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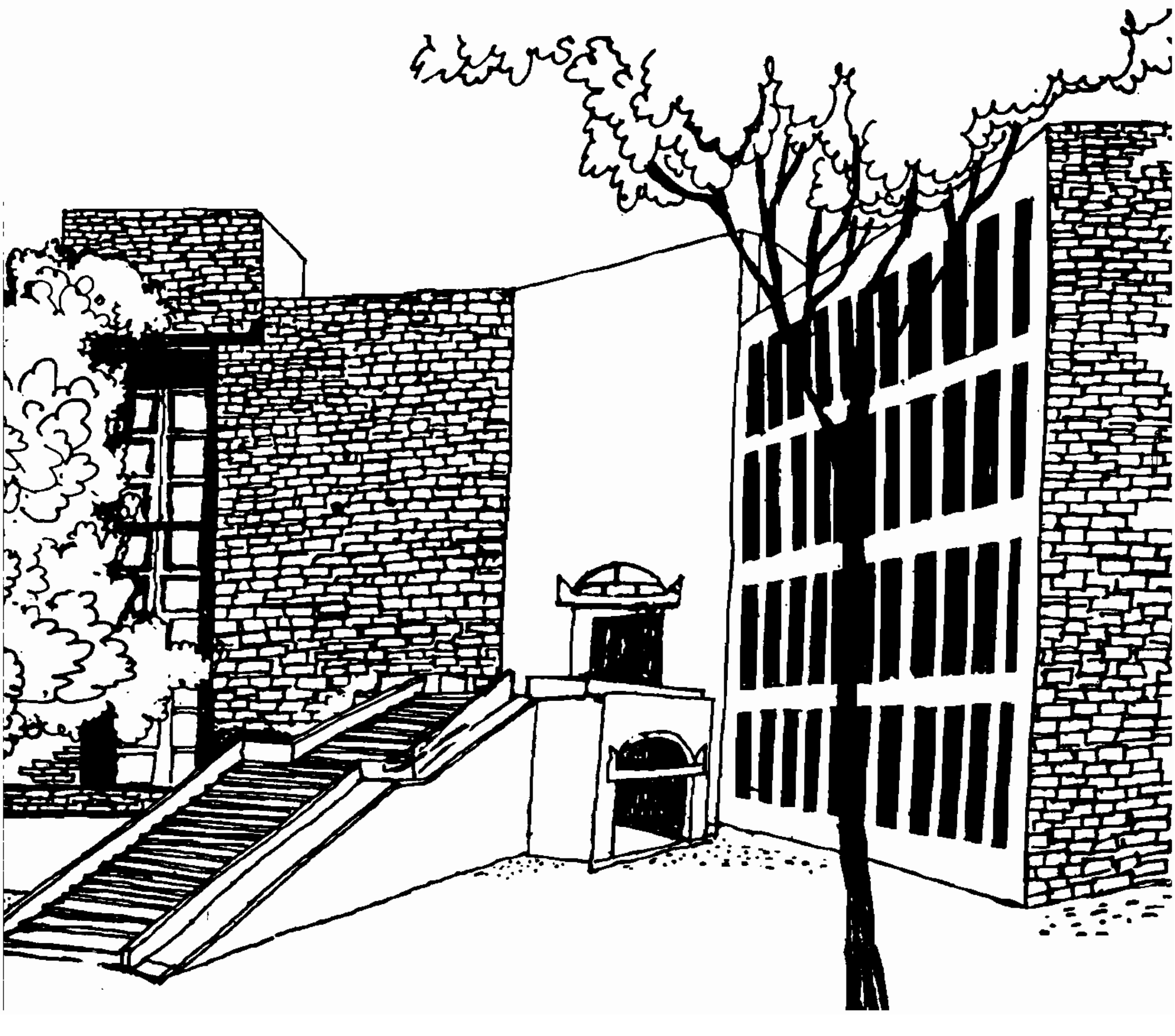

AXIOMATIC CHARACTERIZATION OF SOME EXTENSIONS

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Axiomatic Characterization of Some Extensions

Somdeb Lahiri
Indian Institute of Management
Ahmedabad 380 015
India
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1. Introduction: The general problem we are interested in this paper is of the following variety: We are given a finite universal set and a linear ordering on it. What is the minimal axiomatic characterization of a particular extension of this linear ordering to the set of all non-empty subsets of the given set?

In Kannai and Peleg [1984] we find the starting point of this literature, which basically asserts that if the cardinality of the universal set is six or more, then there is no weak order on the power set which extends the linear order and satisfies two properties: one due to Gardenfors and the other known as Weak Independence. This result was followed by a quick succession of possibility results in Barbera, Barret and Pattanaik [1984], Barbera and Pattanaik [1984], Fishburn [1984], Heiner and Packard [1984], Holzman [1984], Nitzan and Pattanaik [1984] and Pattanaik and Peleg [1984]. Somewhat later, Bossert [1989] established a possibility result by dropping the completeness axiom for the binary relation on the power set and otherwise using the same axioms as in Kannai and Peleg [1984].

In recent times Malishevsky [1997] and Nehring and Puppe [1999] have addressed the problem of defining an "indirect utility preference". Malishevsky [1997] addresses the integrability problem: given a weak order on the power set, under what conditions is it an indirect utility preference? A similar question is also addressed in Nehring and Puppe

[1999]. In our framework, a binary relation on the power set is an indirect utility extension if given two non-empty sets, the first is as good as the second if the best element (: with respect to the linear order) of the first set is as good as the best element of the second. In this paper, we first provide a minimal set of assumptions which uniquely characterizes the indirect utility extension. Subsequently we invoke a property which is implied by a definition of rational choice due to Puppe [1996] and obtain a second axiomatic characterization of the indirect utility extension. The indirect utility extension is easily observed to be a slight modification of the weak ordering extension due to Barbera and Pattanaik [1984].

In a later section of this paper we consider the problem of axiomatically characterizing the so called "lexicographic" extension. It is similar to the extension considered by Bossert [1989]. However unlike the extension due to Bossert our extension is complete,and though it satisfies Gardenfor's Property it fails to satisfy Weak Independence. Given a set we consider the pair consisting of its best and worst point.Now given two sets the first is atleast as good as the second,if either the best point of the first set is better than the best point of the second or they both share the same best point,in which case the worst point of the first is required to be atleast as good as the worst point of the second. In a way,the decision maker becomes pessimistic only if he/she has not much to choose between the best points of two sets.

Similar results can be found in Pattanaik and Xu [1990] and Puppe [1996].

In a final section we characterize the intersection of the asymmetric parts of all reflexive and transitive binary relations on the set of all non-empty subsets (of the linearly ordered universal set) which satisfy three properties which are rather popular in the literature. We also characterize the intersections of the symmetric parts of the same family of binary relations. It turns out that each of them coincide with the asymmetric and symmetric parts of a rather easily definable member of the same family. This analysis appears deceptively similar to the analysis in section 3 of Dutta [1997].

<u>2. The Model</u>: Let n be a positive integer and let X be the set of first n positive integers. Let [X] denote the set of all non-empty subsets of X. Given $A \in [X]$, let # (A) denote the number of elements in A.

Let $\Re$ be a binary relation on [X]. It is said to be (i) <u>reflexive</u> if $\forall A \in [X]$, $(A, A) \in \Re$; (ii) <u>complete</u> if $\forall A, B \in [X]$ with $A \neq B$, either $(A, B) \in \Re$ or $(B, A) \in \Re$;

(iv) <u>transitive</u> if $\forall A, B, C \in [X]$, $[(A, B) \in \Re, (B, C) \in \Re]$ implies $(A, C) \in \Re$.

Let $I(\Re) = \{(A, B) \in \Re / (B, A) \in \Re\} \}$, $P(\Re) = \{ (A, B) \in \Re / (B, A) \notin \Re \}$ and $W(\Re) =\{ (A, B) \in [X]x[X] / (B, A) \in P(\Re) \}$. $P(\Re) \subset P(\Re')$ implies and is implied by $W(\Re) \subset W(\Re')$.

Given $A \in [X]$, let g(A) be the unique element of A satisfying $g(A) \geq x$ whenever $x \in A$ and let l(A) be the unique element of A satisfying $x \geq l(A)$ whenever $x \in A$.

A binary relation $\Re$ on [X] is said to satisfy

(a) <u>Gardenfor's Property(GP)</u> if $\forall A \in [X]$ and $x \in X \setminus A$, (i) $x > g(A)$ implies $(A \cup \{x\}, A) \in P(\Re)$; (ii) $l(A) > x$ implies $(A, A \cup \{x\}) \in P(\Re)$.

(b) <u>Weak Independence(W.IND)</u> if $\forall A, B \in [X]$ with $(A, B) \in P(\Re)$, if $x \in X \setminus (A \cup B)$ then $(A \cup \{x\}, B \cup \{x\}) \in \Re$.

Kannai and Peleg [1984] show the following:

<u>Theorem 1</u>:- If n >5 , then there does not exist any binary relation on [X] which satisfies reflexivity, completeness, transitivity, GP and W.IND.

The above mentioned result lead to the search for a possibility result for n equal to five, resulting in the paper by Bandopadhyay (1988). Here we provide another different possibility result for n equal to five.Our method of proof suggests an alternative (: and perhaps simpler) approach to the result established in Bandopadhyay(1988) as well.

Given, $A \in [X]$, let M(A) be the unique element of A such that M(A) is greater than or equal to every element in A,and let m(A) be the unique element of A such that m(A) is less than or equal to every element in A.

Given, $p \in N$ a p-dimensional extension function is a function F: $[X] \rightarrow N^p$ such that for all i,j $\in X$ with $i \neq j$, $F(\{i\}) >> F(\{j\})$ if and only if i>j, where given $a,b \in N^p$ (i)$a \geq b$ means $a_k \geq b_k$ for all $k \in \{1,...,p\}$; a>b means $a \geq b$ and $a \neq b$; a>>b means $a_k > b_k$ for all $k \in \{1,...,p\}$.

Let F: $[X] \rightarrow N^p$ be a p-dimensional extension function.Let $\Re_F = \{(A,B) \in [X]x[X] / F(A) \geq F(B)\}$. Clearly, $\Re_F$ is reflexive and transitive although it may not be complete.However, if p=1,then $\Re_F$ is complete as well.

The following two axioms are being adapted to apply to $\Re_F$:

<u>Gardenfors Principle</u> (GP):For all $A \in [X]$ and $y \in X \setminus A$:(i)m(A)>y implies $F(A) >> F(A \cup \{y\})$;(ii)y>M(A) implies $F(A \cup \{y\}) >> F(A)$.

<u>Weak Independence</u> (W.IND): For all $A,B \in [X]$ and $y \in X \setminus (A \cup B)$:[F(A)>>F(B) implies $F(A \cup \{y\}) \geq F(B \cup \{y\})$.

The following result can be found in Bossert(1989).The simple proof is being provided for completeness.

Theorem 2: Let F: $[X] \rightarrow N^p$ be a p-dimensional extension function satisfying GP and W.IND.Then for all $A \in [X]$, $F(A)=F(\{M(A),m(A)\})$.

Proof: For #(A) equal to one or two the theorem is self evident.Hence assume #(A)>2. Let $A=\{j_1,....,j_k\} \in [X]$,with k>2 and $j_i < j_{i+1}$,for all $i \in \{1,...,k-1\}$.Hence $m(a)= j_1$ and $M(A)= j_k$ .By successive applications of GP,$F(\{j_k\}) >> F(\{j_2,....,j_k\})$ and by W.IND, $F(\{j_1\ ,j_k\}) \geq F(A)$.Similarly,by successive applications of GP, $F(\{j_1,....,j_{k-1}\}) >> F(\{j_1\})$ and by W.IND, $F(A) \geq F(\{j_1,j_k\})$.Hence the theorem.

Q.E.D.

Example due to Kannai and Peleg (1984): Let F: $[X] \rightarrow N^2$ be defined by F(A)= $(10^q\, m(A)+M(A),\ 10^q\, M(A)+m(A))$,where q is any positive integer such that $10^q > n$. Then F satisfies GP and W.IND.

In the above example the following observation is implicit:

Observation: Let F: $[X] \rightarrow N^p$ be a p-dimensional extension function.Then,for r equal to factorial p,there exists an r-dimensinal extension function H: $[X] \rightarrow N^r$ such that for all $A,B \in [X]$,F(A)>F(B) if and only if H(A)>>H(B).

Proof of observation: Let Π be the set of permutaions on $\{1,...,p\}$and let $q \in N$ such that $10^q > n$.For $\pi \in \Pi$,let $J_\pi(A)=\Sigma_{i=1,...p}[F_{\pi(i)}(A)]\ 10^{i\cdot q}$.Let r be equal to factorial p and let $k:\{1,...,r\} \rightarrow \Pi$ be any one-one function.Let H: $[X] \rightarrow N^r$ be such that $H_i(A)= J_{k(i)}(A)$,for $i \in \{1,...r\}$.Clearly, H(A)>>H(B) if and only if F(A)>F(B).

<u>Corollary 1 of Theorem 2</u>: Let F: $[X]\rightarrow N^p$ be a p-dimensional extension function satisfying GP and W.IND.Then for all $A\in[X]$ with $\#(A)\geq 2$, $F(A)>>F(A\setminus\{y\})$ implies $y=M(A)$ and $F(A\setminus\{y\})>>F(A)$ implies $y=m(A)$.

<u>Proof</u>:By Theorem 2, $F(A)=F(\{M(A),m(A)\})$,so that if $y\notin\{M(A),m(A)\}$,then $F(A)=F(A\setminus\{y\})$.On the other hand as a consequence of GP,$y=M(A)$ implies $F(A)>>F(A\setminus\{y\})$ and $y=m(A)$ implies $F(A\setminus\{y\})>>F(A)$.Hence the corollary.

Q.E.D.

<u>Proposition 1</u>: Let F: $[X]\rightarrow N^p$ be a p-dimensional extension function satisfying GP.Then for all $i,j,k,r\in X$, $i\geq j\geq r, i\geq k\geq r$ implies $F(\{i,j\})\geq F(\{k,r\})$.Further if either $i>k$ or $j>r$,then $F(\{i,j\})>> F(\{k,r\})$.

<u>Proof</u>:If $i=k$ and $j=r$,there is nothing to prove.Hence assume that either $i>k$ or $j>r$.Suppose $i>k$.Hence $M(\{i,j,k\})=i$ and $m(\{i,j,k\})\leq j$. By GP, $F(\{i,j\})>> F(\{k,j\})$.Now $j\geq r$ implies $F(\{k,j\})= F(\{k,r\})$ if $j=r$,and $F(\{k,j\})>> F(\{k,r\})$ if $j>r$,where the latter follows from GP. Combining the inequalities,we get the desired result for the case $i>k$.A similar conclusion obtains for the case $j>r$.

Q.E.D.

We now prove a partial converse of Theorem 2.

<u>Theorem 3</u>: Let F: $[X]\rightarrow N^p$ be a p-dimensional extension function such that for all $A\in[X]$, $F(A)=F(\{M(A),m(A)\})$.Suppose:

(a)for all $i,j,k\in X$,$[i\geq j > k$ implies $F(\{i,j\})\geq F(\{i,k\})]$;

(b)for all $i,j,k\in X$,$[k>i\geq j$ implies $F(\{k,j\})\geq F(\{i,j\})]$;

(c)for all i,j,k,r,y∈X, with i ≥j , k≥r and y ∉{i,j,k,r},[F({i,j})>> F({k,r}) implies F({i,j,y}) ≥ F({k,r,y})].

Then F satisfies GP and W.IND.

Proof: Follows easily from the following: for y ∈ X :

(i) if A ∈ [X] and y < m(A), then M(A∪{y}) = M(A) and m(A∪{y}) = y;

(ii) if A ∈ [X] and y > M(A), then M(A∪{y}) = y and m (A∪{y}) = m (A);

(iii) if A ∈ [X], then M(A∪{y}) = M({M(A),y}) and m (A∪{y}) = m({m(A),y}).

Q.E.D.

The following theorem is due to Bandopadhyay [1988] :

Theorem 4: Let n=5 and let F : [X] →N be defined as follows :

F(A) = 10 + M(A) if 1 ∈ M(A)

= 33 if A = {2, 4}

= 10 M(A) + m(A), otherwise.

Then F is a 1-dimensional extension function satisfying GP and W.IND.

We will provide an analogous but different result here.

Our proposal is the following : Let n=5 and let G : [X] → N be defined by

G(A) = 50 + m(A) if 5 ∈ A

=33 if M(A), m(A)) = (4,2)

=10 m(A) + M(A), otherwise.

It is easy to see that G is indeed an extension. Further, G(A) = G({M(A), m(A)}) for all A ∈ [X].

Lemma 1 : Let n=5 and let i, j, k, r $\in$ X with i $\geq$ j $\geq$ r, i $\geq$ k $\geq$ r.Then, G({i,j}) $\geq$ G ({k , r}). Further if either i>k or j>r,then G({i,j})>> G({k,r}).

Proof :- Easily verified.

Note :- G({5,1}) = 51 > 33 = G({2,4}). However, if F is as defined in Theorem 4, then F({1,5}) = 15 < 33. Hence the rankings of the non-empty subsets of X given by F and G are indeed different.

Theorem 5 :- G is a 1-dimensional extension satisfying GP and W.IND.

Proof :- That G is a 1-dimensional extension has already been observed. Similarly, (a) and (b) of Theorem 3 are easily verified (:hence G satisfies GP). Thus it remains to show that (c) of Theorem 3 holds as well. Let i, j, k, r, y $\in$ X with i $\geq$ j, k $\geq$ r and y $\notin$ {i, j, k, r}. Suppose G({i, j}) > G({k, r}). Suppose i $\geq$ k and j $\geq$ r. Then M({i, y}) $\geq$ M({k, y}) $\geq$ m({r, y}) and M({i, y}) $\geq$ m({j, y}) $\geq$ m({r, y}). Hence G({i, j, y}) = G({M({i, y}), m({j, y})}) $\geq$ G({M({k, y}), m({r, y})}) (: as a consequence of Lemma 1). However, G({M({k, y}), m({r, y})}) = G({k, y, r}). Hence G({i, y, j}) $\geq$ G({k, y, r}).

Hence assume either (i) i < k and j > r or (ii) i > k and j < r.

(The remaining case is excluded by Lemma 1 and the requirement that G({i, j}) > G({k, r})).

Case 1 :- k = 5: Then G({k, r}) = 50 + r. Now G({i, j}) > G({k, r}) implies i = 5. Hence, G({i, j}) = 50 + j. Thus j > r and i $\geq$ k contradicting both (i) and (ii).Hence Case 1 is ruled out.

Case 2 :- k < 5, i = 5: Thus y < 5.Thus F({i, y, j}) $\geq$ 50 > F ({k, y, r}), since m({r,y})$\leq$ M ({k, y}) <5.

Case 3 :- i < 5, {k, r} = {4,2}:

$\therefore$ G({k, r}) = 33

Now G ({i, j}) > G({k, r}) implies {i, j} ≠ {4, 2}, {i, j} ≠ {3}

$\therefore$ G ({i, j}) = 10j + i.

$\therefore$ 10j + i > 33.

$\therefore$ either j = 3 or j = 4.

Suppose j = 4. Then 5 > i ≥ j implies i = 4.

$\therefore$ {i, j} = {4}. But then i ≥ k, j ≥ r, contradicting (i) and (ii).

Suppose j = 3. Then 5 > i ≥ j implies i = 4 or 3. If i = 4, then i ≥ k, j ≥ r contradicting (i) and (ii). Thus i = 3. Thus {i, j} = [3}.

$\therefore$ G({i, j}) = 33 = G({k, r}),contradicting G({i, j}) > G({k, r}).

Hence Case 3 is ruled out.

<u>Case 4</u> :- k < 5, {k, r} ≠ {4, 2}, {i, j} = {4, 2}:

$\therefore$ G ({k, r}) = 10r + k,

and G ({i, j}) = 33.

$\therefore$ r = 3, 2 or 1.

If r = 3, then k ≥ r implies G ({k, r}) ≥ 33 = G({i, j}), contradicting G({i, j}) > G({k, r}).

Thus r ≠ 3. Thus r < j. Hence not (ii). Hence by (i), r < j = 2, Thus r = 1.

Further k > i = 4 implies k = 5, contradicting k < 5. Hence Case 4 is ruled out.

<u>Case 5</u> : - k < 5, {k, r} ≠ {4, 2}, {i, j} ≠ {4, 2}, i < 5:

Thus G({i, j}) = 10j + i

and G ({k, r}) = 10r + k.

G({i, j}) > G({k, r}) implies j ≥ r. Hence not (ii). Hence by (i) r < j and i < k.

Let $y = 5$, Then $G(\{i, y, j\}) = 50 + j > 50 + r = G(\{k, y, r\})$.

Let $y = 4$. Thus $y \geq k > i$. Then $G(\{i, y, j\}) = G(\{y, j\})$ and $G(\{k, y, r\}) = G(\{y, r\})$

Further $j > r$ implies $G(\{y, j\}) > G(\{y, r\})$.

$\therefore G(\{i, y, j\}) > G(\{k, y, r\})$.

Let $y = 3$. Thus $i < k < 5$ implies $\{i, k, j, r\} \subset \{4, 2, 1\}$ since $3 \notin \{i, k, j, r\}$.

Further $k > i \geq j > r$ implies $k = 4$, $i=j=2$, $r=1$.

$\therefore G(\{i, y, j\}) = 23 > 14 = G(\{k, y, r\})$

Let $y = 2$. Thus $i < k < 5$ implies $\{i, k, j, r\} \subset \{4, 3, 1\}$ since $2 \notin \{i, k, j, r\}$.

Further $k > i \geq j > r$ implies $k = 4$, $i = j = 3$, $r = 1$.

$\therefore G(\{i, y, j\}) = 23 > 14 = G(\{k, y, r\})$.

Let $y = 1$. Thus $1 < k < 5$ implies

$\{i, k, j, r\} \subset \{4, 3, 2\}$ since $1 \notin \{i, k, j, r\}$.

Further $k > i \geq j > r$ implies $k = 4$, $i = j = 3$, $r= 2$, contradicting $\{k, r\} \neq \{4, 2\}$.

Hence, we may conclude that if $G(\{i, j\}) > G(\{k, r\})$ with $i \geq j$ and $k \geq r$ and if $y \notin \{i, j, k, r\}$ then $G(\{i, y, j\}) \geq G(\{k, y, r\})$. Thus by Theorem 3, G satisfies GP and W.IND.

Q.E.D.

Bossert [1989] proves the existence of a unique binary relation on [X] which satisfies all the properties in Theorem 1 other than completeness.

Let $\overline{\Re} = \{(A,B) \in [X] \times [X] / g(A) \geq g(B)\}$. $\overline{\Re}$ is called the indirect utility extension. It is easy to see that $\overline{\Re}$ satisfies reflexivity, completeness, transitivity and W.IND, but does not satisfy GP. However, it satisfies the following property which modifies a similar one due to Barbera [1977]:

Property 1: $\forall$ x, y $\in$ X, [x > y implies ({x}, {x, y}) $\in \mathfrak{R}$ and ({x, y}, {y}) $\in P(\mathfrak{R})$]. Further $\overline{\mathfrak{R}}$ satisfies the following modification of W.IND:

Property 2: (A, B) $\in \mathfrak{R}$ and x $\in$ X \ (A $\cup$ B) implies (A $\cup$ {x}, B $\cup$ {x}) $\in \mathfrak{R}$.

Note: Property 2 implies W.IND.

A property found in Nehring and Puppe [1999] is the following:

Monotonicity (MON): $\forall$ A, B $\in$ [X], B $\subset$ A implies (A, B) $\in \mathfrak{R}$.

It is not difficult to see that $\overline{\mathfrak{R}}$ satisfies (MON).

3. Axiomatic Characterizations of the Indirect Utility Extension:-

Theorem 6: The only transitive binary relation on [X] to satisfy Property 1, Property 2 and MON is $\overline{\mathfrak{R}}$.

Proof: We have already seen that $\overline{\mathfrak{R}}$ satisfies the above mentioned properties. Hence let $\mathfrak{R}$ be transitive and satisfy Property 1, Property 2 and MON. By MON, $\mathfrak{R}$ must be reflexive. Let

A $\in$ [X]. Suppose A = $\{x_1, ..., x_n\}$ where $x_i > x_{i+1}$ for i $\in$ {1, ..., n - 1}. If n = 1, then A = $\{x_1\}$ = {g (A)} and hence (A, {g(A)}) $\in I(\mathfrak{R})$. Hence suppose n $\geq$ 2 and 2 $\leq$ k $\leq$ n. Observe, g(A) = x_1 and $x_1 > x_2$ implies by Property 1, $(\{x_1\}, \{x_1, x_2\}) \in \mathfrak{R}$. Suppose $(\{x_1\}, \{x_1, ..., x_{k-1}\}) \in \mathfrak{R}$. Now $x_{k-1} > x_k$ implies (by Property 1) that $(\{x_{k-1}\}, \{x_{k-1}, x_k\}) \in \mathfrak{R}$. By repeated application of Property 2, we get $(\{x_1, ..., x_{k-1}\}, \{x_1, ..., x_k\}) \in \mathfrak{R}$. Hence by transitivity of $\mathfrak{R}$, we get, $(\{x_1\}, \{x_1, ..., x_k\}) \in \mathfrak{R}$. We have seen that $(\{x_1\}, \{x_1, ..., x_k\}) \in \mathfrak{R}$ for k=1 and 2.

Further since $(\{x_1\},\{x_1,\ldots,x_{k-1}\}) \in \mathfrak{R}$ implies $(\{x_1\},\{x_1,\ldots,x_k\}) \in \mathfrak{R}$, we have by finite mathematical induction that $(\{x_1\}, A) \in \mathfrak{R}$. By (MON), $(A, \{x_1\}) \in \mathfrak{R}$.Thus,$(A, \{g(A)\}) \in I(\mathfrak{R})$. Let $(A, B) \in [X]$. If $g(A) = g(B)$, then $(A, \{g(A)\}) \in I(\mathfrak{R})$ and $(\{g(B)\}, B) \in I(\mathfrak{R})$ implies $(A, B) \in I(\mathfrak{R}) \subset \mathfrak{R}$ (by transitivity of $\mathfrak{R}$). Hence suppose, $g(A) \neq g(B)$. Now, $(g(A), g(B)) \in \mathfrak{R}$ implies by Property 1 and transitivity of $\mathfrak{R}$, that $(\{g(A)\}, \{g(B)\}) \in P(\mathfrak{R})$. Thus $(A, B) \in P(\mathfrak{R}) \subset \mathfrak{R}$, by transitivity of $\mathfrak{R}$, since $(A, \{g(A)\}) \in I(\mathfrak{R})$ & $(\{g(B)\}, B) \in I(\mathfrak{R})$.

Conversely suppose, $(A, B) \in \mathfrak{R}$. Towards a contradiction suppose, $g(B) > g(A)$. By Property 1 and transitivity of $\mathfrak{R}$, $(\{g(B)\}, \{g(A)\}) \in P(\mathfrak{R})$. This combined with transitivity of $\mathfrak{R}$ and $(A, \{g(A)\}), (\{g(B)\}, B) \in I(\mathfrak{R})$ gives, $(B, A) \in P(\mathfrak{R})$, contradicting $(A, B) \in \mathfrak{R}$. Hence $(A, B) \in \mathfrak{R} \leftrightarrow g(A) \geq g(B)$. Completeness of $\mathfrak{R}$ is thus immediate. Thus $\mathfrak{R} = \overline{\mathfrak{R}}$.

Q.E.D.

Logical Independence of Property 1, Property 2 and MON:-Let $X = \{x, y, z\}$ with $x > y > z$.Given $A \subset [X]x[X]$,let $T(A)$ denote the transitive hull of A.

Example 1:- Let $\mathfrak{R} = [X] \times [X]$. $\mathfrak{R}$ satisfies Property 2 and MON, but not Property 1, since $x > y$, and yet $(\{x, y\}, \{y\}) \notin P(\mathfrak{R})$.

Example 2:- $\mathfrak{R} = T(\{(\{x, y\}, \{y\}), (\{x, z\}, \{z\}), (\{y, z\}, \{z\}), (\{x\}, \{x, y\}), (\{y\}, \{y, z\}),(\{x\},\{x,z\})\}) \cup \{(A, B) \in [X] \times [X] / B \subset A\}$.Now $\mathfrak{R}$ satisfies MON and Property 1. However, $(\{x\}, \{x,y\}) \in \mathfrak{R}$, $z \notin \{x, y\}$ and yet $(\{x, z\}, \{x, y, z\}) \notin \mathfrak{R}$. Thus $\mathfrak{R}$ does not satisfy Property 2.

Example 3:- $\mathfrak{R} = T(\{(\{x, y\}, \{y\}), (\{x, z\}, \{z\}), (\{y, z\}, \{z\}), (\{x\}, \{x, y\}), (\{x\}, \{x, z\}),(\{y\},\{y, z\}), (\{x, y, z\}, \{y, z\}), (\{x,y, z\}, \{x, z\}),(\{x,z\},\{x,y,z,\}), (\{x, y\}, \{x, y, z\})\})$. Here $\mathfrak{R}$ satisifies Properties 1 and 2. However, $\{x\} \subset \{x, y\}$ and

$(\{x, y\}, \{x\}) \notin \mathfrak{R}$. Hence $\mathfrak{R}$ does not satisfy MON.

Example 4:- Let $\mathfrak{R} = T(\{(\{x\}, \{x, y\}), (\{y\}, \{y, z\}), (\{x\}, \{x, z\}), (\{x,y\}, \{y\}), (\{x, z\}, \{z\}), (\{y, z\}, \{z\})\})$. $\mathfrak{R}$ satisifies Property 1. However $y \notin \{x, z\}$, $(\{x\}, \{x, z\}) \in \mathfrak{R}$ and yet $(\{x, y\}, \{x, y, z\}) \notin \mathfrak{R}$. Thus $\mathfrak{R}$ does not satisfy Property 2. Since $(\{x, y\}, \{x\}) \notin \mathfrak{R}$ it does not satisfy MON either.

Example 5: Let $\mathfrak{R} = \{ (\{ x \} , \{ x, y \}) , (\{ x, z \} , \{ x, y, z, \}) \}$. $\mathfrak{R}$ does not satisfy Property 1, because, $y > z$ and yet $(\{ y, z \},y) \notin \mathfrak{R}$. Neither does, $(\{ y, z, \} , \{ z \})$ belong to $\mathfrak{R}$. However, $\mathfrak{R}$ satifies Property 2. Since $(\{ x, y \}, \{x \})\notin \mathfrak{R}$, $\mathfrak{R}$ does not satisfy MON.

Example 6 : Let $\mathfrak{R} = T(\{(A, B) \in [X] \times [X] / B \subset A \} \cup \{ (\{x\},\{x,z\}) \})$. $\mathfrak{R}$ satisfies MON. However, $x > y$ and yet $(\{x,y\},\{y\}) \})\notin \mathfrak{R}$. Thus $\mathfrak{R}$ does not satisfy Property 1. Further $y \notin \{x,z\}$, $(\{x\},\{x,z\}) \in \mathfrak{R}$ and yet $(\{x,y\},\{x,y,z\})\notin \mathfrak{R}$. Thus $\mathfrak{R}$ does not satisfy Property2.

Example 7 : Let $\mathfrak{R} = \{(A, B) \in [X] \times [X] / x \notin A \cup B \}$. $\mathfrak{R}$ does not satisfy Property 1,since $x > y$ and yet $(\{x\},\{x,y\}) \notin \mathfrak{R}$. It does not satisfy Property 2, because, $(\{y\},\{y,z\}) \in \mathfrak{R}$,$x\notin \{y,z\}$ and yet $(\{x,y\},\{x,y,z\})\notin \mathfrak{R}$. It does not satisfy MON because, $(\{x,y\},\{y\}) \notin \mathfrak{R}$.

The following property which by rights should be attributed to Puppe[1996] is quite interesting, both in content and by way of implication:

Puppe Property: $(A,A\backslash\{x\}) \in P(\mathfrak{R})$ if and only if $g(A)= x$.

Theorem 7: The only reflexive, complete and transitive binary relation on $[X]$ to satisfy the Puppe Property is $\overline{\mathfrak{R}}$.

Proof :It is easy to see that $\overline{\mathfrak{R}}$ satisfies the Puppe Property.Hence assume that $\mathfrak{R}$ is a reflexive, complete and transitive binary relation on [X] which satisfies the Puppe Property.Let $(A,B) \in \mathfrak{R}$ and let $g(A) = x$, $g(B) = y$. Towards a contradiction suppose $y > x$. Hence, $g(A \cup B) = y$. By successive applications of Puppe Property, transitivity and completeness of $\mathfrak{R}$, we get $(A \cup B, B) \in I(\mathfrak{R})$. Further, $y > x$ and $g(A) = x$ implies $A \subset (A \cup B) \setminus \{y\}$. By the Puppe Property, $(A \cup B, (A \cup B) \setminus \{y\}) \in P(\mathfrak{R})$. If $g((A \cup B) \setminus \{y\})) = x$,then by the above argument $((A \cup B) \setminus \{y\}, A)$ $I(\mathfrak{R})$.Hence transitivity of $\mathfrak{R}$ yields, $(B,A) \in P(\mathfrak{R})$ and thus a contradiction. If not then by repeted application of the same argument we get ($(A \cup B) \setminus \{y\}, A) \in P(\mathfrak{R})$ which leads once again to $(B,A) \in P(\mathfrak{R})$ and thus a contradiction. Hence, $(A,B) \in \mathfrak{R}$ implies that $x \geq y$.

Conversely suppose $x \geq y$ and towards a contradiction suppose $(B,A) \in P(\mathfrak{R})$.However then by the above argument $y \geq x$ as well, so that $x = y$.But then by successive applications of Puppe Property, transitivity and completeness of $\mathfrak{R}$, we get $(A \cup B, B) \in I(\mathfrak{R})$ as well as $(A \cup B, A) \in I(\mathfrak{R})$.Transitivity of $\mathfrak{R}$ implies $(B,A) \in I(\mathfrak{R})$ leading to a contradiction.Hence, $x \geq y$ implies $(A,B) \in P(\mathfrak{R})$. Thus, $\mathfrak{R} = \overline{\mathfrak{R}}$.

Q.E.D.

4.The Lexicographic Extension :- A binary relation on [X] denoted $\mathfrak{R}^*$ is called the lexicographic extension if $\mathfrak{R}^* = \{(A, B) \in [X] \times [X]$/either $g(A) > g(B)$ or $[g(A) = g(B)$ and $l(A) \geq l(B)]\}$.

It is easy to see that $\mathfrak{R}^*$ is reflexive, complete, transitive and satisfies GP. However, it neither satisfies W.IND nor MON.

Example 8:- Let $X = \{x, y, z, w\}$ with all its elements distinct and suppose $x > y > z > w$. Let $A = \{y, w\}$ and $B = \{z\}$. Clearly $(A, B) \in P(\Re^*)$. Further $x \notin A \cup B$ and yet $(B \cup \{x\}, A \cup \{x\}) \in P(\Re^*)$ contradicting W.IND. Neither does $\Re^*$ satisfy MON, since $B \cup \{x\} \subset \{x, y, z, w\}$ and yet $(\{x, y, z, w\}, B \cup \{x\}) \notin \Re^*$.

However, $\Re^*$ is not the only binary relation on [X] to satisfy GP.

Let $\Re_* = \{(A, B) \in [X] \times [X] / \text{either } l(A) > l(B) \text{ or } [l(A) = l(B) \text{ and } g(A) \geq g(B)]\}$.

$\Re_*$ may be called the inverse lexicographic extension.

Example 9:- Let $X = \{x, y, z\}$ with $x > y > z > x$. Now $(\{x, z\}, \{y\}) \in P(\Re^*) \setminus \Re_*$ and $(\{y\}, \{x, z\}) \in P(\Re_*) \setminus \Re^*$. Thus $\Re^* \neq \Re_*$. However, $\Re_*$ satisfies GP.

To narrow down on $\Re^*$ we invoke the following two properties:

Property 3:- $\forall A \in [X]$ and $x, y \in X$,

(i) $l(A) \geq y$ implies $(A \cup \{x\}, \{y, x\}) \in \Re$

(ii) $(\{y\}, A) \in P(\Re)$ implies $(\{y, x\}, A \cup \{x\}) \in \Re$.

Property 4:- $\forall x, y, z \in X$ with $x > y > z$, $(\{x, z\}, \{y\}) \in P(\Re)$.

Note:- $\Re_*$ satisfies Property 3 but not Property 4. $\Re^*$ satisfies both Properties 3 and 4. $\Re_*$ satisfies the following property which $\Re^*$ does not:

Property 5:- $\forall x, y, z, \in X$ with $x > y > z$, $(\{y\}, \{x, z\}) \in P(\Re)$.

Lemma 2:- Let $\Re$ satisfy transitivity, GP and Property 3. Then $\forall A \in [X]$, $(A, \{l(A), g(A)\}) \in I(\Re)$.

Proof:- Let $A = \{x_1, \ldots, x_n\}$ where $x_i > x_{i+1}$ $\forall i \in \{1, \ldots, n-1\}$. By multiple applications of GP and transitivity we get $(\{x_1\}, \{x_1, \ldots, x_{n-1}\}) \in P(\Re)$ and $(\{x_2, \ldots, x_n\}, \{x_n\}) \in P(\Re)$.

By Property 3, $(\{x_1, \ldots, x_n\}, A) \in \Re$ and $(A, \{x_1, x_n\}) \in \Re$. Hence the lemma.

Q.E.D.

Lemma 3:- Let $\Re$ satisfy transitivity, GP and Property 3. Then $\forall$ x, y, z $\in$ X with x > y > z, $(\{x,y\},\{x,z\}) \in P(\Re)$ and $(\{x,z\},\{y,z\}) \in P(\Re)$.

Proof:- By Lemma 1, $(\{x, y, z\}, \{x, z\}) \in I(\Re)$ and by GP, $(\{x, y\}, \{x,y,z\}) \in P(\Re)$. By transitivity, $(\{x, y\}, \{x, z\}) \in P(\Re)$. Further by GP, $(\{x, y, z\}, \{y, z\}) \in P(\Re)$. By transitivity, $(\{x, z\}, \{y, z\}) \in P(\Re)$.

Q.E.D.

Theorem 8: Let $\Re$ be a binary relation on [X] which is reflexive, complete and transitive. Then

(i) $\Re = \Re^*$ if and only if $\Re$ satisfies GP, Property 3 and Property 4;

(ii) $\Re = \Re_*$ if and only if $\Re$ satisfies GP, Property 3 and Property 5.

Proof:- It is easy to see that $\Re^*$ and $\Re_*$ satisfy the desired properties respectively.

(i) Let us suppose that $\Re$ is reflexive, complete, transitive and satisfies GP, Property 3 and Property 4. Suppose $(A, B) \in I(\Re^*)$. Thus g(A) = g(B) and l(A) = l(B). By Lemma 1, $(A, \{g(A), l(A)\}) \in I(\Re)$ and $(B, \{g(B), l(B)\}) \in I(\Re)$. Hence $(A, B) \in I(\Re)$ by transitivity of $\Re$. Thus $I(\Re^*) \subset I(\Re)$.

Now suppose $(A, B) \in P(\Re^*)$.

Case 1: g(A) = g(B). Thus l(A) > l(B). By Lemma 2, $(\{g(A), l(A)\}, \{g(B), l(B)\}) \in P(\Re)$. By Lemma 1 and transitivity of $\Re$, $(A, B) \in P(\Re)$.

Case 2:- g(A) > g(B).

Suppose $g(B) > l(A)$.

Then by Property 4, $(\{g(A), l(A)\}, \{g(B)\}) \in P(\mathfrak{R})$. By GP and reflexivity, $(\{g(B)\}, \{g(B), l(B)\}) \in \mathfrak{R}$. Thus $(\{g(A), l(A)\}, \{g(B), l(B)\}) \in P(\mathfrak{R})$. By Lemma 1 and transitivity of $\mathfrak{R}$, $(A, B) \in P(\mathfrak{R})$.

Now suppose $l(A) = g(B)$.

By GP, $(\{g(A), l(A)\}, \{g(B)\}) \in P(\mathfrak{R})$. By GP and reflexivity, $(\{g(B)\}, \{g(B), l(B)\}) \in \mathfrak{R}$. By transitivity, $(\{g(A), l(A)\}, \{g(B), l(B)\}) \in P(\mathfrak{R})$. By Lemma 2 and transitivity of $\mathfrak{R}$, $(A, B) \in P(\mathfrak{R})$.

Now suppose $l(A) > g(B)$. By GP and reflexivity, $(\{g(A), l(A)\}, \{l(A)\}) \in \mathfrak{R}$. By GP, $(\{l(A)\}, \{l(A), g(B)\}) \in P(\mathfrak{R})$ and $(\{l(A)\}, g(B)\}, \{g(B)\}) \in P(\mathfrak{R})$. By transitivity $(\{l(A)\}, \{g(B)\}) \in P(\mathfrak{R})$. By GP and reflexivity, $(\{g(B)\}, \{g(B), l(B)\}) \in P(\mathfrak{R})$. By Lemma 2 and transitivity, $(A, B) \in P(\mathfrak{R})$. Thus $P(\mathfrak{R}^*) \subset P(\mathfrak{R})$.

Now $I(\mathfrak{R}^*) \cap P(\mathfrak{R}^*) = I(\mathfrak{R}) \cap P(\mathfrak{R}) = \phi$ and $I(\mathfrak{R}^*) \cup P(\mathfrak{R}^*) \cup W(\mathfrak{R}^*) = I(\mathfrak{R}) \cup P(\mathfrak{R}) \cup W(\mathfrak{R}) = [X] \times [X]$ by completeness.

$\therefore I(\mathfrak{R}) = I(\mathfrak{R}^*)$ and $P(\mathfrak{R}^*) = P(\mathfrak{R})$

$\therefore \mathfrak{R}^* = \mathfrak{R}$.

(ii) The proof that $\mathfrak{R} = \mathfrak{R}_*$ if $\mathfrak{R}$ satisfies reflexivity, completeness, transitivity, GP, Property 3 and Property 5 is similar.

Q.E.D.

We have seen that $\mathfrak{R}_*$ $(\neq \mathfrak{R}^*)$ satisfies GP, Property 3 and not Property 4.

$\overline{\mathfrak{R}}$ satisfies Property 3 and Property 4 but not GP.

Let $\mathfrak{R}_1 = \{(A, B) \in [X] \times [X] / g(A) \geq g(B) \ \& \ l(A) \geq l(B)\}$ and $\mathfrak{R}_2 = \{(A, B) \in [X] \times [X] /$ $(A, B) \notin \mathfrak{R}_1 \ \& \ \text{mid}(A) \geq \text{mid}(B)\}$ where if $A = \{x_1, \ldots, x_n\}$ with $x_i > x_{i+1}$ $\forall$ i $\in$ $\{1, \ldots, n-1\}$, $\text{mid}(A) = x_{(n+1)/2}$ if n is odd

$= x_{n/2}$ if n is even.

Here 'mid' is the short form for middle.

Let $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$. $\mathfrak{R}$ satisfies GP and Property 4. However, if $X = \{x, y\ z, w\}$ with $x > y > z > w$, then if $A = \{x, z, w\}$ then $(\{y\}, A) \in P(\mathfrak{R})$. However, $(\{y, w\}, A \cup \{w\}) = (\{y, w\}, A)$ $\notin \mathfrak{R}$. In fact $(A, \{y, w\}) \in P(\mathfrak{R})$. Thus $\mathfrak{R}$ does not satisfy Property 3.

5. The Kernel of Partial Order Extensions Satisfying Simple Monotonicity and Independence :- In this section we try to see what the structure of the intersections of the asymmetric parts of all partial order extensions (defined below) satisfying simple monotonocity and independence and the intersections of the symmetric parts of all order extensions satisfying the same properties appears to be. This section has a deceptive similarity with that of section 3 in Dutta [1997] whose relevant axioms we reproduce here.

Simple Dominance(SD): $\forall$ x, y $\in$ X, [$x > y$ implies $(\{x\}, \{y\}) \in P(\mathfrak{R})$].

Simple Monotonicity (S.MON): $\forall$ x,y $\in$ X with $x \neq y$, $(\{x, y\}, \{y\}) \in P(\mathfrak{R})$.

Independence(IND) if $\forall$ A, B $\in$ [X] and $x \in X \setminus (A \cup B)$: $(A, B) \in \mathfrak{R}$ if and only if $(A \cup \{x\}, B \cup \{x\}) \in \mathfrak{R}$.

A binary relation $\mathfrak{R}$ on [X] which is reflexive, transitive and satisfies SD is called a partial order extension.

<u>Lemma 4</u>:-Let $\mathfrak{R}$ be a transitive binary relation on [X] satisfying SM and IND, and let A,B $\in$ [X] with B $\subset\subset$ A.Then, (A,B) $\in$P($\mathfrak{R}$).

<u>Proof</u>:- Suppose, A,B $\in$ [X] with B $\subset\subset$ A and $\mathfrak{R}$ is a reflexive and transitive binary relation on [X] satisfying SM and IND. It is enough to show that the lemma is true for # (A) = # (B) + 1,since the lemma follows from it by repeated application of transitivity of $\mathfrak{R}$. For # (B) =1,the lemma holds since $\mathfrak{R}$ satisfies SM.Hence,assume that the lemma is true for # (B) = 1,...,r, and let # (B) = r+1.Let B = $\{x_1, \ldots, x_{r+1}\}$ and let A = B $\cup$ {y}.By the induction hypothesis, (A\{ x_{r+1}},B\ { x_{r+1}}) $\in$P($\mathfrak{R}$). By IND, (A,B) $\in$P($\mathfrak{R}$).Hence it is true by a standard induction argument on the cardinality of B.

Q.E.D.

Given, A $\in$[X], let <A> = < $x_1, \ldots, x_r$>,where $\{x_1, \ldots, x_r\}$=A and $x_i > x_{i+1}$ $\forall$ i$\in${1,...,r-1}.

<u>Lemma 5</u>:-Let $\mathfrak{R}$ be a transitive binary relation on [X] satisfying SD and IND, and let A,B $\in$ [X] with (a) <A> = < $x_1, \ldots, x_r$> and <B> = < $y_1, \ldots, y_r$>; (b) $x_i \geq y_i$ $\forall$ i$\in${1,...,r} and $x_i > y_i$ for some i$\in${1,...,r}. Then, (A,B) $\in$P($\mathfrak{R}$).

<u>Proof</u>:- Let A,B and $\mathfrak{R}$ be as in the lemma.It is enough to prove the lemma for the case where $x_i > y_i$ $\forall$ i$\in${1,...,r},since the general case with $\{i / x_i = y_i\} \neq \phi$, follows from it by repeated application of IND and transitivity of $\mathfrak{R}$. For r = 1,the lemma holds by SD.Hence assume that the lemma is true for r=1,...,k and then let r = k+1. By the induction hypothesis ({ $x_1, \ldots, x_r$}, {$y_1, \ldots, y_r$}) $\in$P($\mathfrak{R}$) and by IND, ({ $x_1, \ldots, x_r, y_{r+1}$}, {$y_1, \ldots, y_r, y_{r+1}$ }) $\in$P($\mathfrak{R}$). Further, by SD, ({ x_{r+1} },{y_{r+1} }) $\in$P($\mathfrak{R}$). Hence by repeated application of IND and transitivity of $\mathfrak{R}$, ({ $x_1, \ldots, x_r, x_{r+1}$}, {$x_1, \ldots, x_r, y_{r+1}$ }) $\in$P($\mathfrak{R}$).By transitivity of $\mathfrak{R}$ once again, ({ $x_1, \ldots, x_r, x_{r+1}$}, {$y_1$,

..., y_r, y_{r+1} }) $\in P(\mathfrak{R})$.Hence by a standard induction argument on the cardinality of the sets concerned,we may conclude that the lemma is true.

Q.E.D.

Let, $\mathfrak{R}^S = \{(A,B) \in [X]x[X] /$ (a) $\#(A) \geq \#(B)$; and (b) if $<A> = <x_1, ..., x_r>$ and $<B> = <y_1, ..., y_s>$; (b) $x_i \geq y_i \; \forall \; i \in \{1,...,s\}\}$. It is easy to see that $\mathfrak{R}^S$ is a partial order extension satisfying SM and IND. It is clearly not complete. The following is worth noting:

<u>Lemma 6</u>:-Let $\mathfrak{R}$ be a partial order extension satisfying SM and IND. Then, $P(\mathfrak{R}^S) \subset P(\mathfrak{R})$ and $I(\mathfrak{R}^S) \subset I(\mathfrak{R})$.

<u>Proof</u>:- Since, $I(\mathfrak{R}^S) = \{ (A,A) / A \in [X]\}$ and since $\mathfrak{R}$ is reflexive, clearly, $I(\mathfrak{R}^S) \subset I(\mathfrak{R})$. Hence let us suppose $(A,B) \in P(\mathfrak{R}^S)$. Thus, (a) $\#(A) \geq \#(B)$; (b) if $<A> = <x_1, ..., x_r>$ and $<B> = <y_1, ..., y_s>$; (b) $x_i \geq y_i \; \forall \; i \in \{1,...,s\}$; and (c) $x_i > y_i$ for some $i \in \{1,...,s\}$. If r=s, then $(A,B) \in P(\mathfrak{R})$, by Lemma 4. Hence assume $r > s$. Thus, $(\{x_1, ..., x_r\}, \{x_1, ..., x_s\}) \in P(\mathfrak{R})$, by Lemma 4. If $x_i = y_i \; \forall \; i \in \{1,...,s\}$,then $(A,B) \in P(\mathfrak{R})$.If $x_i > y_i$ for some $i \in \{1,...,s\}$,then $(\{x_1, ..., x_s\}, \{y_1, ..., y_s\}) \in P(\mathfrak{R})$, by Lemma 4.Thus, by transitivity of $\mathfrak{R}$, $(A,B) \in P(\mathfrak{R})$.

Q.E.D.

In view of the fact that $\mathfrak{R}^S$ is a partial order extension satisfying SM and IND and Lemma 5,we may conclude the following:

<u>Theorem 9</u>: (a) $\cap$ { $P(\mathfrak{R})$ /$\mathfrak{R}$ is a partial order extension satisfying SM and IND} = $P(\mathfrak{R}^S)$;

(b) $\cap$ { $I(\mathfrak{R})$ /$\mathfrak{R}$ is a partial order extension satisfying SM and IND} = $I(\mathfrak{R}^S)$.

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