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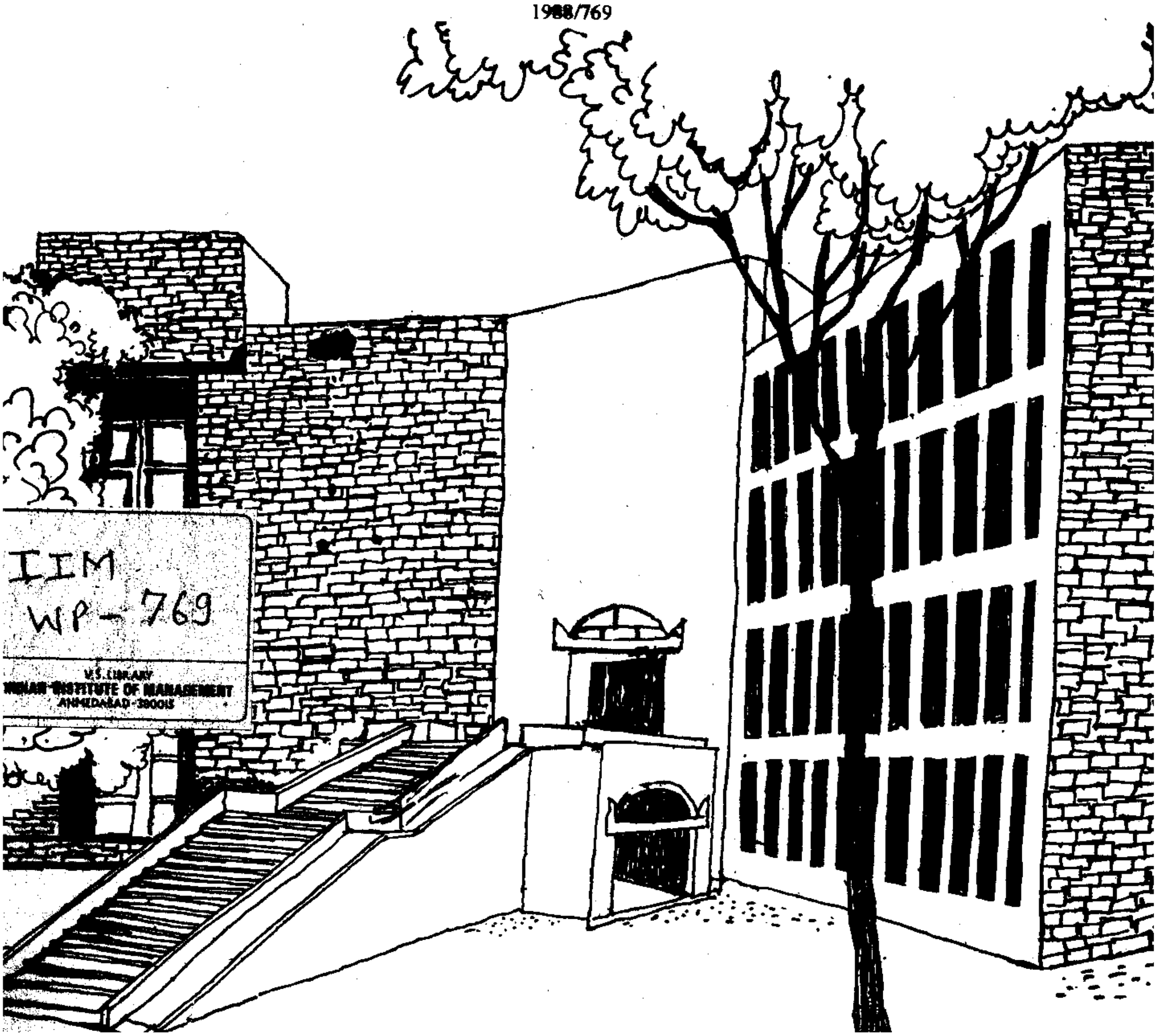

RISK SENSITIVITY IN BARGAINING AND A MONOTONE SOLUTION TO NASH'S BARGAINING PROBLEM

By

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This paper was written when I was visiting the Center for Theoretical Studies, Indian Institute of Science, Bangalore.

I am grateful for useful conversation with Dr. Diptiman Sen.

ABSTRACT

In this paper we show that for a new solution to Nash's bargaining problem, proposed by Lahiri (1988) ("Monotonicity With Respect To The Disagreement Point And A New Solution To Nash's Bargaining Problem", Indian Institute of Management, Ahmedabad, Working Paper No. 724), which satisfies monotonicity with respect to the disagreement point, an increase in risk aversion is to the player's own disadvantage and to the advantage of the opponent in the two person case; to the advantage of all opponents in the multi-person generalization. Thus it parallels results on risk-sensitivity for the Nash and Kalai - Smorodinsky solutions.

1. Introduction :-

In 1950 Nash introduced the two-person bargaining problem. In such a problem two bargainers are involved who can agree upon one of the points in a set S of feasible utility pairs or who can disagree, in which case the pay off is a utility pair d, called the disagreement point. The pair (S,d) determines the problem.

Kihlstrom, Roth, and Schmeidher (1981) and Roth (1979) proved that the Nash and Kalai-Smorodinsky solution of bargaining games with two players have the property of risk sensitiivty. For bargaining over riskless outcomes, an increase in one player's risk aversion changes the solution outcome to the advantage of the other player. Roth and Rothbloom (1982) studied the sensitivity of the two-person Nash solution to changes in risk aversion in the more general case where bargaining concerns both risky and riskless outcomes. They identified a class of situations where it is to the advantage of the opponent and a class of situations where it is to his disadvantage.

Neilsen (1984) generalizes the risk sensitivity property in another direction : Bargaining is over riskless outcomes only, but there may be more than two participants in the game. In this case, the Nash solution does not necessarily predict that an increase in a player's risk aversion helps all his opponents. Some, but not all, may actually be hurt. However, it can be unambiguously concluded that the player whose risk aversion has increased does not gain. The Kalai-Smorodinsky solution for games with more than two players predicts that an increase in a player's risk aversion helps all his opponents and hurts the player himself.

In this paper we consider a new solution which is monotone with respect to the disagreement point as in Lahiri (1988). We consider first the case when there are only two participants in the game and subsequently the case when there are more than two participants is considered.

The basic references for 'n' person generalization of the Nash (1950) solution is Roth (1979) and for the n-person generalization of the Kalai-Smorodinsky (1975) solution is Thomson (1983). Lensberg (1981) adopts a similar extension of the Nash solution. This has led to intensive study of these solutions in ongoing work by Thomson and Lensberg (1983).

2. Increasing Risk Aversion

It will be assumed that the players bargain over a set C of riskless outcomes. Each player has a preference on a class P of probability distributions on C containing the simple distributions (those concentrated at a finite number of points). The preference relation of player i is represented by a utility function ui on C. This means that if p is a distribution in P, the ui is integrable with respect to p, and if p and q are distributions in P, then p is preferred to q if and only if the expected value of ui under p is greater than the expected value under q. Axioms ensuring the existence of ui can be found for examp,le in von Neumann and Morgenstern [1947], Herstein andMilnor [1975], Fishburn [1970], and Shepherdson [1980] for the case where P contains only simple distributions, and in Fishburn [1970], Foldes [1972], and Grandmont [1972] for more general distributions.

Based on Yaari [1969], Kihlstrom and Mirman [1974] and Roth [1979] have defined what it means that one concave utility function Vi on a convex subset C of IR^m is more risk averse than another concave utility function ui. For present purposes, there is no reason to assume that C is a convex subset of IR^m or that ui and vi are concave. So, let C be an arbitrary set, P a class of probability distribution on C containing the simple distributions, and vi and ui two utility functions on C representing two preference relations on P. Call vi *more risk averse* than ui if ui and vi represent the same preference relation on C and if Ep (vi) $\geq$ vi (c) implies Ep (Ui) $\geq$ Ui (c) for all p in P and c in C.

In words, all distributions preferred to a riskless outcome by a decision maker with utility function Vi are also preferred to that same riskless outcome by a decision maker with utility function Ui.

If all distributions in P have certain equivalents for ui, i.e., if Ui(C) is an interval, then Vi is more risk averse than Ui if and only if there exists a concave function k on Ui (C) such that V = ko ui (see, e.g. Roth [1979]).

3. Risk Sensitivity of The New Monotone Solution

An n-person bargaining game is a pair (S,d), where S is a compact convex subset of $\mathbb{R}^n$, d is a point in S, and there is at least one point b in S with $d \ll b$. The set S represents the possible combinations of utility levels that the players can simultaneously reach, and the point d, the disagreement point, represents the utility levels that they end up with if they do not agree on another point.

We shall consider a sub-class of bargaining problems defined below as in Lahiri [1988]:

Let $W = \{(S,d) / S \subseteq \mathbb{R}^n$, S is convex, compact, and $\exists\, x \in S\}$ with $x \gg d\}$.

Let, $\bar{W} = \{(S,d)$ W/ if $x \in S$ and $0 \leq y \leq x$ then $y \in S\}$.

and $\bar{\bar{W}} = \{(S,d) \in \bar{W}/$ $u \in S$ such that $d \gg u\}$.

We shall refer to games $(S,d) \in W$ as comprehensive games and to games $(S,d) \in \bar{\bar{W}}$, as proper compehensive games.

In this paper we consider a solution $F: \bar{\bar{W}} \longrightarrow \mathbb{R}^n$ defined thus:

Let $Z(S) = (Z_1(S), Z_2(S), \ldots\ldots, Z_n(S))$,

where $Z(S) = \min \{X_i / X \in S\}$. Then $\forall (S,d) \in W$

F (S,d) satisfies the following two conditions:

(a) $$\frac{F_i(s,d) - Z_i(S)}{d_i - Z_i(S)} = \frac{F_j(s,d) - Z_j(S)}{d_j - Z_j(S)} \quad \forall i,j \in \{1,\ldots n\}$$

(b) $$\frac{x_i - z_i(S)}{d_i - z_i(S)} = \frac{F_i(S,d) - z_i(S)}{d_i - z_i(S)}, \quad \forall i \in \{1,\ldots,n\},$$

and $X > F(S,d)$ implies $X \notin S$.

We shall assume that the von-Neumann Morgenstern utility functions of the players are normalized so as to be consistent with the following blanket hypothesis:

<u>Blanket</u> <u>Hypothesis</u>:- For all $(S,d) \in \bar{\bar{W}}$, $z_i(S)=0$, $i=1,\ldots,n$.

The conditions defining the above bargaining solution for $n = 2$ are :

<u>Condition</u> <u>1</u>:- $F(S,d) \gg d \;\; \forall (S,d) \in \bar{\bar{W}}$

<u>Condition</u> <u>2</u>:- Let $a_1, a_2 \in IR_{++}$, $b_1, b_2 \in IR$, and (S,d), $(S',d') \in \bar{\bar{W}}_2$ and define $d_i' = a_i d_i + b_i$, $i = 1,2$ and $S' = \{x \in IR^2 / x_i = a_i y_i + b_i,\; i = 1,2,\; y \in S\}$. Then,

$$F_i(S',d') = a_i F_i(S,d) + b_i,\; i = 1,2.$$

<u>Condition</u> <u>3</u>:- If $(S,d) \in \bar{\bar{W}}$ satisfies $d_1 = d_2$ and $(x_1,x_2) \in S$ implies $(x_2, x_1) \in S$, then $F_1(S,d) = F_2(s,d)$.

<u>Condition</u> <u>4</u>:- If $x \gg F(s,d)$ then $x \notin S$.

<u>Condition</u> <u>5</u>:- Let (S,d) and (S',d') satisfy

(a) $d_1 = d_1'$, $d_2 \leq d_2'$

(b) $S \subseteq S'$. Then $F_2(S',d') \geq F_2(S,d)$.

If in addition $S=S'$, then $F_1(S',d') = F_1(S,d)$ with

$$F(S',d') \neq F(S,d) \text{ if } (d_1,d_2) \neq (d_1',d_2').$$

Condition 1 stipulates individual rationality.

Condition 2, requires that the solution should be invariant to positive affine utility transformations.

Condition 3, imposes symmetry.

Condition 4, requires <u>weak</u> <u>Pareto</u> <u>optimality</u>.

Condition 5, is our version of monotonicity with respect to the disagreement point.

In Lahiri [1988] we prove the following :-

<u>Theorem 1</u>:- The function $F: \overline{\overline{W}} \longrightarrow \mathbb{R}^2$ satisfying (a) and (b) is well defined, satisfies Conditions 1 to 5 and is the only function to satisfy these conditions.

The assumption that the players bargain over the riskless outcomes in C means that they are playing a game (S,d), where $S=u(C)$, $u = (u1,...,un)$, and each ui is a utility function on C for player i. In particular, the assumption requires that u(C) be convex. Contrary to what sometimes seems to be implied (Roth [1979], Kihlstrom, Roth and Schmeidler [1981], Roth and Rothblum [1982]), convexity of u(C) does not follow from an assumption that C is a convex subset of some vector space and all ui are concave.

If a player i becomes more risk averse (or if a more risk averse player takes his place), then the i^{th} utility function becomes $Vi = k \circ ui$ for some differentiable concave, strictly increasing function k on ui(C). For any x in X, put $\tilde{x} = (x1, \ldots, xi-1, k(xi), xi, \ldots, xn)$. Put $\tilde{S} = \{\tilde{x} : x \in S\}$. If S is convex, then $(\tilde{S},\tilde{d})$ is an n-person bargaining game. The assumption that $(S,d) \in \overline{\overline{W}} \subset \overline{W}$ guarantees that $\tilde{S}$ is convex.

4. The Situation With Two Players:-

Suppose there are just two players and suppose without loss of generality that player 2 becomes more risk averse or is replaced by a more risk-averse player. By methods similar to the ones used in Theorem 3.3 of Jansen and Tijs [1983], we can show that our solution F satisfying (a) and (b) is continuous with respect to the Hausdorff metric topology on $\underline{\underline{W}}$. In the following theorem we shall show that the new outcome is not preferred to the old outcome by the new player 2 (or, for that matter, by the old player 2). Player 1 stands to gain in the process and it is in this sense that player 2 is at a disadvantage.

<u>Theorem</u> <u>2</u>:- Suppose that $(\hat{S},\hat{d})$ is a game derived from (S,d) by making player 2 more risk-averse. Let $y = F(S,d)$ and let $x \in S$ be chosen such that $\hat{x} = F(\hat{S},\hat{d})$. Then $x_1 > y_1$.

<u>Proof</u> <u>of</u> <u>Theorem</u> <u>2</u>:- Since F is invariant under positive affine transformations, it can be assumed that $d = 0$ and $\hat{k}(0) = \hat{d}_2 = 0 = d_1$. Since $\hat{x} = F(\hat{S},\hat{d})$ and $y = F(S,d)$, it follows that

$$\phi(y_1) = y_1$$

where $u_1 \longmapsto \phi(u_1)$ is the decreasing, concave function for which $\{(u_1, \phi(u_1)) : u_1 \in [\underline{u}_1, \bar{u}_1]\}$ is the set of all weakly Pareto - optimal outcomes of S. We can without loss of generality assume that ϕ is differentiable since differentiable weakly Pareto - optimal boundaries are dense in the class of all continuous weakly Pareto optimal boundaries and F is continuous in the Hausdorff metric-topology.

Clearly $\hat{x}$ satisfies the condition

$$k(\phi(\hat{x}1)) = \hat{x}1$$

where $\hat{x}1 = x1$, $\hat{x}2 = k(\phi(x1)) = k(x2)$ and $\hat{x} = (\hat{x}1, \hat{x}2)$

Let,

$A(u1) = (k(\phi(u1)) - u1$

Clearly, $A(x1) = 0$.

Also,

$$A(y1) = k(\phi(y1)) - y1$$

Now, the concavity and nonnegativeness of k implies

$$\frac{k(a)}{k(b)} - \frac{a}{b} \gtreqless 0$$

according as $\frac{k(a)}{a} - \frac{k(b)}{b} \gtreqless 0$

according as $a \lesseqgtr b$.

Since $\phi(y1) > 1$, $k(\phi(y1)) > \phi(y1)$ as $k(1) = 1$

$\therefore A(y1) > 0$

Further,

$$A'(u1) = k'(\phi(u1)).\ \phi'(u1) - 1 < 0;$$

which is easily observed since $\phi'(.) < 0$.

Thus,

$A(x1) = 0$, $A(y1)>0$ and $A'(u1) < 0$ implies

$x1>y1$ as was required to be proved.

Q.E.D.

In the case where n = 2, i.e., the player who beomes more risk averse has only one opponent, it follows that this opponent does not prefer the old outcome. This result agrees with the result observed for the Nash solution by Kihlstrom, Roth, and Schmeidler [1981].

5. The Situation With More Than Two Players:-

With more than two players an analogous result can be obtained. Nielsen [1984] obtains risk-sensitivity results for the Nash and the Kalai-Smorodinsky solution with more than two players. As in Nielsen [1984] we show that if player i becomes more risk averse or is replaced by a more risk averse player, then the old outcome is not preferred to the new one by any of the opponents, and neither the old player i nor the new player i prefers the new outcome. The old and or the new player i can only be indifferent between the outcomes if all players are indifferent between them.

Theorem 3:- Let $(S,d) \in \overline{\overline{W}}$. Then $(\hat{S},\hat{d})$ $\overline{\overline{W}}$. Let $y=F(S,d)$, and let $z \in S$ be such that $z = F(\hat{S}\ \hat{d})$. Then $z_j \geq y_j$ for $j \neq i$ and $z_i \leq y_i$. If $z_i = y_i$, then $z=y$.

Proof of Theorem 3:- Clearly $(\hat{S},\hat{d})$ $\overline{\overline{W}}$. To see that $\hat{S}$ is convex, let $p, q \in \hat{S}$, t [0,1], and put $v=tp+(1-t)q$. Since k is concave, $t\ k(p_i) + (1-t)\ k\ (q_i) \leq k(v_i)$, so that $d \leq tp + (1-t)\ q \leq v$. But since (S,d) has disposable utility, this implies that $tp+(1-t)\ q \in \hat{S}$. It can be assumed that $d = (1,1)$ $k(1) = 1$, and $k\ (0) = 0$ because F is invariant under positive linear transformations. Then $Z(S) = Z(\hat{S})=0$. Since k is concave, $\frac{k(t)}{t}$ is a decreasing function of 't'.

As before, let $x_1 = \phi(x_2,\ldots\ldots,x_n)$, and let us assume that player 1 is replaced by a more risk-averse player. Let ϕ be the Pareto-optimal surface of S, and as in the proof of Theorem 2 (for the same reasons as before) we may assume that ϕ is a differentiable function.

Clearly,

$$\phi(y_2,\ldots\ldots,yn) = y2 \cdots\cdots = yn$$

Similarly,

$$k(\phi(z2,\ldots\ldots,zn)) = z2 \cdots\cdots = zn$$

Let,

$$A(x2,\ldots\ldots,xn) = k\phi(x2,\ldots\ldots,xn)) - x2$$

$\therefore$ $A(z2,\ldots\ldots,zn) = 0$ and

$$A(y2,\ldots\ldots,yn) = k\ (\phi(y2,\ldots\ldots,yn)) - y2$$

Observe,

$$\frac{k(\phi(y2,\ldots\ldots,yn)}{\phi(y2,\ldots\ldots,yn)} < 1 = \frac{k(1)}{1}$$

$$\therefore\ k(\phi(y2,\ldots\ldots,yn)) < \phi(y1,\ldots\ldots,yn).$$

$$\therefore\ A(y2,\ldots\ldots,yn) = \phi(y1,\ldots,yn) - y2 < 0$$

Further,

$$dA(x\ ,\ldots\ldots,x\ ) = k'(\phi(x\ ,\ldots\ldots,x\ ))\ d\phi(x2,\ldots,xn) - 1$$

Since ϕ is the equation of the Pareto optimal surface,

$$\frac{\partial\phi}{\partial xi} < 0 \quad \forall\ i \in (2,\ldots\ldots,n)$$

$$\therefore\ \frac{\partial A}{\partial xi}(x2,\ldots\ldots,xn) < 0 \quad \forall\ i \quad (2,\ldots\ldots,n).$$

$A(y2,\ldots\ldots,yn) < 0$, $A(z2\ ,\ldots\ldots,zn\ ) = 0$ and

$\frac{\partial dA}{\partial xj}(x2,\ldots,xn)<0$ implies $zj > yj$ for some j $(2,\ldots\ldots,n)$.

But

$$z2 = \ldots\ldots\ldots = zn \text{ and}$$

$$y2 = \ldots\ldots\ldots = yn$$

implies,

$$z_j \geq y_j \;\forall\; j \in \{2,\ldots,n\}.$$

$\therefore$ $z_1 = \phi(z_2,\ldots,z_n)) \leq \phi(y_2,\ldots,y_n) = y_1$, since ϕ is the equation of the Pareto optional surface. Now by going to the limit for the general case, we get $z_1 \leq y_1$ as was required to be proved.

Q.E.D.

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