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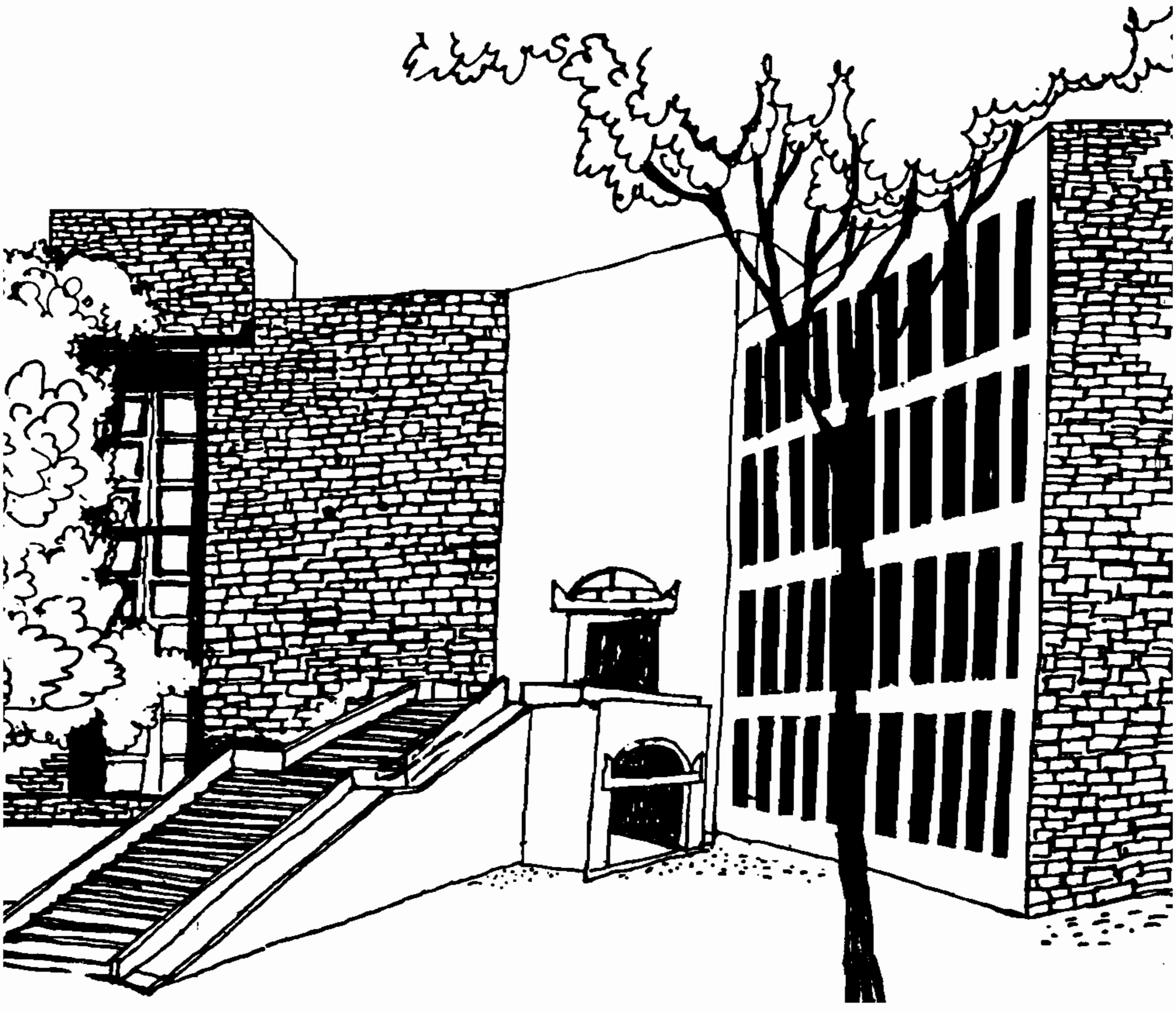

A NEW CONGRUENCE AXIOM AND TRANSITIVE RATIONAL CHOICE

By

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Abstract

Rationality in choice theory has been an abiding concern of decision theorists. A rationality postulate of considerable significance in the literature is the weak congruence axiom of Richter [1971] and Sen [1971]. It is well known that in discrete choice contexts of the classical type [i.e. all nonempty finite subsets of a given set comprise the set of choice problems], this axiom is equivalent to full rationality. The question is: whether a weakening of the weak congruence axiom would suffice to imply full rationality? This is the question we take up in this paper.

We propose a weaker new congruence axiom which along with the Chernoff Axiom implies full rationality. The two axioms are independent. We also study interesting properties of these axioms and their interconnections through examples.

Key Words: Chernoff, Congruence, Binary Relations, Transitivity.

AMS (1991) Classification Numbers : 04A05, 904A06, 90A07, 90A08.

1. Introduction : Rationality in choice theory has been an abiding concern of decision theorists. A rationality postulate of considerable significance in the literature is the weak congruence axiom of Richter [1971] and Sen [1971]. It is well known that in discrete choice contexts of the classical type [i.e. all nonempty finite subsets of a given set comprise the set of choice problems], this axiom is equivalent to full rationality. The question is: whether a weakening of the weak congruence axiom would suffice to imply full rationality? This is the question we take up in this paper.

   We propose a weaker new congruence axiom which along with the Chernoff Axiom implies full rationality. The two axioms are independent. We also study interesting properties of these axioms and their interconnections through examples.

2. An Overview of Full Rationality : In this section we closely follow Suzumura (1983) and Moulin (1985).

Let X be a nonempty universal set and given a non-empty subset S of X, let [S] denote the set of all non-empty finite subsets of S. A choice function (on X) is a function

$C : [X] \rightarrow [X]$ such $\phi \neq C(S) \subset S \; \forall \; S \in [X]$.

Given a choice function C, we define binary relations:

$$R_c = \bigcup_{S \in [X]} [\, C(S) \times S \,]$$

$$R^c = \bigcup_{x,y \in x} [C(\{x, y\}) \times \{x, y\}]$$

$$R_c^* = \bigcup_{S \in [X]} [C(S) \times (S \setminus C(S))].$$

A choice function C is said to satisfy the Weak Axiom of Revealed Preference (WA) if $(x,y) \in R_c \rightarrow (y,x) \notin R_c^*$.

A choice function C is said to satisfy the Weak Congruence Axiom (WCA) [of Richter (1971) and Sen (19171)] if $(x,y) \in R_c$, $x \in S$, $y \in C(S) \rightarrow x \in C(S)$.

A choice function C is said to satisfy Arrow's Axiom (AA) [Arrow, 1959] if S, $T \in [X]$, $S \subset T$ and
$\phi \neq C(T) \cap S \rightarrow C(S) = C(T) \cap S$.

A binary relation R on X is said to be

(a) reflexive if $(x,x) \in R \; \forall \; x \in X$

(b) complete if $x \neq y$, $x,y \in X \rightarrow$ either $(x,y) \in R$, or $(y, x) \in R$

(c) transitive if $\forall x,y,z \in X$, $(x,y) \in R \;\&\; (y,z) \in R \rightarrow (x,z) \in R$

Given a binary relation R on X and $S \in [X\}$ let $G(S,R) = \{x \in S \,/\, (x,y) \in R \; \forall \; y \in S\}$.

A choice function C is said to be full rational (FR) if there exits a reflexive, complete and transitive binary relation R on X: $C(S) = G(S,R) \; \forall \; S \in [X]$.
The following results, available in Moulin (1985), are obtained from received theory : WA (: which is equivalent to WCA), AA and FR are all equivalent to one another.
A choice function C is said to satisfy the Chernoff Axiom

(CA) if $\forall$ S, T $\in$ [X], S $\subset$ T $\rightarrow$ C(T) $\cap$ S$\subset$C(S).

Clearly CA is weaker than AA. Let us provide an example to show that CA does not imply rationality, let alone full rationality.

Example 1 : Let X = {x, y, z}, C(X) = {x}, C({x, y}) = {x, y}, C({x, z}) = {x}, c({y, z}) = {y}, c({a}) = {a} $\forall$ a $\in$ X. Clearly C satisfies CA. Towards a contradiction assume that there exists a binary relation R on X : C(S) = G(S,R) $\forall$ S $\in$ [X].
Thus C({y, z}) = {y} $\rightarrow$ (y, z) $\in$ R and C({x, y}) = {x, y} $\rightarrow$ (y, x) $\in$ R.
Since C({y}) = {y} $\rightarrow$ (y,y) $\in$ R, we have (y, a) $\in$ R $\forall$ a $\in$ X. Thus y $\in$ G(X,R).
However, y $\notin$ C(X).

The following observation is immediate :

Given a choice function C if there exists a binary relation R on X such that C(S) = G(S, R) $\forall$ S $\in$ [X] then R = R^c.

It is easy to see that if C satisfies CA then C(S) $\subset$ G(S, R^c) $\forall$ S $\in$ [X].

3. A New Congruence Axiom : In this section we propose and derive consequences of a new congruence axiom.

A choice function C is said to satisfy New Congruence Axiom (NCA) if (x,y) $\in$ R^c, x$\in$S, y$\in$C(S) $\rightarrow$ x$\in$C(S).

Since $R^c \subset R_c$ we must have WCA $\rightarrow$ NCA.

However, the converse need not be true :

Example 2 :- Let X = {x, y, z, w} ; let C(X) = {x}, C({x, y, z}) = {x, y}, C({x, y, w}) = {x}, C({y, z, w}) = {w}, C({x, z, w}) = {x}, C({x, y}) = {x}, C({x, z}) = {x}, C({y, z}) = {y}, C(w, z}) = {w}, C({x, w}) = {x}, C({y, w}) = {w}, C({a}) = {a} $\forall$ a $\in$ X. Here C satisfies NCA; however C does not satisfy WCA. This is because (y, x) $\in$ R_c , x $\in$ C (X) and yet y $\notin$ C(X).

It should be noted that CA and NCA are independent.

Example 3 :- Let X = {x, y, z}; let C(X) = {x}, C({x, y}) = {x, y}, C({y, z}) = {y}, C({x, z}) = {x}, C({a}) = {a} $\forall$ a $\in$ X. C satisfies CA as is easily verified. However, C does not satisfy NCA : (y, x) $\in$ R^c, x $\in$ C(X) but y $\notin$ C(X).

Example 4 :- Let X = {x, y, x} : let C(X) = {x, y}, C({x, y}) = {y}, C({y, z}) = {y}, C({x, z}) = {x}, C({a}) = {a} $\forall$ a $\in$ X. C satisfies NCA as is easily verified. However, C does not satisfy CA: x$\in$C(X) but x $\notin$ C({x,y}).

Theorem 1:- A choice function C is full rational if and only if C satisfies CA and NCA.

Proof :- It is easy to see that if C is fully rational then C satisfies CA and NCA. Now suppose C satisfies CA and NCA. By CA, $C(S) \subset G(S, R^c)$ $\forall$ $S \in [X]$. Now suppose $x \in G(S, R^c)$. Thus $x \in S$ and $(x, y) \in R^c$ $\forall$ $y \in S$. Thus $(x,y) \in R^c$ $\forall$ $y \in C(S)$. By NCA, $x \in C(S)$. Thus $G(S, R^c) \subset C(S)$. Hence $C(S) = G(S, R^c)$. Remains to show that R^c is transitive (:since it is always reflexive and complete). Let $(x,y) \in R^c$ and $(y,z) \in R^c$. Consider $S = \{x, y, z\}$. If $x \in C(S)$, then $x \in C(\{x,z\})$ by CA and hence $(x, z) \in R^c$. If $y \in C(S)$, then $(x,y) \in R^c$ and NCA implies $x \in C(S)$. Thus $(x,z) \in R^c$. If $z \in C(S)$, then $(y, z) \in R^c$ and NCA implies $y \in C(S)$. But then, $y \in C(S)$ and $(x,y) \in R^c$ implies $x \in C(S)$. Hence $(x,z) \in R^c$. Thus R^c is transitive. This proves the theorem.

Q.E.D.

The following axiom is of considerable interest in the literature on rational choice theory:

A choice function C is said to satisfy generalized Condorcet property (GC) if $G(S, R^c) \subset C(S)$ $\forall$ $S \in [X]$.

It is well known that GC along with CA implies acyclic rationality, which is really the minimal rationality requirement one can require for a choice function. On the other hand NCA along with CA implies full rationality, which in some senses is the maximal rationality requirement one can require for a choice function. It is not very difficult to see that NCA implies GC. However, the converse need not be true, as it is shown in the following example.

Example 5 :- Let $X = \{x, y, z\}$. Let $C(\{a\}) = \{a\}$ for all 'a' in X, $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x, z\}$, $C(\{y, z\}) = \{y\}$, $C(X) = \{x\}$. C satisfies GC but does not satisfy NCA as the comparison between x and z reveals.

Remark 1 :- In Aizerman and Aleskerov (1995) it is asserted that any choice function can be expressed as a finite union of choice functions satisfying (GC). This is true only if we allow choice functions to be empty valued. Otherwise it is not true as it is shown in the following example.

Example 6 :- Let $X = \{x, y, z\}$; Let $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x\}$, $C(\{x, y, z\}) = \{y\}$, and $C(S) \neq \phi$ for all other $S \in [X]$. Towards a contradiction suppose there exists $C_1, \ldots, C_P$ satisfying GC such that $C(S) = \bigcup_{i=1}^{P} C_i(S) \forall S \in [X]$. Clearly $C_i(\{x, y\}) = \{x\}$ $\forall$ i and $C_i(\{x, z\}) = \{x\}$ $\forall$ i. By GC, $x \in C_i(X)$ $\forall$ i and thus $x \in C(X)$, contradicting our definition of C.

Remark 2 :- It is interesting to note that in general a choice function satisfying CA need not be expressible as a finite union of choice functions satisfying AA.

Example 7 :- Let X = {x, y, z} ; C(X) = {y}, C({x, y}) = {x, y}, C({x, z}) = {z}, C({y, z}) = {y}, C({a}) = {a} $\forall$ a $\in$ X. Suppose towards a contradiction that $C(S) = \bigcup_{i=1}^{p} C_i(S)$ $\forall$ S $\in$ [X]. where $C_1,\ldots,C_P$ satisfies AA.Thus $C_i(X) = \{y\}$ $\forall$ i and $C_j(\{x, y\}) \ni X$ for some j. Now $y \in C_j(X) \cap (\{x, y\})$.Thus by AA, $C_j(\{x, y\}) = C_j(X) \cap \{x, y\} = \{y\}$ which is a contradiction.

Remark 3 :- If a choice function C satisfies CA and GC then there exists a function $\phi : X \times X \to \mathbb{R}$ such that $C(S) = \{ x \in X / \phi(x, y) \geq o\ \forall y \in S\}$, $\forall$ S $\in$ [X].For, let

$$\phi(x, y) = 1 \quad \text{if } \{x\} = C(\{x, y\})$$

$$\phi(x, y) = 0 \quad \text{if } \{x, y\} = C(\{x, y\})$$

$$\phi(x, y) = -1 \quad \text{if } \{y\} = C(\{x, y\})$$

Let $\widetilde{C}(S) = \{x \in X / \phi(x, y) \geq 0\ \forall y \in S\}$ $\forall$ S $\in$ [X].

Let $x \in \widetilde{C}(S)$. Thus $x \in C(\{x, y\})$ $\forall y \in S$.

By GC, $x \in C(S)$.

Now, let $x \in C(S)$. By CA, $x \in C(\{x, y\})$ $\forall y \in S$.

Thus $\phi(x, y) \geq 0$ $\forall y \in S$. Thus $X \in \widetilde{C}(S)$.

Thus $C(S) = \widetilde{C}(S)$ $\forall$ S $\in$ [X].

Remark 4 :- We have already noted that AA $\to$ CA. It is easy to see that AA $\to$ NCA: let $(x, y) \in R^c$, $x \in S$, $y \in C(S)$.

Thus $C(S) \cap \{x, y\} \neq \phi$

By AA, $C\{x, y\} = C(S) \cap \{x, y\}$.

Thus $x \in C(S)$.

On the other hand CA and NCA $\to$ AA. It is enough to show that S, T $\in$ [X], $S \subset T$ and $C(T) \cap S \neq \phi$ implies $C(S) \subset C(T) \cap S$ (:$C(T) \cap S \subset C(S)$ follows from CA). Let $x \in C(S)$ and towards a contradiction assume $x \notin C(T)$.

Let $y \in C(T) \cap S$. Thus by CA, $C(S) \cap \{x, y\} \subset C(\{x, y\})$ and so $x \in C(\{x, y\})$. Thus $(x, y) \in R^c$. Since $x \in S \subset T$ and $y \in C(T)$, by NCA, $x \in C(T)$ which is a contradiction. Thus $C(S) \subset C(T) \cap S$.

4. The Relative Strengths of CA and NCA : In this section we show that in some senses, NCA is much weaker than CA. This is observed by showing that along with other well known axioms NCA fails where CA succeeds. Towards that end we invoke the following two properties appearing in Aizerman and Aleskerov [1995].

A choice function C is said to satisfy Concordance (CON) if $\forall$ S, T $\in$ [X], $C(S) \cap C(T) \subset C(S \cup T)$.

A choice function C is said to satisfy Outcasting (O) if S, T $\in$ [X], $C(T) \subset S \subset T$ implies $C(T) = C(S)$.

Given a binary relation R on X, let $P(R) = \{(x, y) \in R \,/\, (y, x) \notin R\}$: A binary relation R on X, is said to be acyclic if there does not exist $x^0, x^1, \ldots, x^t \in X$ for some $t \in N$, with $(x^j, x^{j-1}) \in P(R)$ for $j \in \{1, \ldots, t\}$ and $(x^0, x^t) \in P(R)$.

A binary relation R on X, is said to be quasi-transitive if $\forall$ x, y, z $\in$ X, $(x, y) \in P(R)$, $(y, z) \in P(R)$ implies $(x, z) \in P(R)$.

The following results, whose origins can be traced to Prof. Amartya Sen, can be found in Aizerman and Aleskerov [1995].

Theorem 2 : A choice function C is acyclic rational (i.e. $C(S) = G(S, R)$ for all $S \in [X]$, where R is acyclic) if and only if satisfies CA and CON.

Theorem 3 : A choice function C is quasi-transitively rational i.e., ($C(S) = G(S, R)$ for all $S \in [X]$, where R is quasi-transitive) if and only if C satisfies CA, CON and O.

Indeed, if $C(S) = G(S,R)$ for all $S \in [X]$, then R must be acyclic. This is easily verified. However, we show in the following two examples that substituting CA by NCA in the above two theorems will not lead to the desired result.

Example 8 :- Let $X = \{x, y, z\}$, $C(X) = \{x, y\}$, $C(\{x, y\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{a\}) = \{a\}$ for all $a \in X$.C satisfies NCA and CON. However, $\{x, y\} \subset X$, $x \in C(X) \cap \{x, y\}$, but $x \notin C(\{x, y\})$. Thus C does not satisfy CA, whence there does not exist a binary relation R on X such $C(S) = G(S, R)$ for all $S \in [X]$.

Example 9 : - Let $X = \{x, y, z\}$, $C(X) = X$, $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{z\}$, $C(\{y, z\}) = \{y\}$, $C(\{a\}) = \{a\}$ for all $a \in X$. C satisfies NCA, CON and O. However, {x, y} ⊂ x, y ∈ C(X) ∩ {x, y}, but y ∉ C({x, y}). Thus C does not satisfy CA, whence there does not exist a binary relation R on X such that C(S) = G(S, R) for all S ∈ [X].

As observed in Section 3, a choice function is acyclic rational if and only if it satisfies CA and GC. However, Example 8, shows that NCA and GC does not imply (acyclic) rationality.

However, transitivity of the underlying binary relation seems to be a "minimal" requirement for a choice function generated by it to satisfy NCA, as the following theorem suggests:

Theorem 4:- Let C be a choice function such that C(S) = G(S, R) for all S in [X], where R is a binary relation on X. Then R is transitive if and only if C satisfies NCA.

Proof :- That transitivity of R implies C satisfies NCA has been established in Theorem 1. Hence suppose, C as above satisfies NCA. Let x, y, z ∈ X with (x, y) ∈ R and (y, z) ∈ R. Let S = {x, y, z}. If y ∈ C(S), then by NCA, x ∈ C(S) and hence (x, z) ∈ R. Further, x ∉ C(S), y ∉ C(S) would have to imply, {z} = C(S) since C(S) must be non empty. But then the absence of y in C(S) would contradict NCA. Hence (x, z) ∈ R and so R is transitive.

Q.E.D.

NCA looks deceptively similar to the following axiom which can be found in Nehring [1997].

A choice function C is said to satisfy the ß Axiom (ß A) if for all x, y ∈ X, for all S ∈ [X], (x, y) ∈ R^c, (y, x) ∈ R^c, {x, y} ⊂ S implies [x ∈ C (S) ↔ y ∈ C (S)]. The following theorem, lays bare the relationship.

Theorem 5 : Let C be a choice function. Then,

(i) NCA implies ß A
(ii) ß A and CA together implies NCA
(iii) ß A need not imply NCA.

Proof : (i) is obvious. To prove (ii), let C satisfy ß A and CA and let (x, y) ∈ R^c, S ∈ [X], y ∈ C (S) and x ∈ S. If (y, x) ∈ R^c , then by ß A, x ∈ C (S). Further (y, x) ∉ R^c contradicts CA. Hence C satisfies NCA.

To show that ß A need not imply NCA, let X = {x, y, z}, C(X) = {y}, C({x, y}) = {x}, C ({y, z}) = {y}, C({x, z}) = {z}. C satisfies ß A. However, (x, y) ∈ R^c, y ∈ C (X) and x ∉ C (X). Hence C does not satisfy NCA.

Q.E.D.

5. Sen's Axiom and NCA :- The following axiom is due to Sen [1971]:

A choice function C is said to satisfy Sen Axiom (SA) if $\forall$ A, B $\in$ [X] : [A $\subset$ B & C(A) $\cap$ C(B) $\neq \phi$} implies C(A) $\subset$ C(B).

Example 2 above shows that NCA need not imply SA, since {x, y, z} $\subset$ X, C(X) $\cap$ C({x, y, z}) = {x} $\neq \phi$ and yet C ({x, y, z}) $\not\subset$ C(X) since C ({x, y, z}) = {x, y} $\not\subset$ {x} = C(X). Neither does CA imply SA as the following example reveals:

Example 10 :- Let X = {x, y, z} ; let C(X) = {x} and C(S) = S $\forall$ S $\in$ [X] with S $\neq$ X. Clearly C satisfies CA, but does not satisfy SA : {x, y} $\subset$ X and x $\in$ C({x, y}) $\cap$ C(X) and yet C({x, y}) = {x, y} $\not\subset$ {x} = C(X). However, the following holds :

Proposition 1 :- CA and NCA together imply SA.

Proof : Let C satisfy CA and NCA and let S, T $\in$ [X] with S $\subset$ T and C(S) $\cap$ C(T) $\neq \phi$. Let x $\in$ C(S) and let y $\in$ C(S) $\cap$ C(T).

Thus {x, y} $\subset$ S and x $\in$ C(S) $\cap$ {x, y}.

By CA, x $\in$ C({x, y}). Further, y $\in$ C(T), x $\in$ C(S) $\subset$ S $\subset$ T and (x, y) $\in$ R^c.

Hence by NCA, x $\in$ C(T). Thus C(S) $\subset$ C(T).

Hence C satisfies SA.

Q.E.D.

Example 9 above shows that SA does not imply CA.

Example 11 :- Let X = {x, y, z}, let C(X) = {x, z}, C({x,y}) = {x}, C({x, z}) = {x}, C({y, z}) = {y} and C({a, y}) = {a} $\forall$ a $\in$ X. Clearly C satisfies SA. However C does not satisfy NCA since z $\in$ C(X), y $\in$ X, (y, z) $\in$ R^c but z $\notin$ C(X).

However the following is true :

Proposition 2 :- CA and SA implies NCA.

Proof :- Let C satisfy CA and SA and let S $\in$ [X], x $\in$ S, y $\in$ C(S) and (x,y) $\in$ R^c. Thus {x, y} $\subset$ S. By CA, y $\in$ C(S) $\cap$ {x, y} $\subset$ C({x, y}). Thus C({x, y}) = {x, y}. Now {x, y} $\subset$ S and, y $\in$ C({x, y}) $\cap$ C(S) $\neq \phi$. By SA, C({x, y}) $\subset$ C(S). Thus, x $\in$ C(S). Hence C satisfies NCA.

Q.E.D.

In view of Proposition 1, Proposition 2 and Theorem 1, the following well known result due to Sen [1971] is immediate:

Theorem 6: - A choice function C is full rational if and only if C satisfies CA and SA.

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