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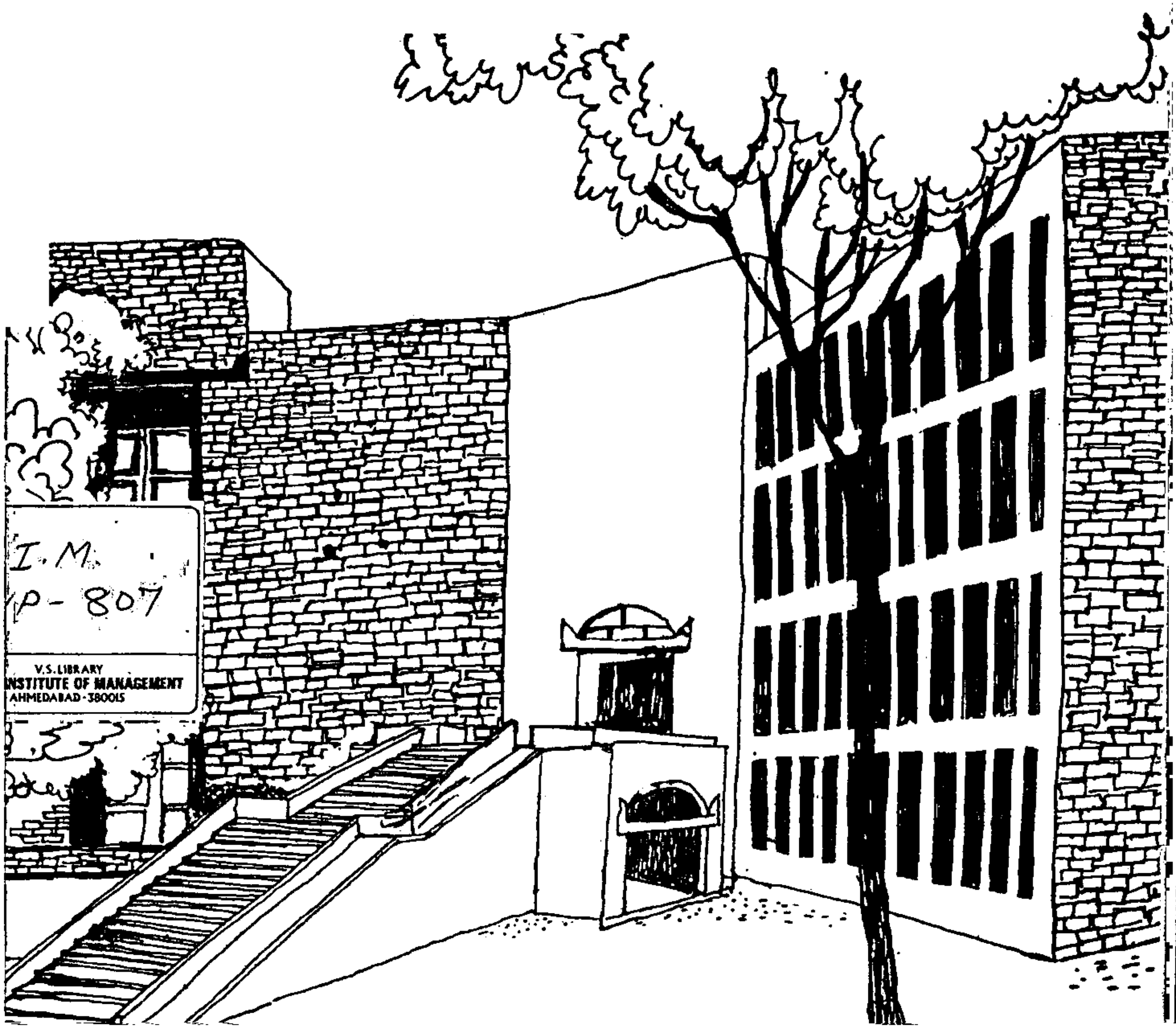

THE IMPOSSIBILITY OF ANONYMOUS ORDINAL SOLUTIONS FOR TWO PERSON BARGAINING PROBLEMS

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 056
INDIA

ABSTRACT

In this paper we prove the non-existence of pure bargaining solutions which satisfy Pareto optimality, Anonymity and Invariance With Respect to Allowable Ordinal Transformations.

1. Introduction:

Consider an arbitrator responsible for helping two players to cooperate in game situations. For any bargaining problem which the players might face, he must be prepared to recommend a fair cooperative agreement for the two players. The received theory of bargaining assumes that the preferences of the players over distinct outcomes is representable by a cardinal utility function. A notable exception in the paper by Myerson (1977), where he shows that the egalitarian solution of Kalai (1977), is invariant under some ordinal transformation of the preferences of the players.

Our purpose in this paper is to prove that if the arbitrator was to impose a mild anonymity requirement then it would be impossible to find a bargaining solution which is Pareto optimal and invariant with respect to 'allowable' order preserving transformations.

2. Definitions and Notations:

In this paper, we will follow the axiomatic approach to the bargaining problem as initiated by Nash (1950). We restrict our attention to two-person bargaining problems.

Formally, a (two-person) bargaining game S is a proper subset of the plane $\mathbb{R}^2$ satisfying

1) S is closed, convex and $\sup\{x_i / x \in S\} \in \mathbb{R}$ for all $i \in \{1,2\}$;

2) $0\ (= (0,0)) \in S$ and $x > 0$ for some $x \in S$;

3) S is comprehensive, i.e. for all $x \in S$ and $y \in \mathbb{R}^2$, if $y \leq x$ then $y \in S$.

Let B denote the family of all bargaining games. When interpreting an $S \in$ B, one must think of the following game situation. Two players (bargainers) may cooperate and agree on a feasible outcome x in S , giving utility x_i to player i = 1,2, or they may fail to cooperate, in which case the game ends in the disagreement outcome 0. So for any $S \in$ B, the disagreement outcome is fixed at 0. Closedness of S is required for mathematical convenience; convexity stems from allowing lotteries in an underlying bargaining situation or concave and monotone utility functions in fair division problems. Further, it is assumed that S is bounded from above, but not from below, since we allow free disposal of utility. The requirement $x > 0$ for some $x \in S$ serves to give each player an incentive to cooperate. Not all of the restrictions in (1) - (3) are necessary for all of our results, but assuming them simplifies matters and, moreover, none of them goes against intuition.

A (two-person) <u>bargaining solution</u> is a map $\phi : B \to \mathbb{R}^2$ assigning to each $S \in$ B an outcome $\phi(S) \in S$ and such that Axiom 0 holds:

Axiom 0 : $\phi(S)$ depends only on (the shape of) S.

Axiom 0 is crucial in view ofthe result obtained by Shapley (1969), that if we allow for arbitrary order preserving transformations of the utility scales of the players and require the bargaining solution to remain invariant under such transformations then Axiom 0 is violated. Thus in order to establish our result we will need to restrict the class of allowable order preserving transformations. We shall prove that even by restricting the class of order preserving transformations we cannot hope to do much better.

An order-preserving transformation of the reals is a function $g : \mathbb{R} \to \mathbb{R}$ such that g is one-to-one , onto, g (0) = 0 and $x > y$ implies $g(x) > g(y)$. It can easily be shown that an order-preserving transformation must be a continuous function.

An ordinal transformation is a pair of order preserving transformations (g, h).

Let $S \in B$. We shall say that (g,h) is an allowable ordinal transformation if and only if it is an ordinal transformation which satisfies the additional condition:

$$(*) \left\{ \begin{array}{l} S = \{(g(x), h(y) \,/\, (x,y) \in S\} \\ \text{implies } g(x) = x,\ h(y) = y \text{ for all} \\ (x,y) \in \mathbb{R}^2 \end{array} \right.$$

For such allowable ordinal transformations Axiom 0 is trivially satisfied. Hence on this subclass of ordinal transformations Shapley's (1969) theorem has little to say.

On the other hand as Shubik (1987) points out, there are some nonlinear groups of utility - scale transformations that do not run afoul of Shapley's Bargainers Paradox. This raises the interesting possibility of intermediate utility types, between ordinal and cardinal. However, as we show subsequently we cannot hope to be very optimistic about such possibilities.

For $S \in B$, let $P(S) := \{ x \in S /$ for all $y \in S$, if $y \geq x$, then $y = x \}$ denote the Pareto optimal subset of S.

Let $\phi : B \to \mathbb{R}^2$ be a bargaining solution. The following two axioms will play an important role.

Axiom 1 (Pareto Optimality, PO) : $\phi(S) \in P(S)$ for all $S \in B$.

Axiom 2 (Invariance With Respect of Allowable Ordinal Transformations, IAOT): Let (g,h) be an allowable ordinal transformation. Then, $(g \circ \phi_1(S), h \circ \phi_2(S)) =$
$\phi(\{(g(x), h(y)) / (x,y) \in S\})$ whenever, $\{(g(x), h(y)) / (x,y) \in S\} \in B$.

Axiom 3 (Anonymity, AN) : Let $\pi : \{1,2\} \to \{1,2\}$ be a function such that $\pi(1) = 2$, $\pi(2) = 1$. For $(x, y) \in \mathbb{R}^2$ denote $(x, y)_\pi = (y, x)$ and for $A \subset \mathbb{R}^2$ let
$A_\pi = \{(y,x) \in \mathbb{R}^2 / (x,y) \in A\}$
Then $\phi(S_\pi) = (\phi(S))_\pi \; \forall S \in B$.

We adopt another notation : for a finite number of vectors (x^1,y^1), (x^2,y^2), ..., (x^l,y^l) in $\mathbb{R}^2$,

$$S((x^1,y^1), (x^2,y^2), \ldots\ldots, (x^l,y^l))$$
$$= \{(x,y) \in \mathbb{R}^2 ; (x,y) \leq (\bar{x}, \bar{y}) \text{ for some } (\bar{x},\bar{y}) \in \text{Conv.} \{(x^1,y^1), (x^2,y^2), \ldots\ldots, (x^l,y^l)\}\}$$

Axiom 4 (Pureness of Solutions, PS) : Given,
$S((x^1,y^1), (x^2,y^2), \ldots\ldots, (x^l,y^l)) = S \in B$,
$\phi(S) \in \{(x^1,y^1), (x^2,y^2), \ldots\ldots, (x^l,y^l)\}$.

The terminological convention of denoting such outcomes, and hence the solution, as pure, is in contrast to defining a convex combination of two such outcomes as mixed. The significance of the ensuing results, in problems of social choice is therefore quite obvious.

3. Main Results:

In this section we state and prove the main theorem of this paper. The content of this theorem is that there does not exist any solution satisfying Axioms 0 to 4 on the class of games B.

Theorem 1 : There does not exist a function $\phi : B \to \mathbb{R}^2$ such that $\phi(S) \in S \;\forall\; S \in B$ satisfying Axioms 0, 1, 2, 3 and 4.

Proof : Let $S \equiv S((x^1,y^1), (x^2,y^2))$, with $(x^1,y^1), (x^2,y^2) \in P(S)$. Suppose towards a contradiction that there exists a solution $\phi : \mathcal{B} \to \mathbb{R}^2$ such that $\phi(S) \in S \; \forall \; S \in B$ satisfying Axioms 0, 1, 2, 3 and 4. Without loss of generality assume that $\phi(S) = (x^1, y^1)$ and $x^1 > x^2$. Hence $y^1 < y^2$.

Consider,

$S' = S\left((\underline{x}^1, \underline{y}^1), (\underline{x}^2, \underline{y}^2)\right)$ with
$(\underline{x}^1, \underline{y}^1), (\underline{x}^2, \underline{y}^2) \in P(S')$ and $\underline{x}^1 > \underline{x}^2$.
Suppose $S' \neq S$. Clearly there exist order preserving transformations $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ such that $g(x^1) = \underline{x}^1$, $g(x^2) = \underline{x}^2$, $h(y^1) = \underline{y}^1$, $h(y^2) = \underline{y}^2$, with other points on the parceto frontier being defined linearly.

Thus by axiom 2, $\phi(S') = (\underline{x}^1, \underline{y}^1)$
By axiom 3, $\phi(S'_\pi) = (\underline{y}^1, \underline{x}^1)$.

Since S can be obtained from S'_π by an allowable ordinal transformation, which assigns x^2 to $\underline{y}^1$, x^1 to $\underline{y}^2$, y^2 to $\underline{x}^1$, y^1 to $\underline{x}^2$, other points on the Parceto frontier being defined linearly, we get by Axiom 2, that
$\phi(S) = (x^2, y^2)$ and a contradiction. This proves the theorem.

Q.E.D

Conclusion:

In this paper, we prove a theorem which asserts the non-existence of bargaining solutions under seemingly innocuous assumptions. Shapley (1969) proves the non-existence of a bargaining solution which is invariant under ordinal transformations. Shubik (1987) however asserts that there are bargaining solutions which are invariant under order-preserving non-linear transformations. We show that the class of such non-linear transformations could definitely not be very large, if in addition we require anonymity of bargaining solutions.

References :

1. R. Myerson (1977) : "Two-Person Bargaining Problems and Comparable Utility", Econometrica, Vol. 45, Pgs. 1631-1637.

2. J.F. Nash (1950): "The Bargaining Problem", Econometrica, Vol.18, Pgs. 155-162.

3. L. Shapley (1969) : "Utility Comparison and the Theory of Games", in La Decision. Paris : Edition du Centre' Nationale de la Recherche Scientifique, France, pp. 251-263.

4. M. Shubik (1987) : "Game Theory in the Social Sciences : Concepts and Solutions", MIT Press, Cambridge, Mass.