline{\Omega}$ = {w/w has exactly $\bar{k}$ elements} and, $y \in C(T_1,...,T_n)$. Thus $\bar{k} \leq r(y;T_1,...,T_n)$.

Now $r(x;T_1,...,T_n) = r(x;S_1,...,S_n) - 1 \geq r(y; S_1,...,S_n) = r(y;T_1,...T_n) \geq \bar{k}$.

Thus $x \in C(T_1,..,T_n)$ which is a contradiction.

Thus $x,y \in C(S_1,...,S_n)$ with $x \neq y \rightarrow r(y;S_1,...,S_n) = r(x;S_1,...,S_n)$.

Thus $C(S_1,...,S_n)$ consists only of the first past the post winners.

Q.E.D.

5. Oligarchy

We have already characterized a federation operator, using the axioms of monotonicity and neutrality with regard to options. It is worthwhile investigating what additional property would be required in order to characterize an oligarchy. It turns out the following assumption is sufficient.

Independence of Irrelevant Voters : Let $x,y \in C(S_1,...,S_n)$, $(S_1,...,S_n) \in \Sigma^N$. Let $j \in N \setminus \{i \in N/x \in S_i$ and $y \in S_i\}$. If $(T_1,...,T_n) \in \Sigma^N$ with $T_i = S_i \ \forall\ i \neq j$, $T_j = S_j \setminus \{x,y\}$, then $x,y \in C(T_1,...,T_n)$.

Theorem 5 : A voting rule satisfies monotonicity, neutrality with regard to options and independence of irrelevant voters if and only if it is an oligarchy.

Proof : It is easy to see that an oligarchy satisfies the desired properties. Hence assume C is a voting rule which satisfies the properties mentioned in the theorem. By Theorem 1, C must be a federation operator i.e.

$$C(S_1,...,S_n) = \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} S_i, \ \forall\ (S_1,...,S_n) \in \Sigma^N,$$

where $\Omega = \{w_1,...,w_q\}$, $\phi \neq w_j \subset N$.

We claim $w_i = w_j \ \forall\ i, j$. Suppose and that too without loss of generality that w_1 is not a subset of w_2 and w_2 is not a subset w_1.

Suppose $h \in w_2 \setminus w_1$.

$$\text{Let } S_i = \{x\} \ \forall\ i \in w_1 \setminus w_2$$
$$= \{x,y\} \ \forall\ i \in w_1 \cap w_2$$
$$= \{y\} \ \forall\ i \in w_2 \setminus w_1$$
$$= \phi \ \forall\ i \in N \setminus (w_1 \cup w_2)$$

Thus $C(S_1,S_2,...,S_n) = \{x,y\}$.

Let $(T_1,...,T_n) \in \Sigma^N$ with $T_i = S_i \ \forall\ i \neq h$ and $T_h = S_h \setminus \{x,y\}$.

Thus $T_h = \phi$. Thus $C(T_1,...,T_n) = \{x\}$, contradicting independence of irrelevant voters.

Thus, either $w_1 \subset w_2$ or $w_2 \subset w_1$.

By minimality of the decisive coalitions, $w_1 = w_2$.

w_1 and w_2 being arbitrary, we conclude that C is an oligarchy.

Q.E.D.

6 Weakly Ordinal Voting Operators

In this section, we are interested in asserting rationalizability properties of voting operators.

A voting operator $C : \Sigma^N \to\to X$ is said to be binary if there exists a binary relation R on X such that for all $(S_1,\ldots, S_n) \in \Sigma^N$, $C(S_1,\ldots,S_n) = \left\{x \in \bigcup_{i=1}^{n} S_i \,/\, (x,y) \in R \,\forall\, y \in \bigcup_{i=1}^{n} S_i\right\}$.

Proposition 1 : Let $C(S_1,\ldots,S_n) = \bigcup_{i=1}^{n} S_i \;\forall\; (S_1,\ldots,S_n) \in \Sigma^N$. Then there exists a reflexive, complete and transitive binary relation R on X such that $C(S_1,\ldots,S_n) = \left\{x \in \bigcup_{i=1}^{n} S_i \,/\, (x,y) \in R \,\forall\, y \in \bigcup_{i=1}^{n} S_i\right\}$

Proof : Take $R = X \times X$.

Q.E.D.

For $A \subset X$ and R a binary relation on X, let $G(A,R) = \{x \in A/(x,y) \in R \;\forall\; y \in A\}$.

Note : A binary relation R on X is said to be (a) reflexive if $(x,x) \in R \;\; \forall\, x \in X$; (b) complete if $\forall\, x,y \in X$, $x \neq y$, either $(x,y) \in R$ or $(y,x) \in R$; (c) transitive if $\forall\, x,y,z \in X$, $[(x,y) \in R \;\&\; (y,z) \in R] \to (x,z) \in R$. A binary relation R which satisfies (a),(b) and (c) is called an ordering.

Suppose $n \geq 2$ and X has atleast two elements.

Theorem 6 : Let $C: \Sigma^N \to\to X$ be a voting operator such that for some $(S_1,\ldots,S_n) \in \Sigma^N$, $C(S_1,\ldots,S_n) \neq \bigcup_{i=1}^{n} S_i$. Then there does not exist any binary relation R on X such that $C(T_1,\ldots,T_n) = G(\bigcup_{i=1}^{n} T_i, R)$ whenever $(T_1,\ldots,T_n) \in \Sigma^N$.

Proof : Towards a contradiction assume the theorem is false. Suppose $x \in C(S_1,\ldots,S_n)$ and $y \in \bigcup_{i=1}^{n} S_i \setminus C(S_1,\ldots S_n)$. Thus there exists $z \in \bigcup_{i=1}^{n} S_i$ such that $(y,z) \notin R$.

Let $(T_1,\ldots,T_n) \in \Sigma^N$ with $T_1 = \{y\}$ and $T_i = \{y,z\} \;\forall\; i \in \{2,\ldots,n\}$. Since $\bigcap_{i \in N} T_i = \{y\} \neq \phi$, $y \in C(T_1,\ldots,T_n) = G(\bigcup_{i=1}^{n} T_i, R)$. Hence $y \in G(\{y,z\},R)$. Thus $(y,z) \in R$ contradicting $(y,z) \notin R$. This proves the theorem.

Q.E.D.

Hence there is no other voting operator, other than the trivial one indicated in Proposition 1 which is binary. Hence we must abandon our search for a binary voting operator.

Deb[1983] has proposed the concepts of weakly binary and weakly ordinal choice functions. We modify those concepts here to fit in with our concept of a voting operator.

A voting operator $C: \Sigma^N \rightarrow\rightarrow X$ is said to be weakly ordinal if there exists a binary relation R or X such $\phi \neq G(\bigcup_{i=1}^{n} S_i, R) \subset C(S_i,...,S_n)\ \forall\ (S_1,...,S_n) \in \Sigma^N \backslash \{(\phi,...,\phi)\}$.

A voting operator $C: \Sigma^N \rightarrow\rightarrow X$ is said to be weakly ordinal if there exists an ordering R or X such $\phi \neq G(\bigcup_{i=1}^{n} S_i, R) \subset C(S_1,...,S_n)\ \forall\ (S_1,...,S_n) \in \Sigma^N \backslash \{(\phi,...,\phi)\}$.

The following property (a*) uniquely characterizes all weakly ordinal voting operators provided X is finite.

Property (a*) : For all $(S_1,...,S_n) \in \Sigma^N \setminus \{(\phi,...\phi)\}$, there exists $x_0 \in C(S_1,...,S_n)$ (possibly depending on the ballot profile) such that $(T_1,...,T_n) \in \Sigma^N \setminus \{(\phi,...,\phi)\}$, $x_0 \in \bigcup_{i=1}^{n} T_i \subset \bigcup_{i=1}^{n} S_i$ implies $x_0 \in C(T_1,...,T_n)$.

Theorem 7 : Let X be finite. Then a voting operator is weakly ordinal if and if only it satisfies Property (a*).

Proof : Let $C : \Sigma^N \rightarrow\rightarrow X$ be weakly ordinal with respect to the ordering R. Given $(S_1,...,S_n) \in \Sigma^N \setminus \{(\phi,...,\phi)\}$, $G(\bigcup_{i=1}^{n} S_i,R) \neq \phi$. Let $x_0 \in G(\bigcup_{i=1}^{n} S_i,R)$.

Then $(T_1,...,T_n) \in \Sigma^N \setminus \{(\phi,...,\phi)\}$, $\bigcup_{i=1}^{n} T_i \subset \bigcup_{i=1}^{n} S_i$ and $x_0 \in \bigcup_{i=1}^{n} T_i$ implies $x_0 \in G(\bigcup_{i=1}^{n} T_i, R) \subset C(T_1,...,T_n)$.

Thus C satisfies Property (a*).

Now suppose C satisfies Property (a*). Let $\phi \neq A \subset X$. Let $(S_1,...,S_n) \in \Sigma^N$ with $\bigcup_{i=1}^{n} S_i = A$. By Property (a*), there exists $x_0 \in C(S_1,...,S_n)$ such that if $(T_1,...,T_n) \in \Sigma^N$ with $\bigcup_{i=1}^{n} T_i = A$, then $x_0 \in C(T_1,...,T_n)$.

Let $D(A) = \bigcap_{\bigcup_{i=1}^{n} S_i = A} C(S_1,\ldots,S_n)$; hence $D(A) \neq \phi$.

Thus $D : \Sigma \setminus \{\phi\} \to\to X$ is a correspondence such that for all $\phi \neq A \subset X$, $\phi \neq D(A) \subset A$.

Further by Property (a*), given $\phi \neq A \subset X$, there exists $x_0 \in D(A)$ such that $x_0 \in B \subset A$ implies $x_0 \in D(B)$.

However, by Theorem 2.10 in Deb [1983], this latter property is necessary and sufficient for the existence of an ordering R and X such that for all $\phi \neq A \subset X$, $\phi \neq G(A,R) \subset D(A)$.

Thus $G(\bigcup_{i=1}^{n} S_i, R) \subset D(\bigcup_{i=1}^{n} S_i) \subset C(S_1,\ldots,S_n)$ for all $(S_1,\ldots,S_n) \in \Sigma^N \setminus \{(\phi,\ldots,\phi)\}$.

Q.E.D.

Remark : All the voting rules axiomatically characterized in this paper satisfy the following property :

Consistency : $\forall\ (S_1,\ldots,S_n) \in \Sigma^N$ and $\forall\ \phi \neq M \subset N$,

$$T_i = S_i \cap C(S_1,\ldots,S_n)\ \forall\ i \in M$$

$$= S_i\ \forall\ i \in N \setminus M$$

$$\to C(T_1,\ldots,T_n) = C(S_1,\ldots,S_n).$$

Hence they may be called consistent voting rules. An example of a voting rule which does not satisfy consistency is the voting rule which selects only those candidates who get the second highest number of votes (: which in principle may be the empty set).

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