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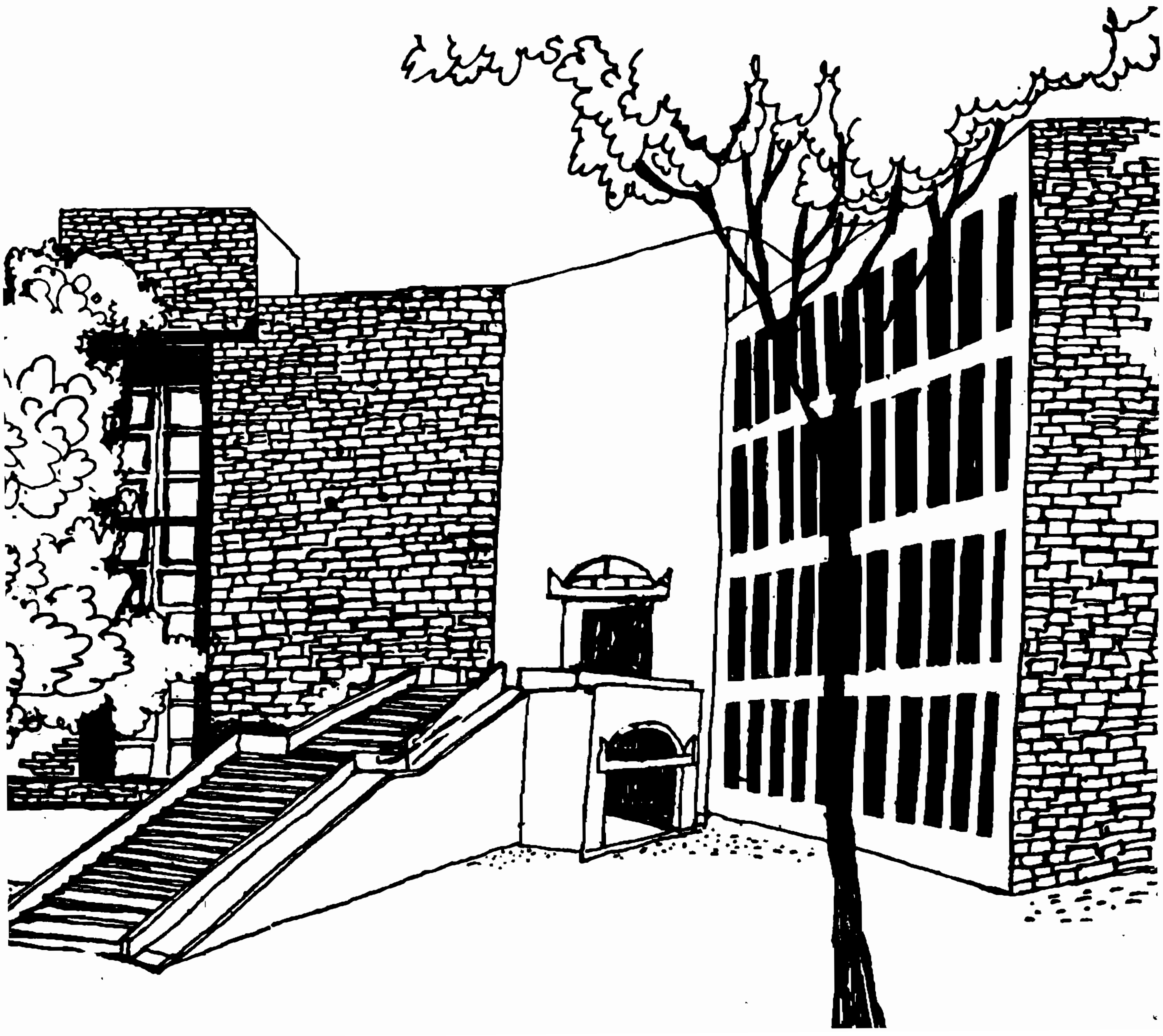

# AXIOMATIC ANALYSIS OF VOTE AGGREGATORS

By

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# Axiomatic Analysis of Vote Aggregators

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**Abstract.** In this paper a model for the aggregation of ballot profiles is considered.In this framework some new results are obtained:the characterization of oligarchic aggregators,the characterization of the plurality aggregator and the non existence of a vote aggregator which is anonymous and yet preserves proximity. In the context of there being precisely two candidates from which voters are required to choose from we provide axiomatic characterizations of the majority vote aggregator and the two single valued selections from it.

## 1 Introduction

The conventional model of voting is one where a finite set of agents announce their rankings of a finite set of alternatives and then a social welfare correspondence aggregates these announcements into a social ranking of alternatives.The seminal work of Arrow dealt with the observation that if the social ranking is such that (i)given two alternatives if one is preferred over the other unanimously then the social ranking also ranks the alternatives similarly; and (ii) the relative social ranking between two alternatives depends only on the relative individual rankings between these two alternatives. then the social ranking is nothing but the ranking of a single individual i.e. the dictator.A rather complete survey of the literature on social welfare correspondences can be found in Aleskerov (1999).

In recent times the more realistic possibility of each individual in a society casting a ballot and a voting operator aggregating the ballots into elected outcomes has been modelled in Lahiri (1999,2001).In Aczel and Roberts (1989), one is introduced to the idea of a merging function which aggregates ballots which are singletons into a singleton outcome.This is definitely a more realistic model of democratic exercises as we see it in practise. Quesada (2000) does a detailed analysis of the manipulability properties of merging functions, i.e. the existence of a voter who can affect the outcome of the merging function unilaterally,irrespective of who all the others vote for.However,even though singleton ballots are a realistic premise for analysis,it is difficult to be theoretically sound and yet exclude the possibility of more than one elected outcome.Thus for instance,under plurality it is quite possible that two candidates receive the maximum number of votes.To accomodate such possibilities,we introduce the concept of a vote aggregator. A vote aggregator is required to satisfy the rather innocuous assumption called unanimity;i.e. if every one votes for the same candidate then that candidate is elected.Further,we require that a vote aggregator never selects a candidate who does not secure even a single vote. Hence, the analysis in this paper is about aggregating ballot profiles, where a ballot is always a singleton.This is ultimately what democracy is all about.

Apart from providing a model for the analysis, we prove some theorems in this paper, which characterize some vote aggregators. Our vote aggregators are analogous to the vote aggregators discussed in Aizerman and Aleskerov (1986,1995).The first vote aggregator we axiomatically characterize is the federation aggregator,which basically allows a finite set of coalitions to unilaterally elect any outcome. We next characterize axiomatically those federation aggregators where coalitions can unilaterally elect outcomes if and only if they have a requisite number. A real world example of such a vote aggregator is the electoral process used in electing members of the *Rajya Sabha* i.e. the upper house of the Indian Parliament. An electoral college comprising of parliamentarians who are themselves elected on the basis of universal adult franchise, must cast a certain minimum number of votes in favour of a candidate for the latter to gain entry into the *Rajya Sabha*. Subsequently we discuss the axiomatic characterization of an oligarchy, where the ability to unilaterally elect an outcome is invested in a single coalition. Once again, an oligarchy is a realistic possibility as for instance, the security council of the United Nations. Further, we characterize axiomatically, the plurality aggregator. This vote aggregator, is the basis of Westminster style parliamentary democracies as practised for instance in India and U.K.The axiomatic characterizations of oligarchic aggregators and the plurality aggregator reported here are quite different from the axiomatic characterizations of oligarchic operators and the plurality aggregator discussed in Lahiri (1999,2001).

An interesting special case of the above model is one where a voter can vote for one out of precisely two candidates. The resulting ballot profile is called a Boolean ballot profile and the vote aggregator whose domain is the set of all Boolean ballot profiles is called a Boolean Vote Aggregator (BVA for short).The framework we develop is a generalization of the concept of a simple game. If names candidates do not matter in deciding how the aggregation procedure works i.e. we assume neutrality, then our framework is exactly that of a simple game. A state of the art survey of progress in the study of simple games can be found in Taylor and Zwicker (1999).In fact Taylor and Zwicker (1999) require simple games to satisfy a property called monotonicity as well. We provide an axiomatic characterization of the vote aggregator which selects those candidates who secure at least fifty percent of the votes. Such a vote aggregator is called the majority vote aggregator. Our axiomatic characterization closely resembles that of May (1952) for aggregating individual preferences over two alternatives to social preferences by the method of majority decision. We show that the assumption called minimal responsiveness which was used in the axiomatic chraracterization of plurality rule, fails to uniquely characterize the majority vote aggregator when it is combined with anonymity, neutrality and non emptiness. However, positive responsiveness due to May (1952) does achieve the purpose. Subsequently, we observe that there is no single valued vote aggregator which satisfies anonymity and neutrality. This compels us to modify our definition of neutrality to require that the names of candidates do not matter in the absence of a tie. We call this property quasi neutrality and we show that the only single valued Boolean vote aggregators which satisfy anonymity, quasi neutrality and positive responsiveness are the two single valued selections from the majority vote aggregator.

In a final section of this paper we consider a property due to Baigent (1987) called proximity preservation.In the conventional model of voting theory it was proposed by Baigent that aggregation procedures should be proximity preserving in the sense that given three preference profiles if the second is closer to the first than the third according to an additively separable metric then the second social ranking should also be closer to the first social ranking compared to the third social ranking.In this framework distance between profiles is measured as the sum of distances between the preferences of each agent. In this paper we assume a metric on the space of ballot profiles which is strongly partially congruent. In a recent work by Grafe and Grafe (forthcoming) another result of a similar nature is available.It is proved there, that there does not exist any metric on preferences which keeps distances.Our main result indicates that a similar (though different) result is true for vote aggregators i.e. there does not exist a metric on the set of all subset of candidates and another on the set of profiles, the latter being partially congruent and the former preserving proximity with respect to the latter (:all terms are defined at the appropriate place in this paper). Since social welfare functions require that the candidates for whom the voters cast their votes are binary relations, our present paper achieves a modest generalization of all the existing results of a similar nature due to Baigent (1987) and Grafe and Grafe (forthcoming).This is possible because a merging correspondence (i.e. a function which assigns to each ballot profile a set of candidates,not necessarily from among the ones who have been voted for,and hence a generalization of a vote aggregator)for which we prove the results does not impose any restriction on the physical characteristics of the candidates. An interesting special case of our metric is the distance between two profiles measured as the sum of the distances between the candidates of each agent on the two ballot profiles. In this framework we obtain the result that there is no merging correspondence (and hence no vote aggregator) which satisfies anonymity and the proximity preservation property. Two similar results, one about social welfare functions and the other about social decision functions can be found in Baigent (1987).However, not only is the context of our analysis different, but the method of proof bears little resemblance to the ones available in the work just cited.

The analytical framework in which aggregation rules are studied in this paper is similar to a device which is referred to in classical choice theory as a choice function. A comprehensive survey of rational choice theory ( i.e. the theory concerned with specifying conditions on a choice function under which there exists a binary relation of a desired type whose "best" elements from a given set of alternatives, coincide with the elements chosen by the choice function) till the mid nineteen eighties is available in Moulin (1985).

## 2. The Model

Let n be a natural number. Let N = {1,...,n} be the set of agents or voters. Let X be a non-empty, finite universal set of alternatives. Let P(X) denote the power set of X, i.e. the set of all subsets of X.

Let $X^N$ denote the set of all functions from N to X. Any element $S = (S_1,...,S_n) \in X^N$, is called a <u>ballot profile</u>. Given $S \in X^N$, the range of S, denoted range(S) = { a $\in$X / there exists $i \in N$, with $S_i = a$}. Let $\Delta$ = { $S \in X^N$ / there exists $x \in X$ such that if $\forall$ $i \in N$ : $S_i = x$}.

A vote aggregator is a function $C : X^N \rightarrow P(X)$ such that for all $S \in X^N$ : (1) $C(S) \subset$ range (S); (2) if there exists $x \in X$ such that if $\forall$ $i \in N$ : $S_i = x$, then C(S)= {x}.

Thus an element which appears on no ballot is never chosen and an element which appears on the ballot of every individual is invariably chosen. The latter property is known as unanimity. As a consequence of our unanimity it easily follows that given any $x \in X$, there exists $S \in X^N$ such that {x} = C(S) : simply take $\forall$ $i \in N$, $S_i = x$.

Given $T \in X^N$ and $x \in X$, let $r(x;T) = |\{i \in N / x \in T_i\}|$ i.e. the cardinality of the set $\{i \in N / x \in T_i\}$. In the sequel we will be considering the following properties of vote aggregators:

**Monotonicity :** Let $x \in C(S)$ and let S and $T \in X^N$ with $\{i \in N / x = S_i\} \subset \{i \in N / x = T_i\}$. Then $x \in C(T)$.

**Neutrality with regard to options :** Let $x, y \in X$ and $S,T \in X^N$. Suppose $\{i \in N/x = S_i\} = \{i \in N/y = T_i\}$. Then $x \in C(S) \leftrightarrow y \in C(T)$.

**Context independence :** Let $x \in X$ and $S,T \in X^N$. Suppose $\{i \in N/x = S_i\} = \{i \in N/x = T_i\}$. Then, $x \in C(S) \leftrightarrow x \in C(T)$.

**Anonymity :** Let $\eta : N \rightarrow N$ be a onto function and suppose $S,T \in X^N$ with $T_{\eta(i)} = S_i$ $\forall$ $i \in N$. Then C(S) = C(T).

**Remark :**

a) Neutrality with regard to options implies context independence;

b) All the vote aggregators axiomatically characterized in this paper satisfy the following property :

Let $C : X^N \rightarrow P(X)$ be a vote aggregator.

**Consistency :** $\forall$ $S \in X^N$ and $\forall$ $\phi \neq M \subset N$, if C(S) = {x}, then C(T) = C(S), where

$$T_i = x, \ \forall \ i \in M,$$

$$= S_i \ \ \forall \ i \in N \setminus M.$$

Hence they may be called consistent vote aggregators. An example of a vote aggregator which does not satisfy consistency is the vote aggregator which, in the absence of unanimity, selects only those candidates who get the second highest number of votes ( and, which in principle may be the empty set). It is easy to see that consistency follows from context independence.

<u>Definition</u> : C is said to be a federation aggregator if there exists $\Omega = \{w_1,...,w_q\}$, a collection of nonempty subsets of N, such that

$$\forall S \in X^N : C(S) = \bigcup_{j=1}^{q} \bigcap_{i \in w_j} \{S_i\}.$$

Definitions :a) C is said to be an oligarchy if C is a federation aggregator with $\Omega = \{ w_1 \}$.
b) C is said to be a k-votes aggregator ( : where 'k' is a positive integer with $k \leq n$) if C is a federation aggregator with $\Omega = \{ w \subset N / w$ has exactly k elements$\}$. (A k-votes aggregator selects only those elements which appear on at least k-ballots.)
c) C is said to be dictatorial if there exists $i \in N$ ( : called a dictator) such that $\forall S \in X^N : C(S) = \{S_i\}$.

## 3 Characterization of Federation,k-votes and Oligarchic Vote Aggregators

**Theorem 1 :** A vote aggregator satisfies monotonicity and neutrality with regard to options if and only if it is a federation aggregator.
**Proof :**It is easy to verify that a federation aggregator satisfies monotonicity and neutrality with regard to options. Hence assume C is a vote aggregator satisfying monotonicity and neutrality with regard to options. We will show that it is a federation aggregator.

Let $x,y \in X$ with $x \neq y$ and let w be any subset of N such that $[\forall i \in w : T_i = \{ x \}; \forall i \notin w : T_i = y]$ implies $[C(T) = \{ x \}]$. Such a subset will be called a decisive set for x against y. By unanimity, N is always a decisive set for x against y. By neutrality with regard to options, if a set is decisive for x against y, then it is decisive for z against w , where $z,w \in X$ with $z \neq w$. A minimal decisive set is any decisive set such that it does not contain any proper subset which is also a decisive set. Let $\Omega = \{ w_1,\ldots,w_q)$ be the collection of minimal decisive sets. By monotonicity,

$$\forall T \in X^N, \bigcup_{j=1}^{q} ( \bigcap_{i \in w_j} \{T_i\}) \subset C(T)$$

(: Let $x \in \bigcup_{j=1}^{q} ( \bigcap_{i \in w_j} \{T_i\})$ ; thus there exists $w_m$ such that

$x \in \bigcap_{i \in w_m} \{T_i\}$ ; thus $x \in C(U)$ where $U_i = T_i$ if $i \in w_m$

= y otherwise.

By monotonicity $x \in C(S)$ where $S_i = T_i \ \forall i$.
Thus, $x \in C(T_1,\ldots,T_n)$.)

Let $T \in X^N$ and suppose $x \in C(T) \setminus (\bigcup_{j=1}^{q} \bigcap_{i \in w_j} \{T_i\})$

Let $w = \{ i \in N / x = T_i\}$.
Clearly, there does not exist $w_j \in \Omega$ such that $w_j \subset w$.
Let $S_i = x$ if $i \in w$
= y otherwise.
Clearly, $x \notin C(S)$. Hence, by monotonicity, $x \notin C(T)$, which is a contradiction.

Thus, $C(T) \subset \bigcup_{j=1}^{q} \bigcap_{i \in w_j} \{T_i\}$. Thus C is a federation aggregator.

Q.E.D.

As a corollary to Theorem 1, we have the following theorem :

**Theorem 2 :** A vote aggregator satisfies monotonicity, neutrality with regard to options and anonymity if and only if it is a k-votes aggregator.

**Proof :** It is easy to see that a k-votes aggregator satisfies the desired properties. Hence assume C is a vote aggregator satisfying monotonicity, neutrality with regard to options and anonymity. By Theorem 1, there exists $\Omega = \{ w_1, \ldots, w_q \}$, $\phi \neq w_j \subset N$ $\forall$ j = 1,...,q such that

$\forall S \in X^N: C(S) = \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} \{S_i\})$ . Let, $\phi \neq w \subset N$ such that cardinality of w is equal to the cardinality of $w_1$.

Let $T \in X^N$ such that

$\forall i \notin w \cup w_1 : T_i = S_i$

$\forall i \in w \cup w_1 : T_{\rho(i)} = S_i$

where $\rho : N \to N$ is onto such that

$\forall i \notin w \cup w_1 : \rho(i) = i$

$\rho(w) = w_1$

$\rho(w_1) = w$.

By anonymity, C(T) = C(S).

Thus, $w \in \Omega$.

Thus, C is a k-votes aggregator.

Q.E.D.

The following axiom is essentially due to Ilyunin, Popov and Elkin as in [11].

Weak Neutrality with regard to options: Let $\sigma : X \to X$ be a bijection and let $S, T \in X^N$ with $T_i = \{\sigma(x) / x \in S_i\}$ $\forall i \in \{1, \ldots, n\}$. Then $C(T) = \{\sigma(x) / x \in C(S)\}$.

**Lemma 1 :** Let C be a vote aggregator which satisfies Anonymity and Weak Neutrality with regard to options. Let $S, T \in X^N$ and suppose $y \in C(S)$. Let $x \in X$ with $x \neq y$. Let $r(x;S) = r(y;T)$. Further suppose that for $z \in X \setminus \{x,y\}$, $\{i \in N / S_i = z\} = \{i \in N / T_i = z\}$. Then, $x \in C(T)$.

**Proof :** Let $U \in X^N$, with $U_i = S_i$ if $S_i \in X \setminus \{x,y\}$,

$= y$ if $S_i = x$

$= x$ if $S_i = y$.

By weak neutrality with regard to options and $y \in C(S)$, we get $x \in C(U)$. Now $\forall z \in X$: $r(z;U) = r(z;T)$. Thus, by anonymity, C(U) = C(T). Hence, $x \in C(T)$.□

**Theorem 3 :** A vote aggregator satisfies Monotonicity, Anonymity and Weak Neutrality with regard to options if and only if it is a k-votes aggregator.

**Proof :** That a k-votes aggregator satisfies the above properties is easily verified. Hence let C be a vote aggregator satisfying the above properties. Let $x, y \in X$ with $x \neq y$. Let, $k = \min\{r \in N / S_i = x\ \forall i = 1,\ldots,r;\ S_i = y$, otherwise implies $C(S) = \{x\}\}$.

Thus, $C(T) \subset \bigcup_{j=1}^{q} \bigcap_{i \in w_j} \{T_i\}$. Thus C is a federation aggregator.

Q.E.D.

As a corollary to Theorem 1, we have the following theorem :

**Theorem 2 :** A vote aggregator satisfies monotonicity, neutrality with regard to options and anonymity if and only if it is a k-votes aggregator.

**Proof :** It is easy to see that a k-votes aggregator satisfies the desired properties. Hence assume C is a vote aggregator satisfying monotonicity, neutrality with regard to options and anonymity. By Theorem 1, there exists $\Omega = \{ w_1, \ldots, w_q \}$, $\phi \neq w_j \subset N \ \forall j = 1, \ldots, q$ such that

$\forall S \in X^N: C(S) = \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} \{S_i\})$ . Let, $\phi \neq w \subset N$ such that cardinality of w is equal to the cardinality of $w_1$.

Let $T \in X^N$ such that

$\forall i \notin w \cup w_1 : T_i = S_i$

$\forall i \in w \cup w_1 : T_{\rho(i)} = S_i$

where $\rho : N \to N$ is onto such that

$\forall i \notin w \cup w_1 : \rho(i) = I$

$\rho(w) = w_1$

$\rho(w_1) = w$.

By anonymity, C(T) = C(S).

Thus, $w \in \Omega$.

Thus, C is a k-votes aggregator.

Q.E.D.

The following axiom is essentially due to Ilyunin, Popov and Elkin as in [11].

<u>Weak Neutrality with regard to options</u>: Let $\sigma : X \to X$ be a bijection and let $S,T \in X^N$ with $T_i = \{\sigma(x) / x \in S_i\} \ \forall i \in \{1, \ldots, n\}$. Then $C(T) = \{\sigma(x) / x \in C(S)\}$.

**Lemma 1 :** Let C be a vote aggregator which satisfies Anonymity and Weak Neutrality with regard to options. Let $S,T \in X^N$ and suppose $y \in C(S)$. Let $x \in X$ with $x \neq y$. Let $r(x;S) = r(y;T)$. Further suppose that for $z \in X \setminus \{x,y\}$, $\{i \in N / S_i = z\} = \{i \in N / T_i = z\}$. Then, $x \in C(T)$.

**Proof :** Let $U \in X^N$, with $U_i = S_i$ if $S_i \in X \setminus \{x,y\}$,

$= y$ if $S_i = x$

$= x$ if $S_i = y$.

By weak neutrality with regard to options and $y \in C(S)$, we get $x \in C(U)$. Now $\forall z \in X$: $r(z;U) = r(z;T)$. Thus, by anonymity, C(U) = C(T). Hence, $x \in C(T)$.□

**Theorem 3 :** A vote aggregator satisfies Monotonicity, Anonymity and Weak Neutrality with regard to options if and only if it is a k-votes aggregator.

**Proof :** That a k-votes aggregator satisfies the above properties is easily verified. Hence let C be a vote aggregator satisfying the above properties. Let $x,y \in X$ with $x \neq y$. Let, $k = \min\{r \in N / S_i = x \ \forall i = 1, \ldots, r;\ S_i = y$, otherwise implies $C(S) = \{x\}\}$.

By weak neutrality with regard to options, k is independent of x and y. By anonymity, $k = \min \{r \in N / |\{ i \in N / S_i = x \}| = r, |\{ i \in N / S_i \neq x \}| = n - r$ implies $C(S) = \{x\}\}$

By monotonicity $y \in C(S)$ whenever $|\{i \in N / y = S_i \}| \geq k$.

Now suppose $y \in C(S)$ and towards a contradiction suppose $|\{i \in N / y \in S_i \}| < k$. Thus $|\{ i \in N / S_i \neq x \}| > n - k$. This contradicts the minimality of k. Hence the theorem.

Q.E.D.

We have already characterized a federation aggregator, using the axioms of monotonicity and neutrality with regard to options. It is worthwhile investigating what additional property would be required in order to characterize an oligarchy. It turns out the following assumption is sufficient.

**Existence of Essential Voter :** Given $x \in C(S)$ with $S \in X^N$ there exists $j \in \{i \in N/x = S_i\}$ ( possibly depending on x and S),such that if $T \in X^N$ with $T_i = S_i \ \forall \ i \neq j$ and $T_j \neq x$, then $x \notin C(T)$.

**Theorem 4 :** A vote aggregator satisfies monotonicity, neutrality with regard to options and existence of essential voter if and only if it is an oligarchy.

**Proof :** It is easy to see that an oligarchy satisfies the desired properties. Hence assume C is a vote aggregator which satisfies the properties mentioned in the theorem. By Theorem 1, C must be a federation aggregator i.e.

$$C(S) = \bigcup_{j=1}^{q} \left( \bigcap_{i \in w_j} \{S_i\} \right), \ \forall \ S \in X^N,$$

where $\Omega = \{w_1,\dots,w_q\}$, $\phi \neq w_j \subset N$. We claim $w_i = w_j \ \forall \ i, j$.

Let $x,y \in X$ with $x \neq y$ and let $S \in X^N$ with $S_i = x$ ,if $i \in w_j$ , $S_i = y$, otherwise. Thus, $x \in C(S)$.Let, $w = \{k \in \{i \in N/x = S_i \} /$ if $T \in X^N$ with $T_i = S_i \ \forall \ i \neq k$ and $T_k \neq x$, then $x \notin C(T)\}$. Thus w is a subset $w_j$. Towards a contradiction suppose that w is a proper subset of $w_j$. Let, $h \in w_j \setminus w$ and let

$$\begin{aligned} T_i &= x \ \forall \ i \in w_j \setminus \{h\}, \\ &= y \text{ if } i = h, \\ &= S_i \text{ otherwise.} \end{aligned}$$

Thus, $T_i = S_i \ \forall \ i \neq h$ and $T_h \neq x$. However, $h \notin w$. Thus, $x \in C(T)$. This contradicts, $C(S) = \bigcup_{j=1}^{q} \left( \bigcap_{i \in w_j} \{S_i\} \right), \ \forall \ S \in X^N$ and proves the theorem.

Q.E.D.

A short step from oligarchy is the dictatorial vote aggregator,which is simply an oligarchy comprising a single agent.The following assumption,in addition to the ones proposed in the axiomatic characterization of an oligarchy proves sufficient for our present purpose.

**Existence of a dictated alternative:** $\forall S \in X^N$ with $C(S) \neq \phi$, there exists $x \in C(S)$, $j \in N$ and $y \in X \setminus \{x\}$such that if $T \in X^N$ with $T_i = S_i \ \forall \ i \neq j$, $T_j = y$, then $y \in C(T)$.

The following now easily follows from Theorem 4:

**Theorem 5 :** A vote aggregator satisfies monotonicity, neutrality with regard to options, existence of essential voter and existence of a dictated alternative if and only if it is dictatorial.

**Proof :** The proof follows easily from theorem 4,by requiring every member of the oligarchy to vote for exactly one and the same alternative, say x, the rest voting for a y different from x,and then applying the definition of the existence of a dictated alternative.

Q.E.D.

Theorem 5, clearly portrays the difference between our framework and the framework of received social choice theory.Whereas in the latter framework dictatorship seems to be the natural outcome of any aggregating procedure,in our framework dictatorship appears as the forced outcome of an analytically belaboured procedure,since the existence of a dictated alternative is not as natural an assumption,as some of the others that we have used in theorem 5,or for that matter elsewhere in this paper.

## 4. Characterization of the Plurality Aggregator

A vote aggregator $C : X^N \to P(X)$ is said to be the plurality aggregator if $\forall T \in X^N : C(T) = \{x \in X / r(x;T) \geq r(y;T)\ \forall y \in X\}$. Brams and Fishburn (1983) were the first to analytically discuss approval voting (:which we call plurality aggregator) although not in the same formal framework as proposed by us. Let [X] denote the set of all non-empty subsets of X, i.e. $[X] = P(X) \setminus \{\phi\}$.

We need two other properties to characterize this aggregator :

**Non-emptiness : Non-emptiness :** For all $S \in X^N, C(S) \in [X]$.

**Minimal-responsiveness :** Let $S \in X^N$ and suppose $x, y \in C(S)$ with $x \neq y$. Suppose $x = S_1$. Let $T \in X^N$ ,with $T_1 \in C(S) \setminus \{x\}$ and $T_i = S_i\ \forall\ i \neq 1$.

Then: (a) $T_1 \in C(T)$;(b) $x \notin C(T)$.

We are now in a position to characterize the plurality aggregator.

**Theorem 6 :** A vote aggregator satisfies anonymity,weak neutrality with regard to options, non-emptiness and minimal responsiveness if and only if it is the plurality aggregator.

**Proof :** It is easy to verify that the plurality aggregator satisfies the above mentioned properties. Hence assume C is a vote aggregator which satisfies the above properties. Let $S \in X^N$ and suppose $C(S) = \{x_1,\ldots,x_q\}$.

Let $w_j = \{i \in N / x_j = S_i\}$.

Thus $C(S) = \bigcup_{j=1}^{q} (\bigcap_{i \in w_j} \{S_i\})$. Note $\{x_j\} = \bigcap_{i \in w_j} \{S_i\}$.

By anonymity and an argument similar to that used in the proof of Theorem 2, there exists a positive integer 'k' such that $\Omega = \{w \subset N / w \text{ has exactly 'k' elements}\}$.

Since, C satisfies non-emptiness, $\{x \in X / r(x;S) \geq r(y;S)\ \forall\ y \in X\} \subset C(S)$. Let $x, y \in C(S)$ with $x \neq y$ (: if C(S) is a singleton then by the above it selects the plurality aggregator winner).Thus, $r(y;S) \geq k$. Suppose $r(y; S) < r(x;S)$. Without loss of generality and by anonymity assume, $x = S_1$. Let $T_1 = y$ and $\forall\ i \neq 1$:$T_i = S_i$.It is easy to check that owing to minimal responsiveness and a replication of the argument, $C(T) = \bigcup_{w \in \bar{\Omega}} (\bigcap_{i \in w} T_i)$, where $\bar{\Omega}$ = {w/w has exactly $\bar{k}$ elements},$y \in C(T)$ and $x \notin C(T)$.

Thus $r(x;S)-1 = r(x;T) < \bar{k} \leq r(y;T)$.

Now $r(x; S)+1 > r(y; S)+1 = r(y;T) \geq \bar{k}$. .

Thus, $\bar{k}+1 > r(x; S)$ and $r(x; S)+1 > \bar{k}$. Thus $r(x; S) = \bar{k}$.Thus, $\bar{k}+1 = r(x; S)+1 > r(y; S)+1 = r(y;T) \geq \bar{k}$, implies $r(y;T) = \bar{k}$.Thus, $r(y; S) = \bar{k}-1 = r(x;T)$.By anonymity, weak neutrality with regard to options and Lemma 1, $y \in C(S)$ implies $x \in C(T)$ and a contradiction. Thus $x,y \in C(S)$ with $x \neq y \rightarrow r(y;S) = r(x;S)$.Thus C(S) consists only of the plurality winners.

Q.E.D.

**Note:**The plurality aggregator does not satisfy monotonicity.Neither does it satisfy context independence.However it does satisfy consistency.

## 5. Characterization of Majority Vote Aggregator

Let $X=\{0,1\}$. Any element $S = (S_1,\ldots,S_n) \in X^N$, is now called a <u>(Boolean) ballot profile</u>. Let E denote the ballot profile such that $\forall\ i \in N$ :$E_i = 1$ and O denote the ballot profile such that $\forall\ i \in N$ :$O_i = 0$. A vote aggregator on $X = \{0,1\}$ is called a Boolean Vote Aggregator (BVA).
Weak Neutrality with regard to options has a particularly simple structure for BVA's:

**Neutrality:** For all $S \in X^N$ : $C(E-S) = \{1-x / x \in C(S)\}$.

<u>Definition</u> :
C is said to be a Majority Vote Aggregator (MVA) if $\forall\ S \in X^N$ : $C(S) = \{x \in X / r(x;S) \geq n/2\}$.

**Proposition 1 :** (a) MVA satisfies Non-emptiness, Monotonicity, Neutrality and Anonymity. (b) Let C be a BVA such that $\forall\ S \in X^N$ :C(S)= range (S). Then, C satisfies Non-emptiness, Monotonicity, Neutrality and Anonymity.
**Proof :** The obvious proof is being omitted.
Q.E.D.

A BVA C is said to satisfy:
**Positive Responsiveness** if $\forall\ S,T \in X^N$ : [$S \geq T$, $S \neq T$ and $1 \in C(T)$] implies [ $\{1\} = C(S)$];
**Negative Responsiveness** if $\forall\ S,T \in X^N$ : [$T \geq S$, $S \neq T$ and $0 \in C(T)$] implies [$\{0\} = C(S)$].

**Proposition 2 :** Let C be a BVA.
(a) If C satisfies Neutrality and Positive Responsiveness then it also satisfies Negative Responsiveness.
(b) If C satisfies Neutrality and Negative Responsiveness then it also satisfies Positive Responsiveness.
**Proof :** We will only prove (a), the proof of (b) being entirely analogous. Hence let C be a BVA satisfying Neutrality and Positive Responsiveness, and let $S,T \in X^N$ with $[T \geq S$, $S \neq T$ and $0 \in C(T)]$. Let $T^1 = E - T$ and let $S^1 = E - S$. Thus, $S^1 \geq T^1$, $S^1 \neq T^1$. By Neutrality, $1 \in C(T^1)$. By Positive Responsiveness, $\{1\} = C(S^1)$. By Neutrality, $\{0\} = C(S)$. Thus, C satisfies Negative Neutrality.
Q.E.D.

The following proposition is easily verified:
**Proposition 3 :** MVA satisfies both Positive and Negative Responsiveness.

In the sequel the following proposition will be found useful:

**Proposition 4 :** Let $S,T \in X^N$ and let $x \in X$. Suppose $r(x;S) \geq r(x;T)$. Then, there exists $U \in X^N$, with $r(x;T) = r(x;U)$ and $\{i \in N / x = U_i\} \subset \{i \in N / x = S_i\}$.
**Proof :** Let $\{i \in N / x = S_i\} = \{i_1,\ldots,i_k\}$ and $\{i \in N / x = T_i\} = \{h_1,\ldots,h_q\}$. By hypothesis, $k \geq q$. Let $U \in X^N$, be such that $\{i \in N / x = T_i\} = \{i_1,\ldots,i_q\}$. Then, $r(x;T) = r(x;U)$ and $\{i \in N / x = U_i\} \subset \{i \in N / x = S_i\}$.
Q.E.D.

**Proposition 5 :** Let C be a BVA satisfying Anonymity and let $S,T \in X^N$, with $r(1;S) = r(1;T)$. Then $C(S) = C(T)$.
**Proof :** Let $\{i \in N / 1 = S_i\} = \{i_1,\ldots,i_k\}$ and $\{i \in N / 1 = T_i\} = \{h_1,\ldots,h_k\}$. Let $\eta : N \to N$ be defined thus : $\eta(i_q) = h_q$ for $q = 1,\ldots,k$; $\eta(i) = i$, if $i \in N \setminus \{i_1,\ldots,i_k\}$. Thus, $\eta$ is an onto function and $T_{\eta(i)} = S_i \ \forall i \in N$. By Anonymity, $C(S) = C(T)$.
Q.E.D.

**Lemma 2:** Let C be a BVA satisfying Non-emptiness, Neutrality, Anonymity and Monotonicity. Let $S \in X^N$, $x \in X$ and suppose, $r(x;S) \geq r(1-x;S)$. Then, $x \in C(S)$.
**Proof :** Let C,S and x be as above and suppose towards a contradiction that $x \notin C(S)$. By Nonemptiness, $\{1-x\} = C(S)$. Let $T = E - S$. By Neutrality $\{x\} = C(T)$. Now, $r(1-x;T) = r(x;S) \geq r(1-x;S) = r(x;T)$. By Proposition 4, there exists $U \in X^N$, with $r(x;T) = r(x;U)$ and $\{i \in N / x = U_i\} \subset \{i \in N / x = S_i\}$. By Anonymity and Proposition 5, $\{x\} = C(T) = C(U)$. By Monotonicity, $x \in C(S)$, contradicting $x \notin C(S)$ and thereby proving the lemma.
Q.E.D.

**Lemma 3:** Let C be a BVA satisfying Positive Responsiveness and Neutrality. Then C satisfies Monotonicity.
**Proof :** Let C be a BVA satisfying Positive Responsiveness and Neutrality. Let $1 \in C(S)$ and let S and $T \in X^N$ with $\{i \in N / 1 = S_i\} \subset\subset \{i \in N / 1 = T_i\}$. Thus, $[T \geq S$, $T \neq S$ and

$1 \in C(S)$]. By Positive Responsiveness, {1}= C(T). Thus, $1 \in C(T)$.Now, let $0 \in C(S)$ and let S and $T \in X^N$ with $\{i \in N / 0 = S_i\} \subset \subset \{i \in N / 0 = T_i\}$.By Proposition 2(a), C satisfies Negative Responsiveness. Thus, [$S \geq T$, $T \neq S$ and $0 \in C(S)$] and Negative Responsiveness implies {0}= C(T).Thus, C satisfies Monotonicity.

Q.E.D.

In the context of a BVA satisfying Neutrality Minimal Responsiveness has the following alternative definition :

**Minimal Responsiveness :** Let $S \in X^N$ and suppose C (S) = {1,0}.Suppose $S_1=0$. Let $T \in X^N$, with $T_1 =1$ and $T_i = S_i \; \forall \; i \neq 1$.Then C(T) = {1}.

Example: Let n =3 and let C be a BVA defined thus: C(E) ={1}, C(O) = {0}, and for $S \in X^N \backslash \{E,O\}$, let $C(S) = \{x \in X / r(S;x) = 1\}$.C satisfies Non-emptiness, Neutrality, Anonymity and Minimal Responsiveness. However, C does not satisfy Monotonicity; for if $S \in X^N$ is defined by $S_1= 1, S_2=S_3=0$, then C(S) = {1}. However, if $T \in X^N$ is defined by $T_1= 1=T_2, T_3=0$, then C(T) = {0}.This happens in spite of the fact that $\{i \in N / 1 = S_i\} \subset \{i \in N / 1 = T_i\}$.

We are now in a position to characterize MVA.

**Theorem 7 :** A BVA satisfies Non- emptiness, Anonymity, Neutrality and Positive Responsiveness if and only if it is MVA.

**Proof :** Propositions 1(a) and 3 show that MVA satisfies Non- emptiness, Anonymity, Neutrality and Positive Responsiveness. Thus suppose C is a BVA satisfying Non-emptiness, Anonymity, Neutrality and Positive Responsiveness.

Case 1: n is an even integer.

In this case if $S \in X^N$, with r(S;1) = r(S;0), then by Lemmas 1 and 2, C(S) = {0,1}. By Positive Responsiveness, r(S;1) > r(S;0) implies C(S) = {1}. By Positive Responsiveness and Proposition 2(a), r(S;0) > r(S;1) implies C(S) = {0}. Thus, C is MVA.

Case 1: n is an odd integer.

In this case if $S \in X^N$, then either r(S;1) > r(S;0) or r(S;0) > r(S;1).

Suppose r(S;1) = (n+1)/2. Thus, $1 \in C(S)$.Suppose towards a contradiction that C(S) = {0,1}.Let T = E – S. Then, by Neutrality, C(T) = {0,1} and r(T;1) = (n-1)/2 < (n+1)/2 = r(S;1). By Proposition 4, there exists $U \in X^N$, with r(x;T) = r(x;U) and $\{i \in N / x = U_i\} \subset \subset \{i \in N / x = S_i\}$. By Anonymity and Proposition 5, C(U) = C(T) = {0,1}. Now, $S \geq U$, $S \neq U$ and $1 \in C(U)$.Thus, by Positive Responsiveness, C(S) = {1}.By Positive Responsiveness, for all $S \in X^N$: [r(S;1) > r(S;0)] implies [ C(S)={1}]. By Neutrality, for all $S \in X^N$: [r(S;0) > r(S;1)] implies [C(S) ={0}]. Thus, C is MVA.

Q.E.D.

**Note:** If n is an odd integer then MVA is always a single valued correspondence. However if n is as odd integer then clearly MVA is not single whenever a ballot profile has an equal number of ones and zeroes.

**Single Valued :** For all $S \in X^N$, C(S) is a singleton.

We now state and prove a result of some interest.

**Proposition 6:** Let n be an even integer. Then, there does not exist any BVA which satisfies Anonymity, Neutrality and is Single Valued.

**Proof:** Towards a contradiction suppose that C is a BVA satisfying Anonymity, Neutrality and is Single Valued. Let $S \in X^N$ be defined thus: $S_i = 1$ for i = 1,...,n/2; $S_i = 0$ for i = (n/2)+1,...,n. Let T = E-S. Let C(S) = {x}. By Neutrality, C(T) = {1-x}≠{x}. However by Anonymity and Proposition 5, C(T) = C(S) = {x}≠{1-x}leading to a contradiction. This proves the proposition.

Q.E.D.

However there are single valued BVA's which are anonymous and satisfy the following property:

**Quasi neutrality:** For all $S \in X^N$ with r(S;1)≠r(S;0): C(E-S)={1-x/x∈ C(S)}.

Say that a BVA C is the Upper Majority Vote Aggregator (UMVA) if : for all $S \in X^N$: (a) [r(S;1) ≥ r(S;0)] implies [C(S) = {1}]; (b) [r(S;0) > r(S;1)] implies [ C(S) = {0}].
Say that a BVA C is the Lower Majority Vote Aggregator (LMVA) if : for all $S \in X^N$: (a) [r(S;1) > r(S;0)] implies [C(S) = {1}]; (b) [r(S;0) ≥ r(S;1)] implies [ C(S) = {0}].
Clearly both UMVA and LMVA are single valued and satisfies Quasi neutrality, Anonymity and Positive Responsiveness. Further, whenever n is an odd integer both UMVA and LMVA coincides with MVA.
The following lemma leads to our next major result:
**Lemma 4:** Let C be Single Valued BVA satisfying Quasi neutrality, Anonymity and Positive Responsiveness. Let $S \in X^N$, x ∈X and suppose, r(x;S) > r(1-x;S). Then, {x}= C(S).
**Proof :** Let C,S and x be as above and suppose towards a contradiction that {x}≠ C(S). Since C is single valued , {1-x}=C(S). Let T = E – S. By Neutrality {x}=C(T). Now, r(1-x;T) = r(x;S) > r(1-x;S) = r (x;T). By Proposition 4, there exists $U \in X^N$, with r(x;T) = r(x;U) and $\{i \in N / x = U_i\} \subset \{i \in N / x = S_i\}$. By Anonymity and Proposition 5, {x}=C(T) = C(U). By Positive Responsiveness, {x}=C(S), contradicting {x} ≠C(S) and thereby proving the lemma.
Q.E.D.
In view of Lemma 4 we can state the following theorem:

**Theorem 8 :** There are exactly two single valued BVA's satisfying Anonymity, Quasi neutrality and Positive Responsiveness. They are UMVA and LMVA.

## 6. Preservation of Proximity

A merging correspondence on X, is a function $C : X^N \to P(X)$ , such that if S ∈Δ then C(x) = {x}, where $x = S_i \forall i \in N$. Clearly, a merging correspondence C on X is a vote aggregator, if and only if $\forall S \in X^I$: C(S)⊂ range (S).A merging correspondence

$C : X^N \rightarrow P(X)$ , such that $\forall\ S \in X^N$ : C(S) is a singleton, is called a merging function .

Let "m" be any metric on P(X).

Let δ be any metric on X and let "r" be any positive real number. Define a (δ, r) induced metric $d_\delta^r$ on $X^N$ as follows : $\forall$ S,T $\in X^N$ : $d_\delta^r(S,T) = \left\{ \sum_{i \in I}^{\delta} [\delta(S_i, T_i)]^r \right\}^{\frac{1}{r}}$ .

A merging correspondence $C : X^N \rightarrow P(X)$ is anonymous if whenever $S,T \in X^I$ and $j,k \in N$ : $[S_i=T_i\ \forall i \in I \setminus \{j,k\}, S_j=T_k, S_k=T_j]$ implies C(S) = C(T).

Given a (δ, r) induced metric $d_\delta^r$ on $X^N$ , a merging correspondence C is said to preserve (δ, r) proximity if $\forall S,T,U \in X^N : [d_\delta^r(S,T) < d_\delta^r(S,U)]$ implies [m(C(S),C(T)) ≤ m(C(S),C(U))].

The following concepts originate in Grafe and Grafe (forthcoming).

Let d be a metric on $X^N$ .It is said to be congruent at (x,y) if (i) x,y $\in$X and x≠y;(ii)$\forall$ S,T,U $\in X^N$ : [range (S) = range (T) = range (U) = {x,y}] &[$\{i \in N/S_i = T_i\} \subset\subset \{ i \in N/ S_i = U_i \}$] implies [d(S,T) < d(S,U)].It is said to be partially congruent if there exists x,y $\in$X such that it is congruent at (x,y).It is said to be globally congruent if it is congruent at (x,y) whenever x,y $\in$X and x≠y.

Let $C : X^N \rightarrow P(X)$ be a merging correspondence. It is said to preserve distances with respect to d, where d is a metric on $X^N$ if (i) d is partially congruent; and (ii) $\forall S,T,U \in X^N$: $[d(S,T) < d(S,U)]$ implies [m(C(S),C(T)) ≤ m(C(S),C(U))].

**Theorem 9** : Let d be any partially congruent metric on $X^N$ .Then there exists a vote aggregator which is anonymous and preserves distances.

**Proof** : Let C(S) = range (S) $\forall S \in X^N$ . Then C is anonymous and preserves distances with respect to any metric m on P(X) and any partially congruent metric d on $X^N$ .□

Let d be a metric on $X^N$ .It is said to strongly congruent at (x,y) if (i) x,y $\in$X and x≠y;(ii)$\forall$ S,T,U $\in X^N$ : [range (S) $\subset$ range (T) = range (U) = {x,y}] &[$\{i \in N/S_i = T_i\} \subset\subset \{ i \in I/S_i = U_i\}$] implies [d(S,T) < d(S,U)].It is said to be strongly partially congruent if there exists x,y $\in$X such that it is congruent at (x,y).It is said to be strongly globally congruent if it is congruent at (x,y) whenever x,y $\in$X and x≠y.

Q.E.D.

Example of a strongly globally congruent metric: Let δ be any metric on X and let "r" be any positive real number. Then a (δ, r) induced metric $d_\delta^r$ on $X^N$ is strongly globally congruent.

Let $C : X^N \rightarrow P(X)$ be a merging correspondence. It is said to preserve proximity with respect to a metric d, if (i) d is strongly partially congruent; and (ii) $\forall S,T,U \in X^N$ : $[d(S,T) < d(S,U)]$ implies [m(C(S),C(T)) ≤ m(C(S),C(U))].

The following lemma is a simple consequence of the definitions:

**Lemma 5** : Let $C : X^N \rightarrow P(X)$ be a merging correspondence which satisfies anonymity. Let d be a metric on $X^N$. If d is weakly congruent at (x,y) then $\forall$ S,T,U $\in X^N$ : [range (S) $\subset$ range (T) = range (U) = {x,y}] &[ #{i∈N/ $S_i$=$U_i$ } < # { i∈N/ $S_i$=$T_i$ }] implies [d(S,T) < d(S,U)].

**Theorem 10** :- Let n be atleast two and let, d be a strongly partially congruent metric on $X^N$. Then there is no merging correspondence which is anonymous and preserves proximity with respect to d.

**Proof** :- Suppose C satisfies anonymity and d is a metric on $X^N$ which is strongly congruent at (x,y) for some $x \neq y$.

Case 1 :- n is an even number.

Let S∈$X^N$ with $S_i$ = x ∀i∈N,

T∈$X^N$ with $T_i$ = y ∀i∈N,

U∈$X^N$ with $U_i$ = x if i∈N and i is odd

= y if i∈N and i is even

W∈$X^I$ with $W_i$ = y if i∈N and i is odd

= x if i∈N and i is even.

By anonymity, C(U) = C(W) and C(S)={x}≠{y}= C(T).

Now ,since d is strongly congruent at (x,y), [range (S) $\subset$ range (W) = range (U) = {x,y}] &[{i∈N/ $U_i$=$W_i$ } ⊂⊂ { i∈N/ $U_i$= $W_i$ }] implies [d(S,U) < d(W,U)].If C(U)≠{x}, then m(C(U),C(S)) > 0, but m(C(U),C(W))= 0.This is contrary to C preserving proximity. Hence suppose C(U) = {x}.Thus, C(U)≠{y}. Now ,since d is strongly congruent at (x,y), [range (T) $\subset$ range (W) = range (U) = {x,y}] &[{i∈N/ $U_i$=$W_i$ } ⊂⊂ { i∈N/ $U_i$=$T_i$ }] implies [d(T,U) < d(W,U)]. But, m(C(U),C(T)) > 0 and m(C(U),C(W))=0.This is contrary to C preserving proximity.

Case 2 :- n is an odd number.

In this case n-1 is an even number greater or equal to two.

Let S∈$X^N$ with $S_i$ = x ∀i∈N,

T∈$X^N$ with $T_i$ = y ∀i∈N,

U∈$X^N$ with $U_i$ = x if i∈N and i is odd

= y if i∈N and i is even

W∈$X^N$ with $W_i$ = y if i∈N, i is odd and i < n

= x if i∈N and[i is even or i =n].

By anonymity, C(U) = C(W) and C(S)={x}≠{y}=C(T).

Now ,since d is strongly congruent at (x,y), [range (S) $\subset$ range (W) = range (U) = {x,y}] &[{i∈N/ $U_i$=$W_i$ } ⊂⊂ { i∈N/ $S_i$=$U_i$ }] implies [d(S,U) < d(W,U)].If C(U)≠{x}, then m(C(U),C(S)) > 0, but m(C(U),C(W))=0.This is contrary to f preserving proximity. Thus, C(U) = C(W) = {x}.

Let R∈$X^N$ with $R_i$ = x if i∈N, i is odd and i < n

= y if i∈N and [i is even or i = n]

V∈$X^N$ with $V_i$ = y if i∈N and i is odd

= x if i∈N and i is even.

By anonymity, C(R) = C(V).

Now ,since d is strongly congruent at (x,y), [range (T) ⊂ range (R) = range (V) = {x,y}] &[{i∈N/ $R_i$=$V_i$ } ⊂⊂ { i∈I/ $V_i$=$T_i$ }] implies [d(T,V) < d(R,V)]. But m(C(R),C(V))=0. Since C preserves proximity, m(C(V),C(T)) = 0.Thus C(V) = {y}.

Now ,since d is strongly congruent at (x,y), [range (W) = range (R) = range (V) = {x,y}] &[#{i∈I/ $R_i$=$V_i$ } = 1 < (n-1) = # { i∈I/ $W_i$=$V_i$ }],Lemma 5 implies [d(V,W) < d(R,V)].But m(C(R),C(V))=0 < m({x},{y}) = m(C(V),C(W)) .This is contrary to C preserving proximity. This proves the theorem.

Q.E.D.

**It follows as a consequence of Theorem 10, that there is no vote aggregator which satisfies anonymity and preserves proximity.**

**Theorem 11** :- Suppose n is atleast two.Then there does not exist any merging correspondence which satisfies anonymity and (δ, r) proximity.

**Proof :-** Suppose C satisfies anonymity and (δ, r) proximity.Let d = $d_\delta^r$.Then C preserves proximity with respect to the strongly congruent metric d, contradicting the conclusion of Theorem 10. This proves the theorem. □

**It follows as a simple consequence of Theorem 11 that there does not exist any merging correspondence which satisfies anonymity and preserves (δ, r) proximity.**

**Note :** The results of this section are all valid even if X is an infinite set.

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