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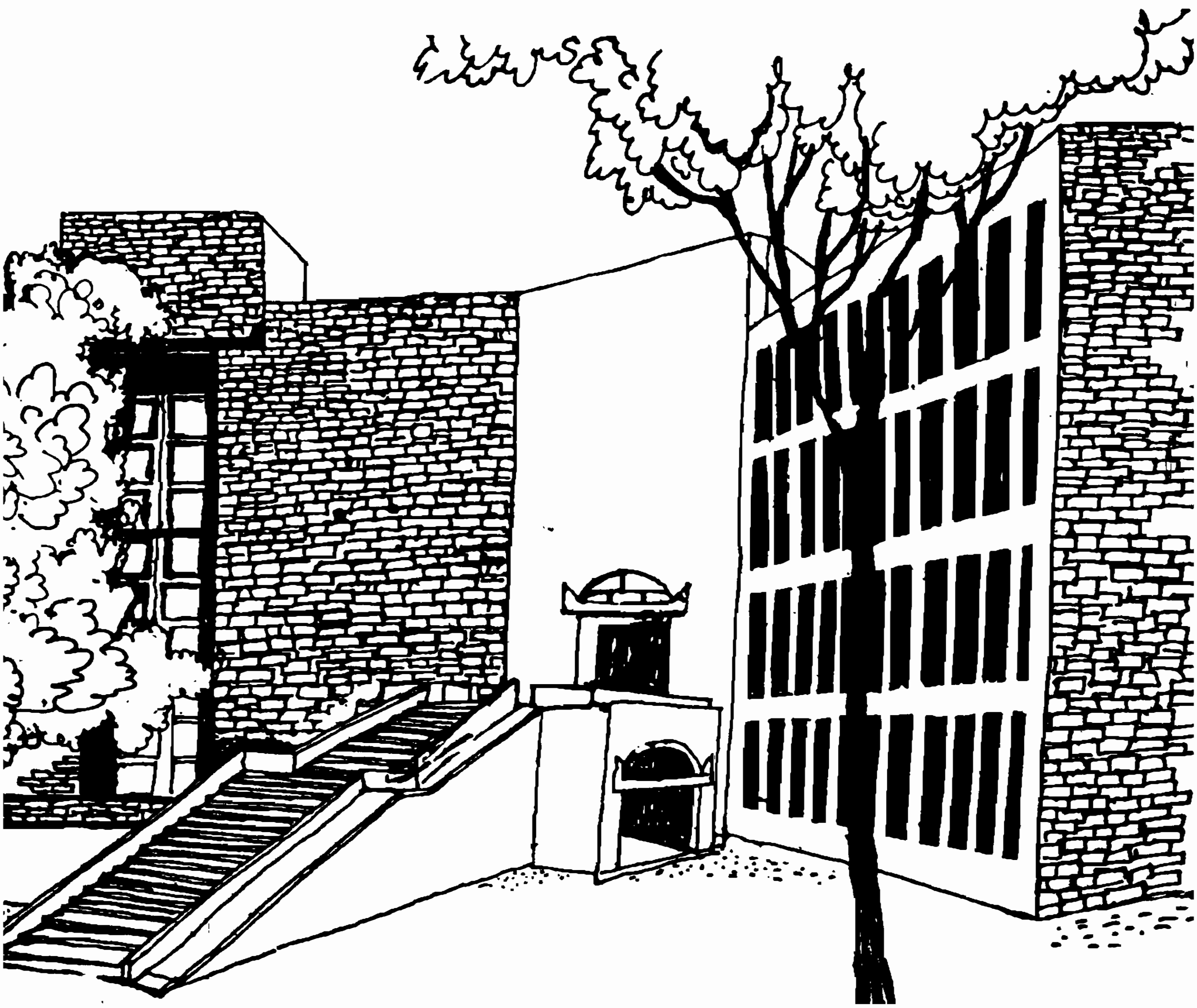

GENERALIZED SHAPLEY VALUE FOR GAMES WITH A COALITION STRUCTURE

By

Somdeb Lahiri

W.P. No. 1393
September 1997

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Abstract

In cooperative games with transferable utility, there is usually no restriction on the possible coalitions that can materialize. A significant departure from this tradition, occurs in Moulin [1995], where the concept of admissible coalitions arise.

In this paper we consider cooperative games with admissible coalitions, requiring that both the grand coalition, as well as, the null coalition are always admissible. We call such games, 'games with a coalition structure'. We define the concept of a core for such games and introduce a generalization of the notion of Shapley value. We define this generalized Shapley value to be the unique value satisfying the Dummy Player Property, Anonymity and Linearity. All these properties have been adapted from the standard context to our framework in such a way, that the existence of a unique value satisfying these properties is guaranteed.

We subsequently consider specific coalition structures and obtain closed form solutions for the generalized Shapley value in each case.

1. Introduction:- In cooperative games with transferable utility, there is usually no restriction on the possible coalitions that can materialize. A significant departure from this tradition, occurs in Moulin [1995], where the concept of admissible coalitions arise.

In this paper we consider cooperative games with admissible coalitions, requiring that both the grand coalition, as well as, the null coalition are always admissible. We call such games, 'games with a coalition structure'. We define the concept of a core for such games and introduce a generalization of the notion of Shapley value. We define this generalized Shapley value to be the unique value satisfying the Dummy Player Property, Anonymity and Linearity. All these properties have been adapted from the standard context to our framework in such a way, that the existence of a unique value satisfying these properties is guaranteed.

We subsequently consider games with connected coalitions on a line. We consider three variants of such coalitions. In the first coalition structure, the first player (in the natural ordering of numbers) must always belong to a non-empty admissible coalition. In the second coalition structure, the last player must always belong to a non-empty admissible coalition. In the third coalition structure, every non-empty admissible coalition must contain either the first or last player. In all the above cases, the only other condition that a non-empty coalition must satisfy is that it must contain all the players located between any two that it contains. In each of the above cases, we obtain closed form solutions for the generalized Shapley value. It turns out, that this value depends on the greedy algorithm and the reverse greedy algorithm. In an appendix to the paper we develop the F-Shapley value for games with a coalition structure which consists of all sets which were admissible under the three previous situations, and further, is closed under intersections.

We also obtain the generalized Shapley value for games with a bilateral coalition structure. These are games where the player set is partitioned into two sets and the coalitions that can form are singletons or a pair with an element from each of the two sets of the grand coalition. It is very close in spirit to marriage games, where singletons correspond to unmarried individuals, pairs correspond to a married couple comprising of a man and a woman, and the grand coalition corresponds to an entirely cooperative living arrangement amongst all the players (of course, then, in principle smaller coalitions can form; this we ignore in the present context. Thus the interpretation in terms of a marriage game is not all that appropriate).

In the final section of the paper we consider coalition structures which is essentially a partition and derive the very simple Shapley value for games with such a coalition structure.

The entire analysis reveals the complete dependence of the generalized Shapley value on the underlying coalition structure.

For a general survey of results in cooperative game theory, we refer the reader to Owen [1982].

2. The Model:- In this section we follow the framework laid out in Moulin [1995] closely.

Let $n \in \mathbb{N}$, be the number of players and let $N=\{1,...,n\}$ denote the player set. Let F be a non-empty collection of subsets of N containing both N and ϕ. For the purpose of this paper, F will be called a coalition structure.

A transferable utility (TU) cooperative game (with respect to F) is a function $v: F \Rightarrow \mathbb{R}$ with $v(\phi) = 0$. Hence forth we shall refer to a TU cooperative game as a game. Let V(F) be the set of all games with coalition structure F.

Given $v \in V(F)$, the core of v denoted C(v) is the following set:

$$C(v) = \{x \in \mathbb{R}^n \, / \, \sum_{i=1}^{n} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \; \forall S \in F\}.$$

At this point we should notice a slight difference in our requirements on coalition structure F and the requirements in Moulin [1995]. In particular, we require N to belong to F whereas singletons may not. Further, in this paper we do not consider any extension of a game from F to the power set; that is why we define F and C (v) as above.

Given a game $v \in V(F)$, a player i will be called a <u>dummy player</u> if the following condition is satisfied:

$\forall S, T \in F$ with $i \notin S$, $i \in T$, $S \subset T$, such that there does not exists a proper subset U of T with $i \in U$, $S \subset U$, we have $v(S) = v(T)$.

Our definition of a dummy player extends the existing one for TU games with respect to the power set, to TU games with respect to an arbitrary F.

We next introduce the concept of a value. A value is a function $f: V(F) \Rightarrow \mathbb{R}^n$ such that $\forall v \in V(F) \; \sum_{i \in N} f_i \,(v) \;\; v(N)$.

We will be interested in the following properties that a value would be required to satisfy:

(i) <u>Dummy Player Property</u>: Given $v \in V(F)$ if i is a dummy player for v, then $f_i(v) = 0$.

(ii) <u>Linearity</u>:

$$\forall v, w \in V(F),\ \forall\ \alpha, \beta \in \mathbb{R}, f(\alpha v + \beta w) = \alpha f(v) + \beta f(w).$$

(iii) <u>Anonymity</u>:- Given $v \in V(F)$ *and* $i, j \in N$ *with* $i \neq j$ if $\forall S \subset N \setminus \{i, j\}$, *with* $S \cup \{i\} \in F$ *and* $S \cup \{j\} \in F$, we have $v(S \cup \{i\}) = v(S \cup \{j\})$, *then* $f_i(v) = f_j(v)$.

Let $\phi \neq S \in F$. Define $u_s : F \Rightarrow \mathbb{R}$ as follows:

$u_s(T) = 1$ *if* $S \subset T$

= 0 otherwise

whenever $T \in F$.

u_s is called the unanimity game for S.

It is easy to see that V(F) is a vector space and $\{u_s : \phi \neq S \in F\}$ is a basis for V(F).

3. <u>The existence of a F-Shapley Value</u>:- We now prove a key theorem of this paper.

<u>Theorem 1</u>:- There exists a unique value ψ which satisfies Dummy Player Property, Linearity and Anonymity.

<u>Proof</u>:- Let ψ satisfy the above three properties and

consider $S \in F$. Let u_S be the unanimity game for S. Let

$x = \psi(u_S)$

It is easy to see that if $i \notin S$, then i is a dummy player.

Hence $x_i = 0 \ \forall \ i \notin S$.

Thus $\sum_{i \in S} x_i = u_S(N) = 1$.

By anonymity $x_i = x_j = 1/|S| \ \forall \ i, j \in S$.

Hence ψ is uniquely determined on $\{u_S : \phi \neq S \in F\}$.

By linearity ψ is uniquely determined on V(F).

<u>Q.E.D.</u>

Given Theorem 1, we now have the following definition.

The F-Shapley value ψ is the unique value to satisfy the Dummy Player Property, Linearity and Anonymity. Note, the exact form of ψ depends on F, which is why we are making no attempts (at least here) to arrive at a closed form solution for ψ.

4. <u>F-Shapley Value for Connected Coalitions on a Line Containing '1'</u>:-

As in Moulin [1995], fix the natural ordering of the agents and consider coalitions of the following form:

$S \in F \leftrightarrow$ *either* $S=\phi$ or there exists $i \in N$ *with* $S=\{1,2,\ldots,i-1,i\}$.

The interpretation that is provided is the following: the players "are located along a road and cooperation within S requires setting a cable connecting all members of S. Agreement of player i is necessary to go through location i." We add to the above interpretation the condition that all non-empty coalitions must contain 1 (the veto player).

The greedy algorithm $G: V(F) \Rightarrow \mathbb{R}^n$ is defined as follows:

$$G_i(v) = v(\{1,2,\dots,i\}) - v(\{1,2,\dots,i-1\}), \text{ for } i \in N \text{ and } v \in V(F).$$

Since $\sum_{i \in S} G_i(v) = v(S) \; \forall \; S \in F, G(v) \in C(v) \; \forall v \in V(F)$.

We are here interested in defining the F-Shapley value ψ.

<u>Theorem 2</u>:- The F-Shapley value ψ is given by

$$\psi_i(v) = \sum_{j=i}^{n} \frac{1}{j} G_j(v) \; \forall i \in N, \; \forall \; v \in V(F).$$

<u>Proof</u>:- Let $v \in V(F)$ and for $\phi \neq S \in F$, let u_S denote the corresponding unanimity game.

Then, $v = \sum_{j=1}^{n} c_{\{1,\dots,j\}} \, u_{\{1,\dots,j\}}$, since $\{u_S / S \in F, S \neq \phi\}$ is a basis for

V_p.

Now $v(\{1,\ldots,j\}) = \sum_{j=1}^{n} C_{\{1,\ldots,j\}}\, u_{\{1,\ldots,j\}}(\{1,\ldots,i\})$

$\sum_{j=1}^{i} C_{\{1,\ldots,j\}}\ \forall\ i\epsilon N$

Since $u_{\{1,\ldots,j\}}(\{1,\ldots,i\}) = 0$ *if* $i<j$.

$\therefore C_{\{1,\ldots,i\}} = v\{1,\ldots,i\} - v(\{1,\ldots,i-1\})$

$= G_i(v)$.

Thus $v = \sum_{j=1}^{n} G_j(v)\ u_{\{1,\ldots,j\}}$

By Linearity,

$\psi_i(v) = \sum_{j=1}^{n} G_j(v)\ \psi_i\ u_{\{1,\ldots,j\}}$.

But $\psi_i\ (u_{\{1,\ldots,j\}}) = \frac{1}{j}\ if\ i \leq j$

$= 0$ if $i > j$

$$\therefore \psi_i\ (v) = \sum_{j=i}^{n} \frac{1}{j} G_j(v).$$

Q.E.D.

5. F-Shapley Value for Connected Coalitions on a Line Containing 'n':-

Now suppose,

$S \in F \leftrightarrow$ *either* $S = \phi$ or there exists $i \in N$ *with* $S = \{i, i+1, \ldots, n\}$.

The interpretation of F here is similar to that in section 4, except that here player 'n' plays the role that player '1' played earlier.

We can now define the reverse greedy algorithm $H: V(F) \Rightarrow \mathbb{R}^n$ as follows:

$H_i(v) = v(\{i, \ldots, n\}) - v(\{i+1, \ldots, n\})$, for $i \in N$ and $v \in V(F)$.

Since $\sum_{i \in S} H_i(v) = v(S) \; \forall \; S \in F$ and $v \in V(F)$,

$H(v) \in C(v) \; \forall \; v \in V(F)$.

Theorem 3:- The F-Shapley value ψ is given by

$$\psi_i(v) = \sum_{j=1}^{i} \frac{1}{(n-j+1)} H_j(v) \qquad \forall \; i \in N \text{ and } \forall \; v \in V(F)$$

Proof:- As before, for $v \in V(F)$, let

$$v = \sum_{j=1}^{n} C_{\{j,j+1,\ldots,n\}} \, u_{\{j,j+1,\ldots,n\}}$$

Now $v(\{i, i+1, \ldots, n\}) = \sum_{j=1}^{n} C_{\{j,j+1,\ldots,n\}} \, u_{\{j,j+1,\ldots,n\}}(\{i, i+1, \ldots, n\})$

$$= \sum_{j=i}^{n} C_{\{j,j+1,\ldots,n\}}$$

$$\therefore\ C_{\{i,i+1,\ldots,n\}} = H_i(v).$$

$$\therefore\ v = \sum_{j=1}^{n} H_j(v)\ u_{\{j,j+1,\ldots,n\}}$$

$$\therefore\ \psi_i\ (v) = \sum_{j=1}^{n} H_j(v)\,\psi_i\ (u_{\{j,j+1,\ldots,n\}})$$

$$= \sum_{j=1}^{i} H_j(v)\,/\,(n-j+1)$$

<u>Q.E.D.</u>

6. <u>F-Shapley Value for Connected Coalitions On a Line Containing either '1' or 'n'</u>:-

Now $S \in F \rightarrow$ either $S = \phi$

or there exists $i \in N$ with $S = \{1, \ldots, i\}$

or there exists $i \in N$ with $S = \{i, \ldots, n\}$.

Here either player '1' or player 'n' is required for an admissible coalition, with the remaining part of the interpretation being the same as before.

Let $v \in V(F)$. Thus $v = \sum_{\phi \neq S \in F} C_S u_S$, where u_S is the unanimity game for $\phi \neq S \in F$.

Now $C_{\{1,\dots,i\}} = G_i(v) \;\forall\; i \in N \setminus \{n\}, v \in V(F)$

and $C_{\{i,\dots,n\}} = H_i(v) \;\forall\; i \in N \setminus \{1\}, v \in V(F)$

Clearly, $C_N = v(N) - \sum_{j=1}^{n-1} G_j(v) - \sum_{j=2}^{n} H_j(v)$.

Thus,

$$v = \sum_{i=1}^{n-1} G_i(v)\, u_{\{1,\dots,i\}} + \sum_{j=2}^{n} H_i(v)\, u_{\{i,\dots,n\}}.$$

$$+ \left[v(N) - \sum_{j=1}^{n-1} G_j(v) - \sum_{j=2}^{n} H_j(v) \right] u_N.$$

Theorem 4:- The F-Shapley value ψ is given by

$$\psi_i(v) = \sum_{j=i}^{n-1} \frac{1}{j} G_j(v) + \sum_{j=2}^{i} \frac{1}{n-j+1} H_j(v)$$

$$+\frac{1}{n}\left[v(N) - \sum_{j=1}^{n-1} G_j(v) - \sum_{j=2}^{n} H_j(v) \right] \textit{ for } i \in N\setminus\{1,n\}.$$

$$= \sum_{j=1}^{n-1} \frac{1}{j} G_j(v) + \frac{1}{n}\left[v(N) - \sum_{j=1}^{n-1} G_j(v) - \sum_{j=2}^{n} H_j(v) \right] \textit{ for } i=1$$

$$= \sum_{j=2}^{n} \frac{1}{(n-j+1)} H_j(v) + \frac{1}{n}\left[v(N) - \sum_{j=1}^{n-1} G_j(v) - \sum_{j=2}^{n} H_j(v) \right] \textit{ for } i=n.$$

Proof:- Easy, since the method is almost the same as before.

Q.E.D.

7. F-Shapley Value for Games With a Bilateral Coalition Structure:-

Let M, W be two non-empty disjoint subsets of N with

$M \cup W = N$

Now,

$S \in F \rightarrow$ *either* $S = \phi$ *or* $S = N$ *or* $S = \{i\}$ for some $i \in N$

or S = {i, j,} for some $i \in M, j \in W$.

Without loss of generality assume

$0 = v(\{i\}) = v(\{j\}) \ \forall \ i \in M, j \in W.$

Given $v \in V(F)$, let $v = \sum_{\phi \neq S \in F} C_S u_S$

Then, $C_{\{i\}} = v(\{i\}) \ \forall i \in N$

$C_{\{i,j\}} = v(\{i,j\}) \ \forall i \in M, j \in W.$

$C_N = v(N) - \sum_{i \in M} \sum_{j \in W} v(\{i,j\})$

Theorem 5:- The F-Shapley value ψ is given by,

$\psi_i(v) = \frac{1}{2} \sum_{j \in W} [v(\{i,j\})] \quad + \frac{1}{n} C_N \quad \forall i \in M$

$$= \frac{1}{2} \sum_{j \in W} [v(\{i, j\})] \quad + \frac{1}{n} C_N \quad \forall i \in W$$

Proof:- Similar to earlier proofs.

Q.E.D.

8. F-Shapley Value for Games With a Partition Coalition Structure:-

Let $\{S_1, \ldots, S_k\}$ be a partition of N and let

$S \in F \leftrightarrow$ *either* $S = \phi$ *or* $S = N$ *or* $S \in \{S_1, \ldots, S_k\}$

Given $v \in V(F)$, let $v = \sum_{\phi \neq S \in F} C_S u_S$

Then, $C_{S_j} = v(S_j)$ and

$$C_N = \left[v(N) - \sum_{j=1}^{k} v(S_j) \right]$$

Theorem 6:- The F-Shapley value ψ is given by,

$$\Psi_i(v) = \frac{1}{|S_m|} v(S_m) + \frac{1}{n}\left[v(N) - \sum_{j=1}^{k} v(S_j)\right]$$

where $i \in S_m$; $i \in N$.

<u>Proof</u>:- Similar to earlier proofs.

<u>Q.E.D.</u>

APPENDIX

$S \in F \sim S = \phi$ or there exists i, j, $\in$ N with $i \leq j$ and

$S = \{i, i+1, \ldots, j-1, j\}$

Without loss of generality assume v ({i}) = 0 forall i belonging to N.

Let $v \in V(F)$.

Then $v(\{1, \ldots, j\}) = \sum_{\substack{\phi \neq S \in F \\ i \in s}} C_s u_s$

Now $C_{\{i\}} = v(\{i\}) \forall i \in N$

and $v(\{i, i+1\}) = v(\{i\}) + v(\{i+1\}) + C_{\{i, i+1\}}$.

Thus, $C_{(\{i, i+1\})} = v(\{i, i+1\})$.

Now for i < j,

$$v(\{i,\ldots,j\}) - v(\{i,\ldots,j-1\}) = C_{\{i,\ldots,j\}} + C_{\{j\}} + C_{\{j-i,j\}}$$

$$= C_{\{i,\ldots,j\}} + v\{j\} + v\{j-i,j\} - v(\{j-1\}) - v(\{j\})$$

$$= C_{\{i,\ldots,j\}} + v(\{j-i,j\}).$$

Thus $C_{\{i,\ldots,j\}} = v(\{i,\ldots,j\}) - v(\{i,\ldots,j-1\})$

$$- v(\{j-1,j\}) \quad \textit{for} \quad i<j.$$

Now, if ψ is the F-Shapley value,

$$\psi(v) = \sum_{i \in S \in F} C_s/|S|$$

$$= \sum_{i<j} \left(\frac{1}{j-i+1}\right) [v(\{i,\ldots,j\}) - v(\{i,\ldots,j-1\}) \quad - v(\{j-1,j\})]$$

$$+ \sum_{j<i} \left(\frac{1}{i-j+1}\right) [v(\{j,\ldots,i\}) - v(\{j,\ldots,i-1\})$$

$$- v(\{i-1,i\})] \quad \text{if } 1 < i < n$$

$$= \sum_{j=2}^{n} \left(\frac{1}{j}\right) [v(\{1,\ldots,j\}) - v(\{1,\ldots,j-1\})$$

$$- v(\{j-1,j\})] \quad \text{if } i = 1$$

$$= \sum_{j=1}^{n-1} \left(\frac{1}{n-j+1}\right) [v(\{j,\ldots,n\}) - v(\{j,\ldots,n-1\}) \qquad - v(\{n-1,n\})]$$

if i = n.

Note,

$$\psi_1(v) = \sum_{j=2}^{n} \left(\frac{1}{j}\right) [G_j(v) - v(\{j-1,j\})]$$

where $G_j(v) = v(\{1,\ldots,i\}) - v(\{1,\ldots,i-1\}) \ \forall \ i=1,\ldots,n$

corresponds to the i^{th} term of the greedy algorithm.

References:-

1. H. Moulin [1995]: "Cooperative Micro Economics: A Game Theoretic Introduction," Prentice Hall, Harvester Wheatsheaf.

2. G. Owen (1982): "Game Theory," 2nd edition (Academic Press, Orlando, IL).