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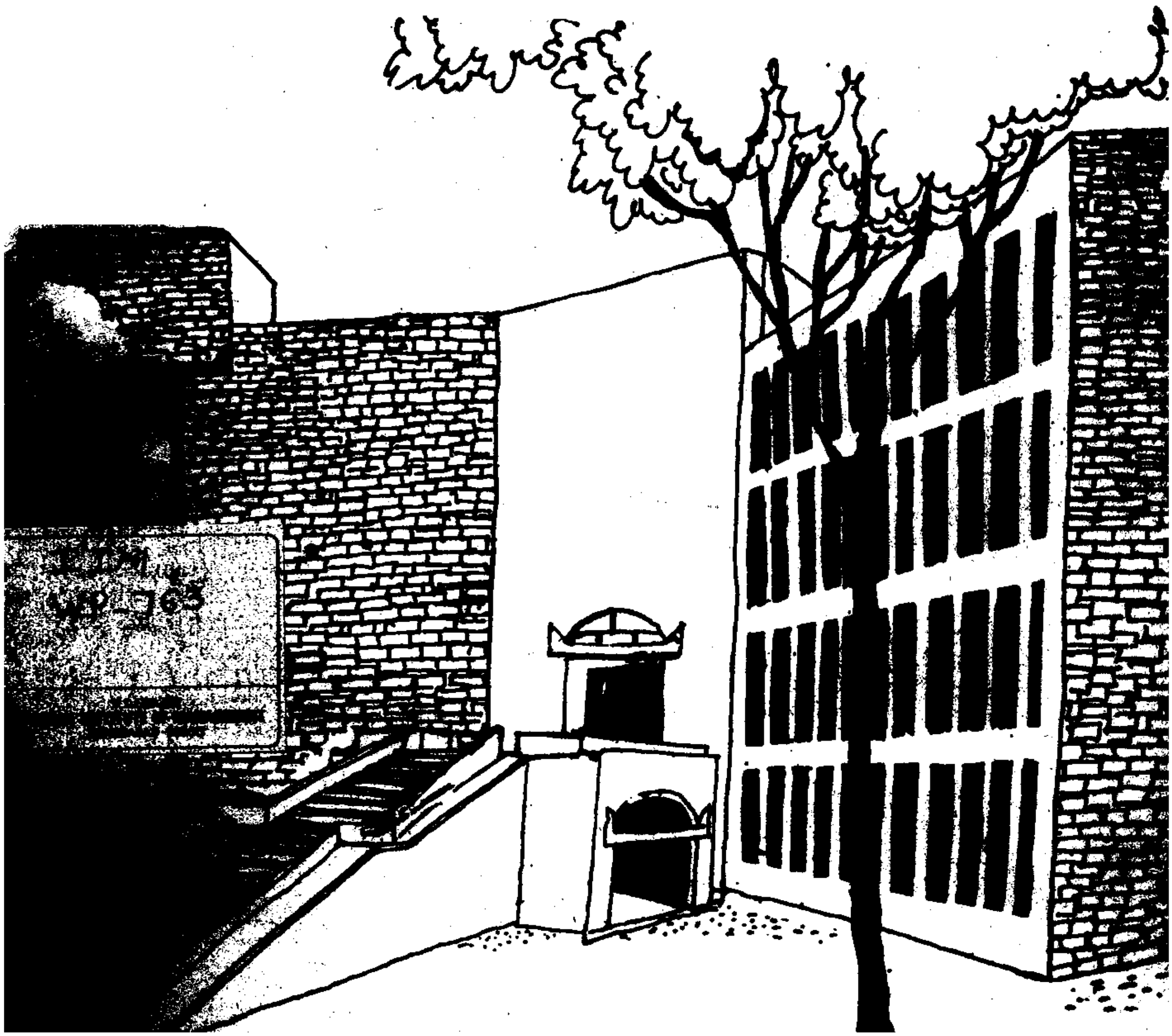

BEHAVIOUR OF INTEGERS: SOME PECULIAR PROPERTIES

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ABSTRACT

The intention of this paper is to highlight certain latent and interesting characteristics of natural numbers and their higher powers vis-a-vis the ultimate sum (single digit sum) of the digits in such integers and their higher powers, their sums and their multiples.

The peculiarities owe their characteristics indeed, to the decimal system of integers. And it is the existence of such peculiarities that this paper endeavours to demonstrate.

While the author is not aware of any practical use to which the interesting property of integers described in this paper could be put to, it may possibly of some interest to those involved with checks and verification of numerical solutions on computers, apart from of course the number theory buffs.

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The intention of this paper is to highlight certain latent and interesting characteristics of natural numbers and their higher powers vis-a-vis the ultimate sum (single digit sum) of the digits in such integers and their higher powers, their sums and their multiples.

The peculiarities owe their characteristics indeed, to the decimal system of integers. And it is the existence of such peculiarities that this paper endeavours to demonstrate.

The understanding of the entire phenomena is likely to be enhanced, if at the outset, we study the contents of Table I.

Table 1 : The first column of the table consists of integers from 1 to 20.

Starting from the integer 10 in the first column, it is clear that in the column represented by S(I), the digits of the integers have been added up viz. $10 \rightarrow 1+0=1$, $11 \rightarrow 1+1=2$, $12 \rightarrow 1+2=3$ and so on.

The second column (represented by $S(I^2)$) gives the ultimate sum of digits of the integers obtained by squaring the corresponding integers in column 1, viz. integer 4; at the crossection of column 2 and row 7, is the ultimate sum of the digits present in the square of 7 i.e. 49 ($7^2 = 49 \rightarrow 4+9 = 13 \rightarrow 4$) and so on.

Having thus understood the structure of the table, we shall proceed to list out the peculiarities presented by the Table A.

TABLE 1

Row No. \ Col. No.		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	I	S(I)	$S(I^2)$	$S(I^3)$	$S(I^4)$	$S(I^5)$	$S(I^6)$	$S(I^7)$	$S(I^8)$	$S(I^9)$	$S(I^{10})$	$S(I^{11})$	$S(I^{12})$	$S(I^{13})$	$S(I^{14})$	$S(I^{15})$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	4	8	7 *1	5	1	2	4	8	7	5	1	2	4	8
3	3	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9
4	4	4	7	1 *2	4	7	1	4	7	1	4	7	1	4	7	1
5	5	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8
6	6	6	9	9	9	9	9	9	9	9	9	9	9	9	9	9
7	7	7	4	1	7	4	1	7	4	1	7	4	1	7	4	1
8	8	8	1	8	1	8	1	8	1	8	1	8	1	8	1	8
9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
10	10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	11	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8
12	12	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9
13	13	4	7	1	4	7	1	4	7	1	4	7	1	4	7	1
14	14	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8
15	15	6	9 *3	9	9	9	9	9	9	9	9	9	9	9	9	9
16	16	7	4	1 *4	7	4	1	7	4	1	7	4	1	7	4	1
17	17	8	1	8	1	8	1	8	1	8	1	8	1	8	1	8
18	18	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
19	19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
20	20	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8

Box 1

Box 2

Note :
a. The first nine rows are recursive (Box 1 repeats itself as Box 2).
b. Represents horizontally recurring cycles.
c. Examples: (*1) is $2^4 = 16 \to 1+6 = 7$. (*2) is $4^3 = 64 \to 6+4 = 10 \to 1+0 = 1$
(*3) is $15^2 = 225 \to 2+2+5 = 9$. (*4) is $16^3 = 4096 \to 4+0+9+6 = 19 \to 1+0 = 1$
Where the sign '$\to$' denotes the operation of arriving at the single digit sum, expressed as operator S in the text.

However, before progressing further, it may be in order to get acquainted to some of the notations used in the paper:

1. The paper deals with natural numbers, symbolised by N.

2. S is an operator meant to represent the ultimate summation of digits in any natural number.

3. $S(N)$ stands for the ultimate summation of the digits in N.

4. $S(I^K)$ (where $I, K \in N$) stands for the ultimate summation of the digits in I^K.

5. Ultimate summation means the summation of the digits, till a single digit number is obtained as the sum, e.g. If $I = 29174$, $S(I) = 2+9+1+7+4 = 23 \rightarrow 2+3 = 5$.

Thus, $\forall\ I, K \in N,\ S(I^K) \in \{1,2,3,\ldots 9\}$

OBSERVATIONS AND PROOFS :

1: The first 9 rows of the table repeat themselves starting at every $(9N+1)$th row.

This is a natural outcome of the decimal system of integers and needs little further elaboration. However, this observation becomes clearer as we analyse the next observation.

2. All $I (I \in N)$ which correspond to a single value of $S(I)$, would again yield a single value of $S(I^K)$, for any value of K in N.

This may be expressed as:

Theorem:

If $S(I_1; I_2, \ldots I_N) = i$ (Where i=Constant and $i \in \{1,2,\ldots 9\}$ and All I's $\in N$)

$S(I_1^K, I_2^K, \ldots . I_N^K) = S(i^K) = j$ (Where j=Constant and $j \in \{1,2,\ldots 9\}$ and I_1, I_2, I_3, etc. are all integers the digits in which add upto the same i)

EXAMPLE :

The numbers 43, 52 and 61 all have their sum of the digits as 7. Thus the numbers 1849, 2704 and 3721 (the squares of the above three numbers respectively) all have their sum of the digits as 4 (ultimate sum) which is the same as the ultimate sum of the square of the number 7 ($7^2 = 49 \rightarrow 4+9 = 13 \rightarrow 4$).

PROOF:

By virtue of the decimal system of integers, the addition of integer 9 to any N leaves the S(N) unchanged, i.e. $S(N) = S(N+9)$.

Now let any $I_P = 9I_L + i$ (Where $I_P, I_L \in N$ and $i \in \{1,2,\ldots 9\}$)

$$(I_P)^K = (9I_L + i)^K \quad (\text{where } K \in N)$$

$$S((I_P)^K) = S((9I_L + i)^K)$$

$$= S((9I_L)^K + {}^KC_1 (9I_L)^{K-1} i + {}^KC_2 (9I_L)^{K-2} i^2 + \ldots + i^K)$$

$$= S((i)^K)$$ (The rest of the integers being multiples of 9, would not affect $S((i)^K)$.)

3. The fact that integers 3 and 6 do not figure anywhere in the table save column 1, can be expressed:

Theorem: If $S(I) = 3,6$

Then $\sqrt[n]{I} \notin N$, where $n \in N - \{1\}$

PROOF:

Now $S(I) = 3$ or $6 \Rightarrow I = 3 \times I_A$, where I_A is not further wholly divisible by 3, as for any such $I \in N$, $I_A \in \{1+(n-1)3\} \cup \{2+(n-1)3\}$, where $n \in N$, and it is obvious that none of the integers in the above set of values of I_A would be wholly divisible by 3.

Now $I = 3 \times I_A$

$(I)^{1/n} = (3)^{1/n} \times (I_A)^{1/n}$

Thus $(I)^{1/n} \in N$ IFF, $(3)^{1/n} \in N$ and $(I_A)^{1/n} \in N$

But $(3)^{1/n} \notin N$

Hence $(I)^{1/n} \notin N$.

(The proof holds true even for $N \in R - \{1\}$, where R represents Real Nos.)

4. Implicit in the table are the additive and multiplicative properties of the function S.

a. The additive characteristic of S is rather obvious and may be expressed as:

Theorem:

$$S(I_1 + I_2 + \ldots + I_N) = S(S(I_1) + S(I_2) + \ldots + S(I_N)) \quad (\text{for all } I\text{'s} \in N)$$

b. The multiplicative property of S may be expressed as:

Theorem:

$$S(I_1 . I_2 \ldots . I_N) = S(S(I_1) . S(I_2) \ldots . S(I_N)) \quad (\text{for all } I\text{'s} \in N)$$

PROOF: Let I_1 and $I_2 \in N$

Let $S(I_1) = i \qquad (i \in \{1, 2, \ldots 9\})$

and $S(I_2) = j \qquad (j \in \{1, 2, \ldots 9\})$

Now $S(I_1 . I_2) = S(I_1 + I_1 + I_1 + \ldots + I_2 \text{ times})$

$= S(S(I_1) + S(I_1) + \ldots + I_2 \text{ times})$

$= S(i + i + i + \ldots I_2 \text{ times})$

$= S(iI_2)$

$= S(I_2 + I_2 + \ldots . + i \text{ times})$

$= S(S(I_2) + S(I_2) + \ldots . + i \text{ times})$

$= S(j + j + \cdots + i \text{ times})$

$= S(ij)$

$= S(S(I_1) S(I_2))$

The result may thus be generated for N number of I's as follows

$$S(I_1 . I_2 . I_3 \ldots . I_N) = S(S(I_1) . S(I_2) \ldots . S(I_N))$$

c. A corolory of the multiplicative property would be :

$S(I^N) = S(S(I))^N = S((S(I))^{N_1}.(S(I))^{N_2})$ (where $N_1 + N_2 = N$ and $N_1, N_2 \in N$)

EXAMPLE:

1. Now $S(38 \times 49 \times 57) = S(106134) = 6$

   Also $S(S(38) \times S(49) \times S(57)) = S(2 \times 4 \times 3) = S(24) = 6$

2. $S(14^6) = S(7529536) = 1$

   Also $S(S(14))^6 = S((5)^6) = S(15625) = 1$

   Also $S((S(14^2)).(S(14^4))) = S((S(196)).(S(38416)))$

   $= S(7 \times 4) = S(28) = 1$

   Also $S((S(14)).(S(14^5))) = S((S(14)).(S(537824)))$

   $= S(5 \times 2) = S(10) = 1$

5. For rows 2 and 5, it will be observed in table 1 that contents of the rows, between columns 1 and 6 repeat themselves in cycles starting at powers which may be represented as 6A+1 (for $A \in N$). This may be expressed as :

Theorem:

If $S(I_1, I_2, \ldots I_N) = 2,5$

$S(I_1^K, I_2^K, \ldots I_N^K) = S(I_1^i, I_2^i, \ldots I_N^i)$ Where $K = 6A+i$

(for all I's, A, $K \in N$ and $i \in \{1, 2, \ldots 6\}$)

PROOF :

$$S(I^K) = S((I)^{6A+i}) = S((I)^{6A}.(I)^i) = S(S(I^{6A})S(I^i))$$

$$= S(S(I^6)S(I^6) \ldots A \text{ times} \times S(I^i)) = S(I^i)$$

(When $S(I) = 2,5$, $S(I^6) = 1$ (see Table A)).

Hence the result.

Similarly it can be proved that :

If $S(I_1, I_2, \ldots I_N) = 4, 7$

$S(I_1^K, I_2^K, \ldots I_N^K) = S(I_1^i, I_2^i, \ldots I_N^i)$ (for all $K, A \in N$, $i \in \{1,2,3\}$)

Where $K = 3A+i$

If $S(I_1, I_2, \ldots I_N) = 8$

$S(I_1^K, I_2^K, \ldots I_N^K) = S(I_1^i, I_2^i, \ldots I_N^i)$ (for all $K, A \in N, i \in \{1,2\}$)

Where K $2A+i$

If $S(I_1, I_2, \ldots I_N) = 1$

$S(I_1^K, I_2^K, \ldots I_N^K) = 1$ (for all $K, A \in N$)

If $S(I_1, I_2, \ldots I_N) = 3, 6$

$S(I_1^K, I_2^K, \ldots I_N^K) = 9$ (for all $K \in N - \{1\}$)

And if $S(I_1, I_2, \ldots I_N) = 9$

$S(I_1^K, I_2^K, \ldots I_N^K) = 9$ (for all $K \in N$)

6. All the above proofs imply that for all I, $S(I^K)$ can be determined, without actually having to find out I^K (for all $I, K \in N$).

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