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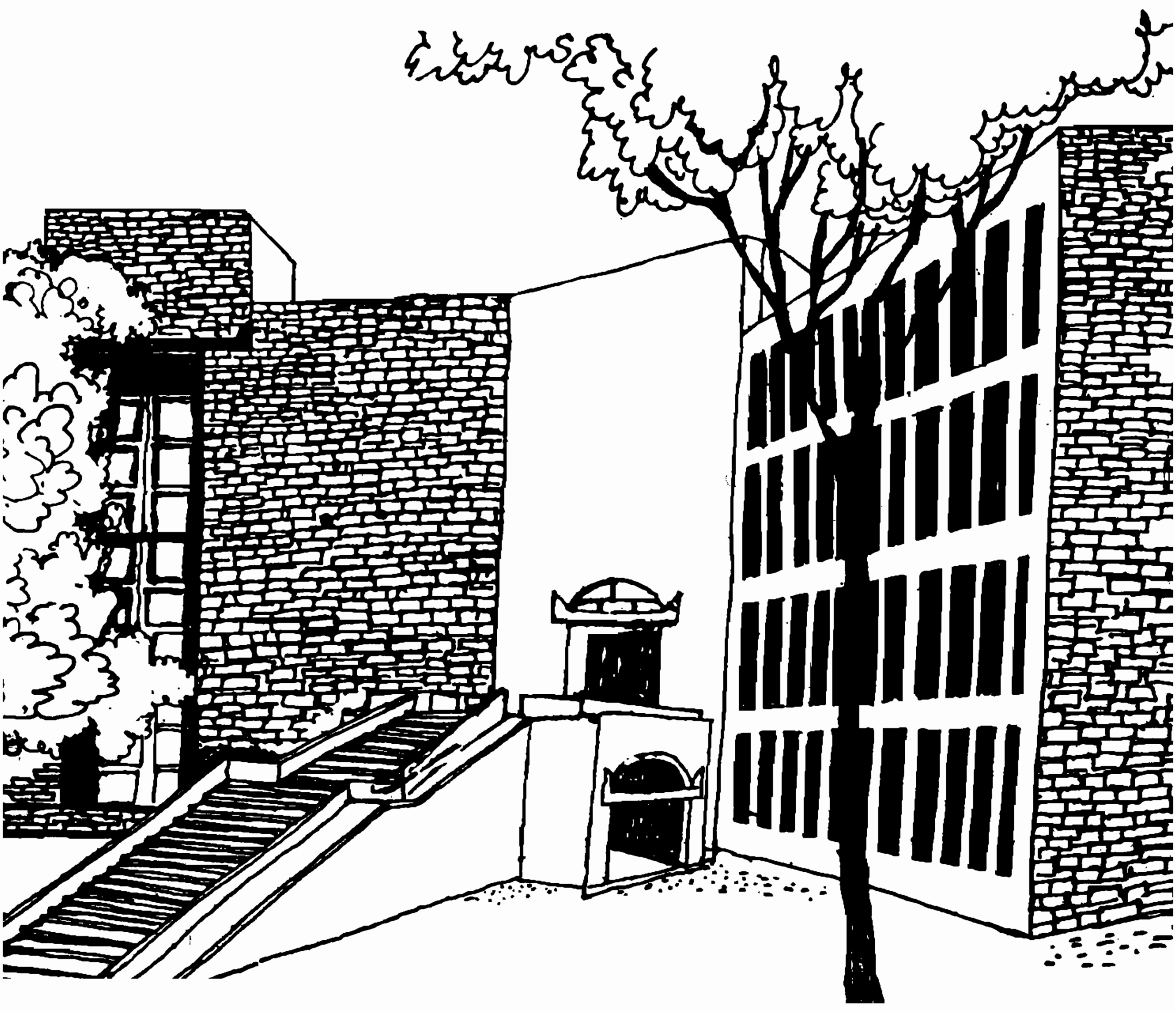

SHARING COSTS AND SHARING REVENUE: THE PROPORTIONAL SOLUTIONS

By

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Abstract

Consider a group of people who have just been awarded a sum of money to undertake a consultancy (or a research) project. The problem that now confronts them is to share the revenue as well as the costs of the overheads involved in undertaking the project.

In this paper we propose a proportional solution to this problem, prove the existence of such a solution (in a special case) and establish its optimality.

1. Introduction :- Consider a group of people who have just been awarded a sum of money to undertake a consultancy (or a research) project. The problem that now confronts them is to share the revenue as well as the costs of the overheads involved in undertaking the project.

A reasoned solution to this problem, compatible with conventional notions of distributive justice, would argue that the person who gets a higher portion of the revenue should also contribute a higher amount towards the cost of the overheads. A step further in this line of reasoning would indicate that costs and revenues should be shared in the same proportion by all the individuals. It is with a view towards formalising this solution concept that we propose the results in this paper.

A formal model for sharing the costs of a public project was initiated in the work of Mas-Colell (1980). However most of the solution concepts that have been developed are confined to the case of the public project being a single public good. A notable departure from this line of activity, where many of the significant concepts originally due to Moulin (1987, 1988) have been extended to the framework developed by Mas-Colell (1980), is the work by Lahiri (1994b).

A large literature on egalitarian revenue sharing in an economy with public projects has grown up with the work of Sato (1985, 1987), Otsuki (1992), culminating in the work of Lahiri (1994a). Most of these papers concentrate on exhibiting the equal income Lindahl equilibrium resource allocation mechanism as compatible (and sometime uniquely compatible) with alternative concepts of distributive justice.

The objective of this paper is not so much to outline a method of distributive justice. We are more interested here in studying an intuitive method of simultaneous costs and revenue sharing which may be acceptable in most budgeting problems. Admittedly, we have not collected data in support of our mechanism. However, it stands to reason that one of the methods that may often be suggested in the process of simultaneous costs and revenue sharing (irrespective of whether it gets accepted in practice), is the method suggested in this paper.

Our main objective in this paper is to prove the existence and optimality of solutions which adhere to the rules of costs and revenue sharing taking place in the same proportion.

2. The Model and Assumptions :- There is given a nonempty, metric space K of projects and a finite collectivity of agents $N=\{1,\ldots,n\}$. Every agent $i \in N$ has preferences on tuples (x,m) of projects and amounts of a unique private good (to be called "money"), represented by a continuous utility function $u_i : K \times \mathbb{R}_+ \to \mathbb{R}$ such that $\forall x \in K$, $m,m' \in \mathbb{R}_+$, $m>m' \Rightarrow u_i(x,m)>u_i(x,m')$. We assume that the finite collectivity of agents is endowed with a positive amount of money $w>0$.

Let $c:K \to \mathbb{R}_+$ be a continuous cost function for the provision of the project.

Definition 1 :- A state is a (n+1) tuple $(x,m_1,\ldots,m_n) \in K \times \mathbb{R}^n_+$. It is denoted by $(x,\underline{m})$.

Definition 2 :- A state $(x,\underline{m})$ is feasible if $c(x) \leq w - \sum_{i \in N} m_i$.

Definition 3 :_ A state $(x,\underline{m})$ is Pareto efficient if it is feasible and if there is no feasible state (x',m') such that $u_i(x',m'_i) \geq u_i(x,m_i)\ \forall i \in N$, with strict inequality for atleast one $i \in N$.

We now define the main concept of this paper.

Definition 4 :- A feasible state $(x,\underline{m})$ is called a proportional ratio equilibrium state if there exists $\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)$ with $\alpha_i \geq 0\ \forall i \in N$ and $\sum_{i \in N} \alpha_i = 1$ such that $\forall i \in N$

(i) $m_i + \alpha_i c(x) < \alpha_i w$

(ii) $x' \in K$, $m'_i \in \mathbb{R}_+$, $m'_i + \alpha_i c(x') \leq \alpha_i w \Rightarrow u_i(x,m_i) > u_i(x',m'_i)$.

An ordered pair $[(x,\underline{m}),\underline{\alpha}]$ where $(x,\underline{m})$ is a proportional ratio equilibrium state and $\underline{\alpha}$ is as above is called a proportional ratio equilibrium.

Our concept of a proportional ratio equilibrium is a minor adaptation to our framework of the concept of a ratio equilibrium due to Kaneko (1977).

3. Efficiency of a proportional ratio equilibrium :-

Theorem 1 :- Let $[(x,\underline{m}),\underline{\alpha})]$ be a proportional ratio equilibrium. Then $(x,\underline{m})$ is a Pareto efficient state

Proof :- Obvious.

Thus a proportional equilibrium, if it exists, is a Pareto efficient state. It remains to prove the existence of a proportional ratio equilibrium, which we do in a special case.

4. Existence of a proportional ratio equilibrium :- Let $K=\mathbb{R}_+$. In addition to what has been assumed in section 2, assume:

(i) $\forall i \in N, \forall m_i \in \mathbb{R}_+, x,x' \in K, x>x' \Rightarrow u_i(x,m_i)>u_i(x',m_i)$

(ii) $u_i : \mathbb{R}^2_+ \to \mathbb{R}$ is quasi-concave.

(iii) $c: \mathbb{R}_+ \to \mathbb{R}$ is continuous, convex and strictly increasing.

Let A be a non-empty, convex subset of $\mathbb{R}^2_+$. Then $B=A \cap \{(x, m_i) \in \mathbb{R}^2_+ / m_i + \alpha_i c(x) \leqq \alpha_i w\}$ is a convex set $\forall \alpha_i \geqq 0$. To see this observe that if $(x,m_i),(x',m'_i) \in B$, then $m_i + \alpha_i c(x) \leqq \alpha_i w$, $m'_i + \alpha_i c(x') \leqq \alpha_i w$.

By the convexity of $c:\mathbb{R}^2_+ \to \mathbb{R}_+$, $(tx+(1-t)x', tm_i+(1-t)m'_i) \in B$.

Claim 1 :- Let $A=\{(x,m_i) \in \mathbb{R}^2_+ / m_i \leqq w+1, x \leqq c^{-1}(w+1)\}$.

Let $(x^*, \underline{m}^*)$ be a feasible state which satisfies $\forall i \in N$

(i) $\alpha_i c(x^*)+m^*_i \leqq \alpha_i w$

(ii) $(x^*, m^*_i) \in A \ \forall i \in N$

(iii) $(x',m'_i) \in A$, $\alpha_i c(x')+m'_i \leqq \alpha_i w \Rightarrow u_i(x^*, m^*_i) \geqq u_i(x',m'_i)$

Suppose $\alpha_i \geqq 0 \ \forall i \in N$, $\sum_{i \in N} \alpha_i = 1$. Then $[(x^*, \underline{m}^*), \underline{\alpha}]$ is a proportional ratio equilibrium.

Proof :- Since $(x^*, \underline{m}^*)$ is feasible, $x^* \leqq c^{-1}(w)$ and $m^*_i \leqq w \ \forall i \in N$.

Suppose there exists $(x',m'_i) \in \mathbb{R}^2_+$ with $\alpha_i c(x')+m'_i \leqq \alpha_i w$ and $u_i(x',m'_i)>u_i(x^*, m^*_i)$.

Clearly $(x',m'_i) \in A^c$ (i.e. the complement of A).

Further $c^{-1}(w+1)>c^{-1}(w)$ since $w+1>w$.

Thus there exists $t \in (0,1)$ such that $(tx'+(1-t)x^*, tm'_i+(1-t)m^*_i) \in A$.

By quasi-concavity and strict monotonicity of u_i, $u_i(tx'+(1-t)x^*, tm'_i+(1-t)x^*)>u_i(x^*, m^*_i)$.

Further $\alpha_i\ c(tx'+(1-t)x^*)+(tm'_i+(1-t)m^*_i)\leq\alpha_i\ w$ by the convexity of c.

This contradicts (iii) and hence establishes the claim.

Q.E.D.

We now proceed to prove the main theorem of this paper.

Theorem 2 :- Under the assumptions of section 2 and section 4, there exists a proportional ratio equilibrium.

Proof :- Let $\Delta=\{\underline{\alpha}\in\mathbb{R}^n_+\ /\sum^n_{i=1}\alpha_i=1\}$.

Let $C=\{(c_1,\ldots,c_n)\in\mathbb{R}^n_+\ /c_i\leq c(w+1)\forall i=1,\ldots,n\}$ and $\forall i\in N$, let $f_i : \Delta \rightarrow\!\!\rightarrow C$ be a correspondence defined as follows:

$f_i(\underline{\alpha})=\{x\in\mathbb{R}_+ / c(x)\leq w+1, 0<\alpha_i w-\alpha_i c(x)\leq w+1$, and $u_i(x,\alpha_i w-\alpha_i c(x))\geq u_i(x',m'_i)\ \forall(x',m'_i)\in\mathbb{R}^2_+, c(x')\leq w+1,\ m'_i=\alpha_i w-\alpha_i c(x')\leq w+1\}$

Since u_i is continuous and the constraint set:

$A\cap\{(x,m_i)\in\mathbb{R}^2_+\ /\alpha_i c(x)+m_i\leq\alpha_i w\}$ (where $A\equiv\{(x,m_i)\in\mathbb{R}^2_+ / c(x)\leq w+1, m_i\leq w+1\}$) is compact, $f_i(\underline{\alpha})\neq\emptyset\ \forall\underline{\alpha}\in\Delta$. Since the constraint set is convex and u_i is quasi-concave, $f_i(\underline{\alpha})$ is convex valued.

Define $g_i : C\mathrm{x}\Delta \rightarrow \Delta$ as follows:

$$g_i(\underline{c},\underline{\alpha})=\frac{\alpha_i+\max\{0,c_i-\sum_{i\in N}\alpha_i c_i\}}{1+\sum_{i\in N}\max\{0,c_i-\sum_{i\in N}\alpha_i c_i\}},\ i\in N.$$

Each g_i is a continuous function and each f_i is upper semi-continuous $\forall i\in N$.

Now consider the correspondence

$[(f_1,\ldots,f_n),(g_1,\ldots,g_n)]: \Delta \mathrm{x}C \rightarrow\!\!\rightarrow \Delta \mathrm{x}C$.

By Kakutani's Fixed Point theorem, there exists $(\alpha^*,c^*)\in\Delta\mathrm{x}C$ such that

$(\alpha^*,c^*)\in[(f_1,\ldots,f_n),(g_1,\ldots,g_n)](\alpha^*,c^*)$.

Thus, $$\alpha^*_i=\frac{\alpha^*_i+\max\{0,c^*_i-\sum_{i\in N}\alpha^*_i c^*_i\}}{1+\sum_{i\in N}\max\{0,c^*_i-\sum_{i\in N}\alpha^*_i c^*_i\}},\ \forall i\in N$$

$\therefore\ \alpha^*_i\sum_{i\in N}\max\{0,c^*_i-\sum_{i\in N}\alpha^*_i c^*_i\}=\max\{0,c^*_i-\sum_{i\in N}\alpha^*_i c^*_i\}\ \forall i\in N$.

Multiply each equation by $c^*_i-\sum_{i\in N}\alpha^*_i c^*_i$ and add:

$\therefore\ \sum_{i\in N}(c^*_i-\sum_{i\in N}\alpha^*_i c^*_i)\max\{0,c^*_i-\sum_{i\in N}\alpha^*_i c^*_i\}=0$.

$\therefore\ c^*_i-\sum_{i\in N}\alpha^*_i c^*_i\leq 0\ \forall i\in N$.

$\therefore\ c^*_i=\sum_{i\in N}\alpha^*_i c^*_i\ \forall i\in N$, since $\alpha^*\in\Delta$.

Further $c^* \in (g_1, \dots, g_n)(\alpha^*)$. Let $\hat{c}=c^*_i \ \forall i\in N$

By appealing to Claim 1, we may conclude that

$[\xi c^{-1}(\hat{c}), (\alpha^*_i w - \alpha_i \hat{c})_{i\in N}, \underline{\alpha}^*]$ is a proportional ratio equilibrium.

Q.E.D.

A proportional ratio equilibrium has several desirable properties. If $\alpha^*_i c$ is conceived as the tax paid by agent i, then the tax structure is progressive: $\alpha^*_i w \geq \alpha^*_j w \Leftrightarrow \alpha^*_i \hat{c} \geq \alpha^*_j \hat{c}$. Further the tax structure is proportional, the proportionality constraint being $\hat{c}/w$. Also, people with a higher portion of the revenue have more after tax income. Finally, the entire process of resource allocation takes place through a decentralized mechanism.

4. Conclusion :- The proportional cost share equilibrium resource allocation mechanism is not being mooted as a mechanism, which implements a fair distribution of resources, in the sense in which fairness and equity are conceived of in the literature. We rather propose it as a workable mechanism for simultaneous costs and revenue sharing, when preferences are given and a progressive (: indeed proportional) tax allocation scheme is desired. Our existence theorem applies to the case of the public project being a public good. It is an open problem at this stage, whether a proof for more general spaces of public projects could be established.

Appendix

In this section we obtain a characterization of proportional ratio equilibrium allocations, when the preferences of each agent are representable by a differentiable utility function of the type studied by Bergstrom and Cornes (1983). Thus let $u_i(m_i, x) = A(x)m_i + B_i(x) \forall (m_i, x) \in \mathbb{R}^2_+$ $\forall i \in N$ where $A: \mathbb{R}_+ \to \mathbb{R}$ is differentiable $B_i: \mathbb{R}_+ \to \mathbb{R}$ is differentiable $\forall i \in N$, $A(x) > 0 \forall x \in \mathbb{R}_+$ and u_i is strictly increasing. In addition we may assume that each u_i $(i \in N)$ is quasi-concave. A set of sufficient conditions for u_i as above to quasi-concave is that A is concave and the function $(m_i, x) \mapsto \frac{m_i + B_i(x)}{A(x)}$ $: \mathbb{R}^2_+ \to \mathbb{R}$ is concave and $B_i(x) \geq 0 \forall y \in \mathbb{R}_+$ (see Campbell and Truchon (1988)). Assume in addition that $c: \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable and strictly increasing. Given, $\alpha^*_i > 0$, the problem faced by agent i is

$A(x)m_i + B_i(x) \to \max$

Subject to $m_i + \alpha^*_i c(x) = \alpha^*_i w$.

A necessary and sufficient condition for m^*_i, x^* to solve the above problem is

$$\alpha^*_i c'(x^*) = \frac{A'(x^*)m^*_i + B'_i(x^*)}{A_i(x^*)} = a(x^*)m^*_i + b_i(x^*)$$

where $a(x^*) = \frac{A'(x^*)}{A(x^*)}$, $b_i(x^*) = \frac{B'_i(x^*)}{A(x^*)}$, $i \in N$.

$$\therefore m^*_i = \frac{\alpha^*_i c'(x^*) - b_i(x^*)}{a_i(x^*)}, \quad i \in N \qquad (*)$$

Since in a proportional ratio equilibrium, $\sum_{i \in N} \alpha^*_i = 1$ and $\sum_{i \in N} m^*_i = w - c(x^*)$, we have

$$c'(x^*) = a(x^*)[w - c(x^*)] + \sum_{i \in N} b_i(x^*). \qquad (**)$$

From (*) and the constraint $m^*_i = \alpha^*_i [w - c(x^*)]$ we have

$\alpha^*_i [w - c(x^*)] a_i(x^*) = \alpha^*_i c'(x^*) - b_i(x^*)$

or $\alpha^*_i = b_i(x^*)/\{c'(x^*) - a_i(x^*)[w-c(x^*)]\}$, $i \in N$ (***)

Thus we solve x^* from (**), α^*_i from (***) and obtain m^*_i from (*) for $i \in N$.

It should be noted that (**) corresponds to the Samuelson condition for Pareto efficiency.

Given our assumptions, it is necessary as well as sufficient that a solution obtained as above corresponds to a proportional ratio equilibrium. In fact at a proportional equilibrium

$$m^*_i = \frac{b_i(x^*)[w-c(x^*)]}{c'(x^*) - a(x^*)[w-c(x^*)]} \quad \forall i \in N.$$

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