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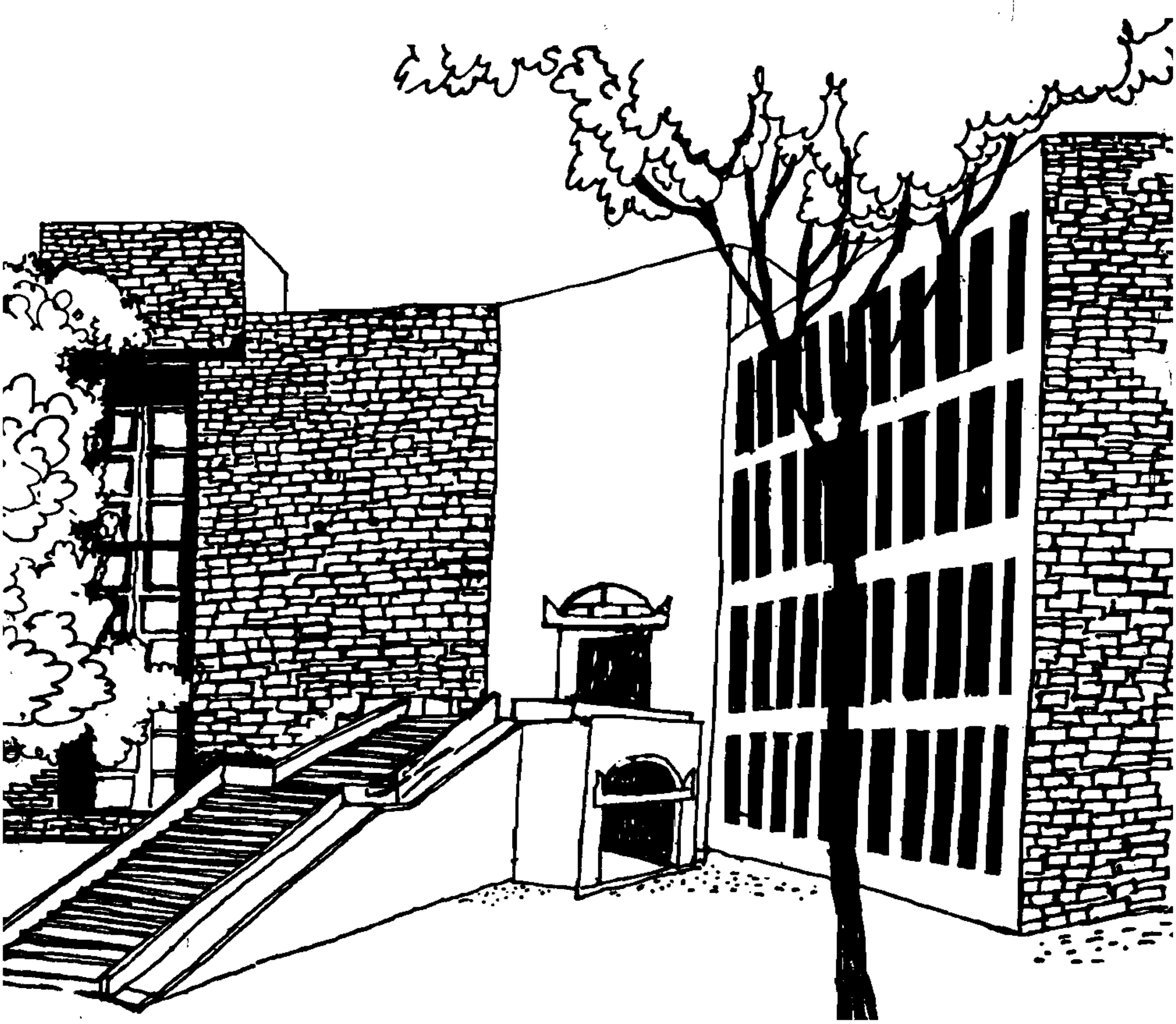

A CONSEQUENCE OF CHERNOFF AND OUTCASTING

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

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Somdeb Lahiri

Indian Institute of Management

Ahmedabad-380 015

India.

e-mail:lahiri@iimahd.ernet.in

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Introduction

The purpose of this paper is to prove by induction the theorem (in Aizerman and Malishevski [1981]) that a choice funtion which satisfies Chernoff's axiom and Outcasting can always by expressed as the union of the solution sets of a finite number of maximization problems. The proof we offer is considerably simpler than the one in Aizerman and Malishevski [1981]. In Moulin [1985], a discussion of a similar result is available. Our framework closely resembles the one of choice theory as enunciated in Moulin [1985]. It is well known that a combination of Chernoff's axiom and Outcasting is equivalent to a property called Path Independence (See Moulin [1985]).

The Framework

Let X be a finite, non empty universal set. If S is any non-empty subset of X, let [S] denote the set of all non-empty subsets of S. A choice function on X is a function $C:[X] \rightarrow [X]$ such that $C(S) \subset S \ \forall \ S \in [X]$.

Given $S \in [X]$, let $|S|$ denote the cardinality of S. C is said to satisfy:

a) Chernoff Axiom (CA), if $\forall$ S, T $\in$ [X], S $\subset$ T implies C (T) $\cap$ S $\subset$ C (S) ;

b) Outcasting (O) , if $\forall$ S, T $\in$ [X], C(T) $\subset$ S $\subset$ T implies C(T) = C(S).

c) Aizerman (A), if $\forall$ S, T $\in$ [X], C(T) $\subset$ S $\subset$ T implies C(S) $\subset$ C(T).

Chernoff Axiom was originally proposed in Chernoff [1954].Outcasting, which occurs under its present nomenclature in Aizerman and Aleskerov [1995],has been attributed to Nash [1950],by Suzumura [1983].Aizerman has been in the literature for a while (for example,see Fishburn [1975]).However,its prominent role was recognized only recently (Aizerman and Malishevsky [1981]).

Clearly, Outcasting implies Aizerman.It is also quite easy to see that Aizerman and Chernoff together imply Outcasting.Hence, a choice function satisfies Aizerman and Chernoff if and only if it satisfies Outcasting and Chernoff.

The issue here is the following theorem in Aizerman and Malishevski [1981] :

<u>Theorem 1</u>: Let C be a choice function on X which satisfies CA and O. Then there exists n $\in \mathbf{N}$ and functions $f_i : X \to \mathbf{N}$, i $\in$ {1, ..., n} such that $\forall$ S $\in$ [X],

$$C(S) = \bigcup_{i=1}^{n} \{x \in S \,/\, f_i(x) \geq f_i(y) \ \forall \ y \in S\}$$

Before we provide a new proof of this theorem, let us provide two examples to show that neither CA nor O is alone sufficient for the above theorem.

<u>Example 1</u> : Let X = { x,y,z}, C(X) = {x}, and C(S) = S $\forall$ S $\in$ [X], S $\subset\subset$ X. Clearly C satisfies CA but not O Towards a contradiction suppose there exists n $\in \mathbf{N}$ and functions $f_i : X \to \mathbf{N}$, i = 1, ..., n such that

$$C(S) = \bigcup_{i=1}^{n} \{a \in S \,/\, f_i(a) \geq f_i(b) \ \forall \ b \in S\} \ \forall \ S \in [X].$$

Then $C(X) = \{x\}$ implies $f_i(x) > \max \{f_i(y), f_i(z)\}$ $\forall$ i.

However, $C(\{x,y\}) = \{x,y\}$ implies $f_i(y) \geq f_i(x)$ for some i, which condtradicts what we obtained before.

Example 2 : Let $X = \{x,y,z\}$, $C(X) = X$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}$ $\forall$ $a \in X$. Clearly C satisfies O but not CA. Towards a contradiction suppose there exist $n \in \mathbf{N}$ and functions $f_i : X \to \mathbf{N}$, i = 1, ..., n such that

$$C(S) = \bigcup_{i=1}^{n} \{a \in S \,/\, f_i(a) \geq f_i(b) \; \forall \; b \in S\} \; \forall \; S \in [X].$$

Then $C(X) = X$ implies there exists $i \in \{1, ..., n\}$ such that $f_i(y) \geq f_i(x)$. However, then $y \in C(\{x,y\})$, contrary to our definition of C.

Proof of Theorem 1 :

We will prove this theorem by induction on the Cardinality of X.

If $|X| = 2$, then there are two possibilities :

a) $C(X) = X$: then define $f : X \to \mathbf{N}$ as follows :

$f(a) = 1 \forall$ $a \in X$.

b) $C(X) \neq X$: then define $f : X \to \mathbf{N}$ as follows :

$f(a) = 2$ if $a \in C(X)$

$= 1$ if $a \in X \setminus C(X)$.

Clearly $C(S) = \{a \in S \mid f(a) \geq f(b)\ \forall\ b \in S\}$.

Hence suppose the theorem is true for $|X| \in \{1,\ldots,m\text{-}1\}$ and suppose $|X| = m \in \mathbf{N}$. Let $C(X) = \{x_1, \ldots, x_p\}$, for some $p \in \mathbf{N}$. For each $x_i \in C(X)$, let $Y_i = X \setminus \{x_i\}$. Then

$$\forall\ (\emptyset \neq)\ S \subset T \subset Y_i,\ C(T) \cap S \subset C(S)$$

$$\forall\ (\emptyset \neq)\ S \subset T \subset Y_i,\ \text{if } C(T) \subset S \text{ then } C(S) = C(T).$$

Let $C_i : [Y_i] \rightarrow [Y_i]$ be defined as follows :

$$C_i(S) = C(S)\ \forall\ S \in [Y_i],\ i \in \{1, \ldots, p\}.$$

By the induction hypothesis $\forall\ i \in \{i, \ldots, p\}$, there exists $m_i \in \mathbf{N}$ and $g_i^j : Y_i \rightarrow \mathbf{N}$, $j = 1,\ldots, m_i$ such that

$$C_i(S) = \bigcup_{j=1}^{m_i} \{a \in S / g_i^j(a) \geq g_i^j(b) \forall b \in S\}, \forall S \in [Y_i].$$

Let $g_i^j(x_i) = [\max\{g_i^j(a) / a \in Y_i\}] + 1$,

$\forall\ j \in \{1,\ldots,m_i\},\ i \in \{1, \ldots, p\}$.

Now suppose $S \in [X]$.

Suppose $S \subset Y_i\ \forall\ i \in \{1,\ldots,p\}$.

$$\text{Then } C(S) = C_i(S) = \bigcup_{j=1}^{m_i} \{a \in S / g_i^j(a) \geq g_i^j(b) \forall b \in S\}\ \forall\ i \in \{1,\ldots,p\}.$$

$$\therefore C(S) = \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \,\forall\, b \in S\}.$$

Hence suppose $S \not\subset Y_i$ for some $i \in \{i,\ldots,p\}$.

Case 1 : $C(X) \subset S$

Then, by (O), $C(S) = C(X)$.

$$\therefore C(S) = \{x_1,\ldots,x_p\} = \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \,\forall\, b \in S\}.$$

Case 2 : $C(X) \not\subset S$.

Let $A = \{i \,/\, x_i \notin S\} \neq \emptyset$

Thus $S \subset Y_i \;\forall\, i \in A$.

By the induction hypothesis,

$$C(S) = C_i(S) = \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \,\forall\, b \in S\}, \forall\, i \in A.$$

Hence,

$$C(S) \subset \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \,\forall\, b \in S\}.$$

Now suppose $i \notin A$. Thus $x_i \in C(X) \cap S$. By CA, $x_i \in C(S)$

$$\therefore \bigcup_{i \notin A} \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \,\forall\, b \in S\} \subset C(S).$$

But, $C(S) = C_i(S) = \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \,\forall\, b \in S\}, \forall\, i \in A.$

$\therefore \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \forall\, b \in S\} \subset C(S).$

Hence $C(S) = \bigcup_{i=1}^{p} \bigcup_{j=1}^{m_i} \{a \in S \,/\, g_i^j(a) \geq g_i^j(b) \forall\, b \in S\}, \forall\, S \in [X].$

The theorem was shown to hold for $|X| = 2$ and has now been shown to hold for $|X| = m$ if it holds for $|X| = m-1$. Hence it is true for all finite non-empty X.

Q.E.D.

Remark: In Moulin [1985],there is a property called Expansion. C is said to satisfy:

Expansion (E) , if $\forall$ S, T $\in$ [X], $C(T) \cap C(S) \subset C(S \cup T)$.

The result due to Schwarz [1976],which we refer to in the introduction as the one available in Moulin [1985] implies the following:

Let C be a choice function on X which satisfies CA ,E and O. Then there exists n $\in \mathbf{N}$ and functions $f_i : X \to \mathbf{N}$, i $\in$ {1, ..., n} such that $\forall$ S $\in$ [X],

(1) $C(S) = \bigcup_{i=1}^{n} \{x \in S \,/\, f_i(x) \geq f_i(y) \;\forall\; y \in S\}$ and (2) $C(S) = \{x \in S / x \in C(\{x,y\}) \forall y \in S\}$. Conversely (1) and (2) imply C satisfies CA,E and O.

The following example shows that (1) above may be satisfied even if C does not satisfy E.

Example 3 : Let X = {x,y,z}, C(X) = {y,z}, C({x,y}) = {x,y}, C({y,z}) = {y,z}, C ({x,z}) = {x,z}, C({a}) = {a} $\forall$ a $\in$ X. Clearly C satisfies CA and O but not E, since $x \in C(\{x,a\}) \forall\, a \in X$ and yet $x \notin C(X)$. Let $f_i : X \to \mathbf{N}$, i = 1, 2 be such that $f_1(y)=3 > f_1(x) = 2 > f_1(z) = 1$ and $f_2(z) = 3 > f_2(x) = 2 > f_2(y) = 1$. However,

$$C(S) = \bigcup_{i=1}^{2} \{a \in S / f_i(a) \geq f_i(b) \ \forall \ b \in S\} \ \forall \ S \in [X].$$

<u>Quasi-Transitive Binary Relations</u>

A binary relation Q on X is any non-empty subset of X x X. Given a binary relation Q on X its <u>asymmetric</u> part denoted $P(Q) = \{(x, y) \in Q / (y, x) \notin Q\}$.A binary relation Q on X is said to be

(i) reflexive if $(x, x) \in Q \ \forall \ x \in X$;
(ii) complete if $x, y \in X$, $x \neq y$ implies $(x, y) \in Q$ or $(y, x) \in Q$;
(iii) quasi-transitive if $\forall \ x, y, z \in X$, $(x, y) \in P(Q)$ and $(y, z) \in P(Q)$ implies $(x,z) \in P(Q)$;
(iv) a quasi order if it is reflexive,complete and quasi-transitive.

We are concerned here with the following theorem, which may be found in Roberts [1979], Aizerman and Malishevsky [1981],Moulin [1985] (and which has been generalized in Lahiri [1999] to the case where the universal set X is possibly infinite) and which now follows as an easy corollary of our Theorem 1:

<u>Theorem 2</u>: Q is a quasi order on X if and only if there exists a positive integer n and functions $f_i:X \rightarrow \Re$, $i \in \{1, \ldots, n\}$ such that $Q = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, \ldots, n\}\}$.

<u>Proof</u>:- It is easy to see that if there exists a positive integer n and functions $f_i:X \rightarrow \Re$, $i \in \{1, \ldots, n\}$ such that $Q = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, \ldots, n\}\}$ then Q is a quasi order.To prove the converse assume that Q is a quasi order. For $S \in [X]$, let $C(S) = \{x \in S / (x,y) \in Q \ \forall \ y \in X\}$.Clearly $C(S) \neq \phi$ whenever $S \in [X]$, since Q is a quasi order. Hence C as defined above is a choice function.Further it is easy to verify that C satisfies CA and O. Hence,by Theorem 1, there exists a positive integer n and functions $f_i : X \rightarrow \Re$ for $i \in \{1, .., n\}$, such that $C(S) = \bigcup_{i=1}^{n} \{x \in S / f_i(x) \geq f_i(y) \ \forall \ y \in S\} \ \forall \ S \in [X]$. Since $(x, y) \in Q$ if and only if $x \in C(\{x,y\})$, and since $x \in C(\{x,y\})$ if and only if $f_i(x) \geq f_i(y)$ for some $i \in \{1, \ldots, n\}$,the proof of the theorem is thereby complete.

<u>Q.E.D.</u>

Stronger Consequences

The following lemma permits to strengthen the two theorems obtained above:

Lemma 1 : Let $f: X \to \Re$ (:the set of real numbers) be given. Then, there exists a positive integer n and one to one functions $f_i: X \to \mathbf{N}$, $i \in \{1, ..., n\}$ such that

$\{ (x, y) \in X \times X / f(x) \geq f(y)\} = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}\}$.

Proof :- Let $\{f(x) / x \in X\} = \{s_1, ..., s_q\}$ where q is a positive integer and $s_j < s_{j+1}$ $\forall j \in \{1, ..., q-1\}$. Let $n_j = |\{x \in X / f(x) = s_j\}|$ and let $n = (n_1)! \times \ldots \times (n_q)!$

Let $g: X \to \mathbf{N}$ be defined as follows:

$g(x) = n_1$, if $f(x) = s_1$

$g(x) = n_1 + \ldots + n_j$,if $f(x) = s_j$

Clearly, $\forall x, y \in X$: [$f(x) \geq f(y)$ if and only if $g(x) \geq g(y)$].

A function $\pi : \{1, ..., n_1 + \ldots + n_q\} \to X$ is called a restricted permutation if $\forall$ $k \in \{1, ..., n_1 + \ldots + n_q\}$: (1) [$\pi(k) \in \{x \in X / f(x) = s_1\}$ if and only $(1 \leq k \leq n_1)$] & (2) [$\pi(k) \in \{x \in X / f(x) = s_i\}$ if and only $(n_{i-1} \leq k \leq n_i$ and $1 < i \leq q)$].Let Π denote the set of all restricted permutations.Since X is finite so is Π. For $\pi \in \Pi$, define $f_\pi : X \to \{1, ..., n_1 + \ldots + n_q\}$ as follows: $\forall x \in X$, $f_\pi(x) = k$ if and only if $\pi(k) = x$. It is now easy to verify that, $\{ (x, y) \in X \times X / f(x) \geq f(y)\} = \{ (x, y) \in X \times X / g(x) \geq g(y)\} = \{ (x, y) \in X \times X / f_\pi(x) \geq f_\pi(y)$ for some $\pi \in \Pi\}$. This proves the lemma.

Q.E.D.

In view of Lemma 1 and Theorems 1 and 2 we have the following:

Theorem 3: Let C be a choice function on X which satisfies CA and O. Then there exists $n \in \mathbf{N}$ and one to one functions $f_i : X \to \mathbf{N}$, $i \in \{1, ..., n\}$ such that

$$\forall S \in [X],\ C(S) = \bigcup_{i=1}^{n} \{x \in S / f_i(x) \geq f_i(y)\ \forall\ y \in S\}.$$

Theorem 4: Q is a quasi order on X if and only if there exists a positive integer n and one to one functions $f_i: X \to \mathbf{N}$, $i \in \{1, ..., n\}$ such that $Q = \{ (x, y) \in X \times X / f_i(x) \geq f_i(y)$ for some $i \in \{1, ..., n\}\}$.

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