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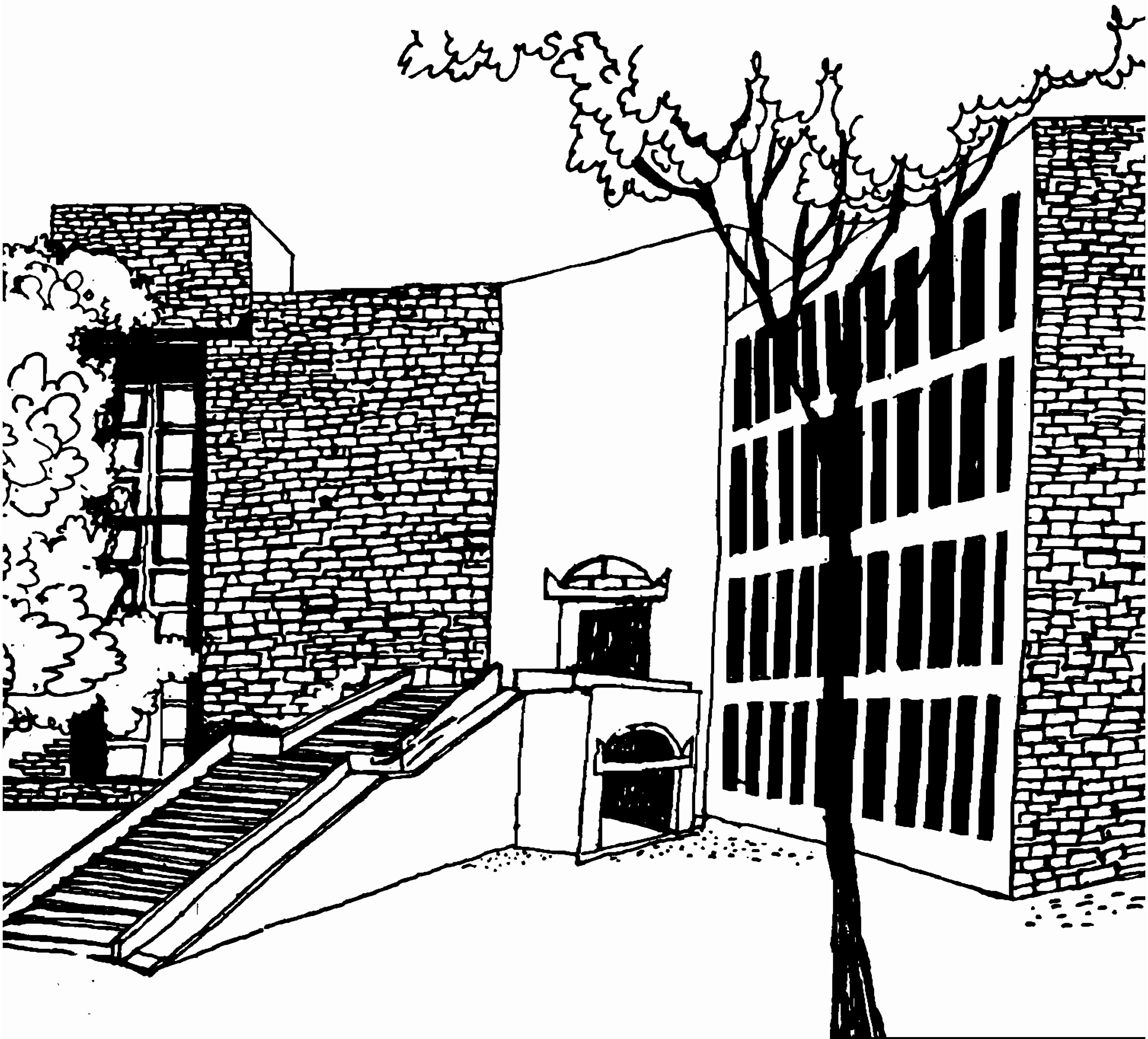

A REDUCED GAME PROPERTY FOR THE EGALITARIAN SOLUTION

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ABSTRACT

In this paper we obtain an axiomatization of the eqalitarian solution using a reduced game property.

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1. Introduction :-

In a recent paper, Peters, Tijs and Zarzuelo [1994] an axiomatic characterization of the Kalai Smorodinsky [1975] solution and a large class of solutions containing the egalitarian solution of Kalai [1977] has been provided, using a reduced game property. A crucial point in the axiomatic characterization of the generalized proportional solution of which the egalitarian solution is a member is that the set of potential players has to be infinite. The other point to note is that even if an anonymity assumption is added to the list, the proposition under discussion (i.e. Theorem 4) does not uniquely characterise the egalitarian solution. Hence, it would be appropriate to suggest that although a large family of solutions containing the egalitarian solution has been characterized in Theorem 4 of Peters, Tijs and Zarzuelo [1994], there is no characterization of the egalitarian solution available on the basis of what has been proved elsewhere in the same paper. For instance the Kalai-Smorodinsky (1975) solution satisfies all the above mentioned properties.

Our objective here is to present an independent characterization of the egalitarian solution, by using the same reduced game property and the independence of irrelevant alternatives assumptions. Our axiomatization draws heavily on Thomson (1983).

2. The Framework :-

We shall use the same notations as in Peters, Tijs and Zarzuelo [1994].

M, a finite subset of the natural numbers, denotes a set of players. R^M_+ denotes the set of all functions from M to R_+ (the non-negative reals). Let $x \in R^M_+$. Then x(i) is denoted by x_i, for all $i \in M$. A bargaining problem for M is a subset S of R^M_+ satisfying the following requirements :

(a) S is non - empty, compact, convex and contains a strictly positive vector.

(b) S is comprehensive, i.e. $y \in S$ whenever $y \in R^M_+$ and $y \leq x$ for some $x \in S$.

Let B^M denote the set of all bargaining problems for M.

Let N be a given set (population) of potential players, whether finite or infinite. Let

$$B_N = \bigcup_{\substack{\emptyset \neq M \subseteq N \\ M \text{ is finite}}} B^M$$

B_N denotes the collection of all bargaining problem for all finite subsets of N.

A solution on B_N is a function

$$F : B_N \longrightarrow \bigcup_{\substack{\emptyset \neq M \subseteq N \\ M \text{ is finite}}} R^M_+$$

that $\forall$ $S \in B_N$, $F(S) \in S$.

We are interested in axiomatically characterizing the egalitarian solution E defined as follows: $\forall$ $S \in B_N$,

$E(S) = \bar{t}\, e_M$ if $S \in B_M$, $\emptyset \neq M \subseteq N$, M finite, where e_M is the vector in R^M_+ with all co-ordinates equal to one and $\bar{t} = \max \{ t \in R_+ \ / \ t\, e_M \in S \}$.

The following properties are easily seen to be satisfied by E :

Weak Pareto Optimality (WPO) :

There does not exist $y \in S$ with $y \gg F(S)$, whenever $S \in B_N$.

Anonimity (AN) : For every finite $M \subseteq N$, all $i, j \in M$, and all $S, T \in B^M$ such that T arises from S by interchanging the i^{th} and j^{th} co-ordinates of the points of S, we have : $F_i(S) = F_j(T)$, $F_j(S) = F_i(T)$ and $F_k(S) = F_k(T)$ $\forall$ $k \neq i, j$.

Homogeneity (HOM) : For every finite subset M of N and every $a \in R^M_{++}$ with $a_i = a_j$ for all $i, j \in M$, we have $F(aS) = a F(S)$ (Here for $a \in R^M_+$, $x \in R^M_+$, ax denotes the vector whose i^{th} co-ordinate $(ax)_i = a_i x_i$, for $S \subseteq R^M_+$; $aS = \{ ax \ / \ x \in S \}$)

Nash's Independence of Irrelevant Alternatives (NIIA) :- For all S, T $\in B^m$. Where M is finite and $M \subseteq N$ if $S \subseteq T$ and $F(T) \in S$, then $F(S) = F(T)$.

Continuity (CONT) :- For all $\emptyset \neq M \subseteq N$, M finite, for all sequences $\{S^v\}$ of elements of B^m, if $S^v \dashrightarrow S \in B^m$, then $F(S^v) \dashrightarrow F(S)$. (In this definition, convergence of S^v to S is evaluated in the Hausdorff topology.

Let L,M be non-empty finite subsets of N with $L \subseteq M$. Let $S \in B^m$. For $x \in R^m_+$, let x_L denote the projection of x on R^L_+. Then S_L denotes the bargaining problem $\{ x_L / x \in S \}$ in B^L. Let $x \in S$, $x \neq 0$, $x_L \neq 0$. Let

$$\lambda(S_L, x_L) = \min \{ \lambda \in R_+ / x_L \in S_L \}$$

The reduced game of S with respect to L and X is the following bargaining problem for L :

$$S^x_L = \lambda(S_L, x_L) S_L$$

It is easy to check that x_L is an element of the weakly pareto optimal subset of S^x_L i.e. $x_L \in W(S^x_L) = \{ y \in S^x_L$ / there is no $Z \in S^x_L$ with $Z \gg y \}$

Reduced Game Property (RGP) : For all non-empty subsets $L \subseteq M$ of N and all $S \in B^m$: if $F_L(S) \neq 0$, then $F(S_L^{F(S)}) = F_L(S)$.

It is easy to check that the egalitarian solution E satisfies RGP.

3. The Characterization Theorems :-

Lemma1 : Let F be a solution on B_N ($|N| \geq 2$) which satisfies NIIA and CONT. Let $\emptyset \neq M \subseteq N$, with $|M| = 2$ and let $S \in B^M$. If $x \in S$, $x \leq 2E(S)$ implies $F(S) = E(S)$, then $F(S)=E(S)$ $\forall S \in B^M$.

Proof :- This is Lemma 4.2 in Thomson and Lensberg [1989].

Theorem 1 :- A solution on B_N ($|N| \geq 2$) satisfies WPO, AN, HOM, NIIA, RGP and CONT if and only if it is the egalitarian solution.

Proof :- Let us check that the above axioms characterize E, since we already know that E satisfies the above axioms.

Let us as in Peters, Tijs and Zarzuelo [1994] first prove that if $|M| = 2$ and $S \in B^M$, then $F(S) = E(S)$ where F satisfies the desired properties.

Let $M = \{i, j\}$ and $S \in B^M$. Let $k \in N \setminus M$ and $E(S) = \bar{\lambda} e_M$, where $\bar{\lambda} \geq 0$. Let $L = \{i, j, k\}$. Construct a set T in R^L_+ as follows :

T = comprehensive convex hull of $\{\bar{\lambda} e_L, S\}$

Clearly $T_M = S$

Let $U = \{x \in R^L_+ / \sum_{i \in L} x_i \leq 3\bar{\lambda}\}$

By AN and WPO, $F(U) = \bar{\lambda} e_L$.

Case 1:- $x \in S \Rightarrow x \leq 2\,E(S)$.

In this case $S \subseteq U$

Thus $T \subseteq U$

Since $\bar{\lambda} e_L \in T$, by NIIA, $F(T) = \bar{\lambda} e_L$

By RGP , $F(T_M^{F(T)}) = \bar{\lambda} e_M$

By HOM, $F(T_M) = \dfrac{\bar{\lambda}}{\lambda(T_M, F_M(T))} e_M$

Thus $F(S) = \dfrac{\bar{\lambda}}{\lambda(T_M, F_M(T))} e_M$

Since F(S) and E(S) are both Weakly Pareto Optimal in S and lie on the diagonal, F(S) = E(S).

Case 2 :- Case 1 does not hold

Then by Lemma 1, F(S) = E(S)

Let now $|M| > 2$ and $S \in B^M$. Let $i, j \in M$. Then

$F_i(S_{\{i,j\}}) = F_j(S_{\{i,j\}})$ by the above

Thus by RGP and HOM, $F_i(S) = F_j(S)$. Since this holds for all $i, j \in M$, we conclude by WPO, F(S) = E(S).

For $|M| = 1$, and $S \in B^M$, F(S) = E(S) by WPO

This proves the theorem.

Weak Reduced Game Property (WRGP): For all non-empty finite subsets L and M of N with $L \subseteq M$ and $|L| = 2$ and all $S \in B^M$

$F(S_L{}^{F(S)}) = F(S)_L$

Theorem 2 :- A solution on B_N ($|N| \geq 2$) satisfies WPO, NIIA, CONT AN, HOM, and the WRGP if and only if it is the egalitarian solution.

Proof :- as in the proof of theorem 1.

Call a solution F on B_N Strongly Individually Rational (SIR) if $F(S) \gg 0$ for all non-empty subsets M of N and all $S \in B^M$.

Lemma 2 :- Let F be a Strongly Individually Rational and Homogeneous solution on B_N satisfying the Reduced Game Property, and let M be a non-empty finite proper subset of N. Let $S \in B^M$. Then $F(S) \in W(S) = \{ X \in S / y \gg x$ implies $y \notin S \}$

Proof : - See Peters, Tijs and Zarzuelo [1994]

Theorem 3 :- Let N be infinite. A solution on B_N satisfies Anomynity, Continuity Homogeneity, Reduced Game Property, Strong Individual Rationality and Nash s Independence of Irrelevant Alternatives Assmption, if and only if it is the egalitarian solution.

Proof : Immediate consequence of theorem 1 and lemma 2

4. Relation with earlies work :-

As pointed out in Peters, Tijs and Zarzuelo [1994], if a solution for B_N satisfies Homogeneity and Reduced Game Property, then it also satisfies the following axiom

Monotonicity with respect to changes in the member of agents (MON) :

For all non-empty finite subsets $L \subseteq M$ of N and all $S \in B^L$, $T \in B^M$, if $S = T_L$, then $F(S) \geq F_L(T)$

(In Lahiri (1990) we discuss some interesting properties of solutions satisfying this axiom.)

They also provide a counter example to show that the converse is not true.

Thomson [1983] characterizes the egalitarian solution using WPO, AN, NIIA, MON and CONT. Thus Thomson's characterization implies Theorem 1, **though not Theorem 2.**

A set $S \in B_N$ is said to be strictly comprehensive if $x \in S$, $y \in S$, $y > x$ implies that there exists $z \in S$ with $z \gg x$. (This definition can be found in Thomson and Lensberg [1989] for instance).

Let $S \in B_N$ $y \in S$ is Pareto optimal in S if $x \in S$, $x \geq y$ implies $x = y$

In the proof of the following theorem we appeal to the fact that for any $S \in B^M$, where $\emptyset \neq M \subseteq N$, M finite, there exists a sequence S^v of sets in B^M, each strictly comprehensive and $S^v \to S$ in the Hausdorff topology.

Theorem 4: Let F be a solution on B_N ($|N|>2$) satisfying SIR, HOM, NIIA, MON and CONT. Then F satisfies R G P

Proof: Suppose $T \in B^M$ is strictly comprehensive for $\emptyset \neq M \subseteq N$, M finite.

Let $L \subseteq M$ and suppose towards acontradiction that,

$0 \neq F_L(T) \neq \lambda(T_L, F_L(T)) F(T_L)$.

Let $\lambda^* = \lambda(T_L, F_L(T))$.

Since $F_L(T) \neq 0$, $\lambda^* > 0$.

Observe $F_L(T) \leq F(T_L)$ by MON.

Since T is strictly comprehensive, there exists $\bar{\lambda} > \lambda^*$ (sufficiently close to λ^*) such that $F_L(T) \nleq \bar{\lambda} F(T_L) = F(\bar{\lambda} T_L)$.
(This is true even if $\lambda^* F(T_L)$ weakly Pareto dominates $F_L(T)$, since $F_L(T)$ is always weakly Pareto optimal in $\lambda^* T_L$. Note the final equality follows from HOM).

Clearly $F_L(T) \in \bar{\lambda} T_L$

Consider $S = \{y \in \mathbb{R}^M_+ / y_L \in \bar{\lambda} T_L\}$
and $S' = T \cap S$

S is the cylinder with base $\bar{\lambda} T_L$ and S' is the intersection of T with the cylinder whose base is $\bar{\lambda} T_L$.

$S' \subseteq T$ and $F(T) \in S'$ => (by NIIA), $F(S') = F(T)$.

Now $S'_L = \bar{\lambda} T_L \Rightarrow F(S'_L) = F(\bar{\lambda} T_L) = \bar{\lambda} F(T_L)$.

Hence by MON, $F(S'_L) = \lambda F(T_L) \geqq F_L(S') = F_L(T)$

But we have $F_L(T) \nleq \lambda F(T_L)$ which leads to a contradiction.

The proof is completed by appealing to CONT and by observing that λ is a continuous function of its arguments.

Q.E.D.

As a consequence of Theorem 3 and theorem 4 we have:

Theorem 5:- Let N be an infinite set. A solution on B_N satisfies SIR, AN, HOM, NIIA, MON and CONT, if and only if it is the egalitarian solution.

In theorem 5, we have replaced the assumption of WPO used in Thomson and Lensberg [1989] earlier. Since SIR by itself does not imply WPO, we may view this as an alternative way of stating an earlier result.

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