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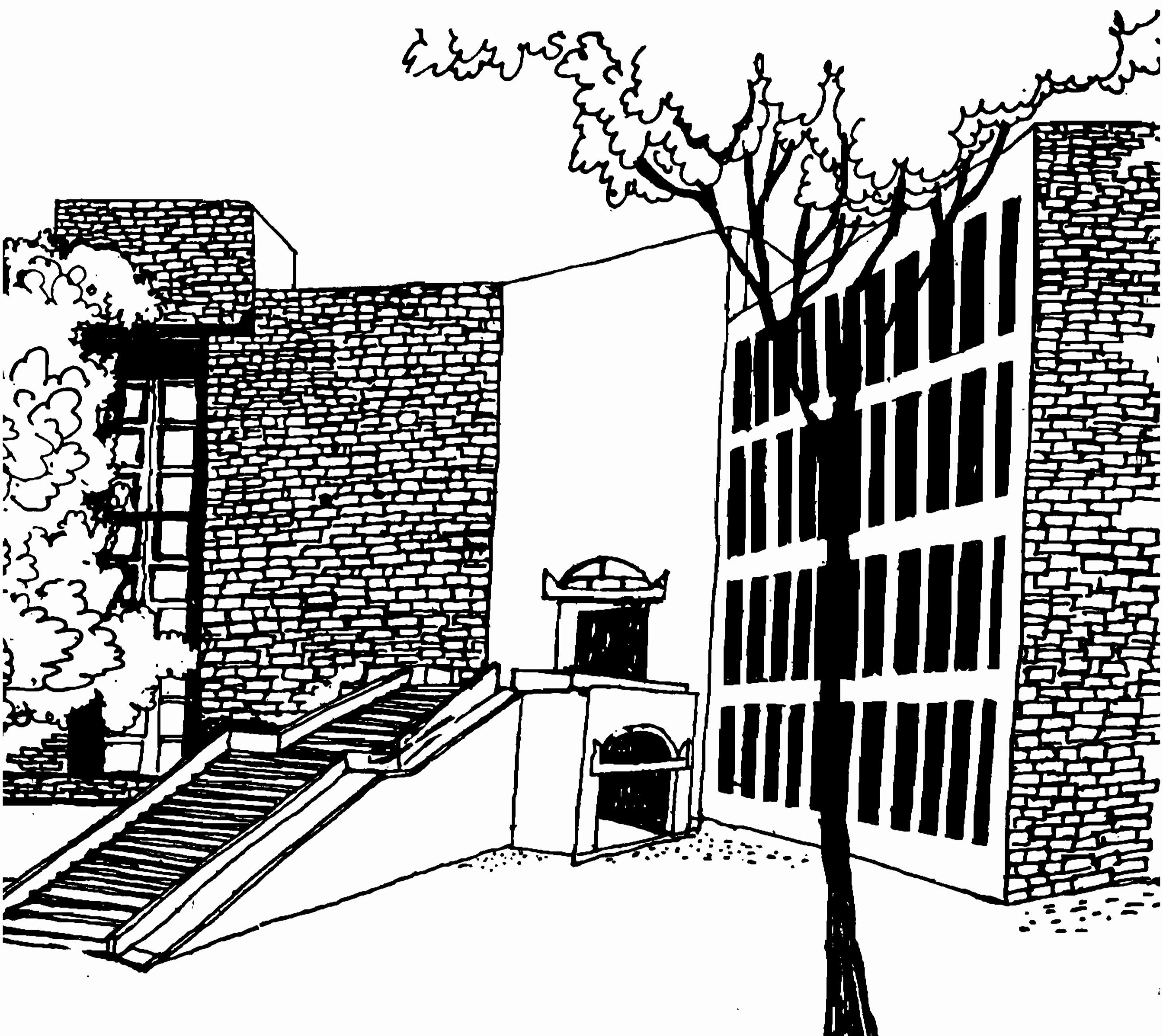

PROBLEMS OF FAIR DIVISION AND THE EGALITARIAN SOLUTION: A RECONSIDERATION

By

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Problems of Fair Division and the Egalitarian Solution : A Reconsideration

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1 Introduction

The present paper attempts to provide simple proofs of two theorems in the literature of axiomatic bargaining with a variable population. Both theorems deal with axiomatic characterizations of the egalitarian solution due to Kalai (1997), in the variable population framework. The egalitarian solution assigns to a bargaining problem (arising out of the problem of dividing a bundle of goods amongst a finite number of agents) the utility allocation which is both Weakly Pareto Optimal and has equal coordinates. There are several exhaustive treatments of the central issues in axiomatic bargaining with a variable population, notably Thomson and Lensberg (1985). The origins of this line of speech can be traced to Thomson (1983a).

The first theorem whose proof we simplify is the one appearing in Thomson (1983b), where the egalitarian solution is characterized by a population monotonicity assumption. Unlike Thomson (1983b), our proof relies heavily on the fact that the set of potential agents is infinite. Thomson (1983b) proves the result even when the set of potential agents is finite.

The second theorem whose proof we simplify is the one appearing in Lahiri (1998), where the egalitarian solution is characterized by a weak reduced game property due to Peters, Tijs and Zarzuelo (1994). Although, we provide this proof for the case where the set of potential agents is infinite, a quick study of the proof reveals that it is equally valid if the number of potential agents is at least three. Thus this is really an alternative proof.

Since this paper is primarily a technical note, we do not provide any intuitive motivations behind our assumptions or results. In any case, elaborate discussions of the properties are available in the literature cited above.

2 The Model

Let $\mathcal{I} \equiv \aleph$ (the set of natural numbers). $\mathcal{I}$ is the set of potential agents. Agents in $\mathcal{I}$ are indexed by the subscript i. $\mathcal{P}$ is the class of nonempty finite subsets of $\mathcal{I}$. Given $P \in \mathcal{P}$, $|P|$ is the cardinality of P and $\Re^P$ (resp. $\Re^P_+$, $\Re^P_{++}$). Given P in $\mathcal{P}$ a bargaining problem (briefly, a problem) for P is any nonempty compact, convex subset S of such $\Re^P_+$ that

(a) S is comprehensive : $0 \le y \le x \in S$, implies $y \in S$;
(b) there exists $x \in S$ such that $x >> 0$.

In the above, given x, $y \in \Re^P$: $x \ge y \leftrightarrow [x_i \ge y_i \; \forall i \in P]$; $x > y \leftrightarrow [x \ge y$ & $x \ne y]$; $x >> y \leftrightarrow [x_i > y_i \; \forall i \in P]$.

Let Σ^P denote the class of all bargaining problems for P and let $\Sigma \equiv \bigcup_{P \in \mathcal{P}} \Sigma^P$.

Let $X = \bigcup_{P \in \mathcal{P}} \Re^P_+$.

A solution on Σ is a function $f : \Sigma \to X$ such that $F(S) \in S \; \forall S \in \Sigma$.

The egalitarian solution on Σ, denoted by $E : \Sigma \to X$ (or simply by E) is defined as follows : given $P \in P$ and $S \in \Sigma^P$, $E(S) = \bar{t}e_p$, where

(i) $\bar{t} = \max\{ t \in \Re_+ / te_p \in S \}$
(ii) e_p is the vector in $\Re^P_+$ with all coordinates equal to 1.

A solution F on Σ is said to satisfy :

a) Weak Pareto Optimality (WPO) if for all $P \in P$ and $S \in \Sigma^P$, $x \in \Re^P_+$, $x >> F(S)$ implies $x \notin S$;

b) Anonymity (AN) if for all P, $Q \in P$ with $|P| = |Q|$ and $S \in \Sigma^P$, $T \in \Sigma^Q$ if $T = \{ \pi(x) / x \in S\}$ then $F(T) = \pi[F(S)]$ where $\pi : P \to Q$ is a one to one function and for $x \in \Re^P_+$, $\pi(x) \equiv y$ where $y_{\pi(i)} = x_i \forall i \in P$.

c) Nash's Independence of Irrelevant Alternatives (NIIA) if for all $P \in \mathcal{P}$, S, $T \in \Sigma^P$ with $S \subset T$, $[F(T) \in S$ implies $F(S) = F(T)]$.

d) Continuity (CONT) if for all $P \in \mathcal{P}$, for all sequences $\{S^\kappa\}$ in Σ^P and S in Σ^P with $\lim_{\kappa\to\infty} S^\kappa = S$ (in the Hausdorff topology), $\lim_{\kappa\to\infty} F(S^\kappa) = F(S)$ (in the Euclidean topology).

e) Monotonicity with Respect to Changes in the Number of Agents (MON) if for all $P,Q \in P$ with $P \subset Q$, for all $S \in \Sigma^P$ and $T \in \Sigma^Q$, if $S = T'$, then for all $i \in P$,

$F_i(S) \geq F_i(T)$, where $T' = \{ x_P / x \in T\}$. (Note : given $x \in \Re^Q$, x_P is the restriction of x to P).

3 The Main Theorem :

Theorem 1 : The only solution on Σ to satisfy WPO, AN, NIIA, CONT and MON is E.

Proof : It is easy to see that E satisfies the above properties. Hence let us assume that F is a solution which satisfies the desired properties. Given $P \in \mathcal{P}$ and $S \in \Sigma^P$, let $u(S) \in \Re^P_+$ be defined as follows : $u_i(S) = \max\{ x_i / x \in S\}$ whenever $i \in P$.

Lemma 1 : Let F satisfy WPO, AN, CONT and MON on Σ. Given $S \in \Sigma^P$ ($P \in \mathcal{P}$), suppose

(a) $u_i(S) = u_j(S)\ \forall\ i, j \in P$.

Then F(S) = E(S).

Proof : Let S satisfy the hypothesis of the lemma. Suppose $\{x \in S / y > x \rightarrow y \notin S\} = \{x \in S / y >> x \rightarrow y \notin S\}$.

Let $E(S) = be_P$ with $b > 0$.

Let $\kappa \notin I \setminus P$ and let $Q = P \cup \{\kappa\}$. Without loss of generality assume $P = \{1,\ldots,\kappa-1\}$. Let $S^0 = S$; let $Q^1 = \{2,\ldots,\kappa\}$ and let $\pi^1 : P \rightarrow Q^1$ be defined by $\pi^1(i) = (i+1) \bmod \kappa$, let $Q^2 = \{1,3,4,\ldots,\kappa\}$ and let $\pi^2 : P \rightarrow Q^2$ be defined by $\pi^2(i) = (i + 2) \bmod \kappa$. In general for $1 \leq j \leq \kappa-1$, let $Q^j = (P \cup \{\kappa\}) \setminus \{ j \}$ and let $\pi^j : P \rightarrow Q^j$ be defined by $\pi^j(i) = (i + j) \bmod \kappa$. Let $S^j = \{\pi^j(x) / x \in S\}$.

Let T be the smallest convex and comprehensive set containing S^0, S^1, $S^2,\ldots,S^{\kappa-1}$ and $\{be_Q\}$.

By WPO and AN, $F(T) = be_Q = E(T)$.
Now $S = S^0 = \{x_P/x \in T\}$.
By MON, $F(S) \geq F_P(T) = be_P$.
But then by WPO of E and by the assumption made on S, $F(S) = be_P = E(S)$.
By CONT, F(S) = E(S), whenever S satisfies the condition of the lemma.

Q.E.D

Proof of Main Theorem Continued : Given $S \in \Sigma^P$, $P \in \mathcal{P}$, let $a = \min\{ u_i(S) / i \in P\}$. Suppose

$\{x \in S / y > x \rightarrow y \notin S\} = \{x \in S / y >> x \rightarrow y \notin S\}$.

If $ae_P = u(S)$, then by lemma 1, $F(S) = E(S)$.
Hence assume, $u(S) > ae_P >> 0$.
Clearly $y >> ae_P \rightarrow y \notin S$.
Thus $y > ae_P \rightarrow y \notin S$.
If $ae_P \in S$, then since $ae_P >> 0$, for all $i \in P$, there exists $x^i \in S$ such that $x_i^i > a$.

But this contradicts definition of a. Thus $ae_P \notin S$.

Let $S(0) = \{x \in S / x_i \leq a \; \forall \; i \in P\}$.
Now $E(S) << ae_P$.
Thus $E(S) = E(S(0)) = be_P$ (say).
By lemma 1,$F(S) = E(S) = E(S(0)) = be_P$.
Further, $be_P \in$ *rel. int.* $\{x \in S(0)/y > x \rightarrow y \notin S\}$.
Let $S(t) = \{x \in S/x_i \leq tu_i(S) + (1\text{-}t)\, u_i(S) \; \forall \; i \in P\}$, $t \in [0, 1]$.
Suppose $F(S) = F(S(1)) \neq be_P$.
By NIIA, $F(S(1)) \in S \setminus S(0)$.

Further, for $t < 1$, $F(S(t)) \in S(0) \rightarrow F(S(t)) = be_P$. Since $be_P \in$ *rel.int.* $\{x \in S(0) / y > x \rightarrow y \notin S\}$, we get a contradiction of the continuity of F.

Thus $F(S) = E(S)$.
By CONT, $F(S) = E(S)$ for all $S \in \Sigma$.

Q.E.D.

Remark : In the above proof we have implicitly assumed that be_Q is Weakly Pareto Optimal in T. Let us now show why this would actually be the case. Let V be the smallest convex comprehensive set containing $S^0, S^1, \ldots, S^{\kappa-1}$ and let $\bar{b}\, e_Q = E(V)$. By WPO and AN, $F(V) = \bar{b}\, e_Q$. Let $d = \max\{b, \bar{b}\}$ and U be the smallest convex comprehensive set containing $S^0, S^1, \ldots, S^{\kappa-1}$ and de_Q. Further, $U_P = S$. Thus by MON, $F(S) = F(U_P) \geq F_P(U) = de_P \geq be_P$. Thus by our assumption, $F(S) = be_P$. Hence $be_P = de_P$. Thus $b = d$. Hence be_Q is Weakly Pareto Optimal in V and also in T.

4. Auxiliary Result :

Once again a solution F on Σ is said to satisfy :

(f) Homogeneity (HOM) if for all $P \in \mathcal{P}$, $S \in \Sigma^P$, and $\lambda > 0$, $F(\lambda S) = \lambda F(S)$. (Here given $x \in \Re^P$ and $\lambda > 0$, $(\lambda x)_i = \lambda x_i \; \forall \; i \in P$; $\lambda S = \{\lambda x / x \in S\}$).

(g) Weak Reduced Game Property (WRGP) if for all P, Q $\in$ *P* with $P \subset Q$ and $|P| = 2$ for all $S \in \Sigma^Q$, $F(S) \neq 0$ implies $F\left(S_P^{F(S)}\right) = F_P(S)$, where $S_P = \{x_P / x \in S\}$, $S_P^{F(S)} = \lambda(S_P, F_P(S))S_P$ and $\lambda(S_P, F_P(S)) = \min\{\lambda \in \Re_+ / F_P(S) \in \lambda S_P.\}$

Theorem 2 : The only solution of Σ to satisfy WPO, AN, HOM, NIIA, WRGP and CONT is E.

Proof : It is easy to check that E satisfies the above properties. Hence, let F be a solution which satisfies the stated properties.

Lemma 2 : Let F satisfy WPO, AN, HOM, WRGP and CONT on Σ. Given $V \in \Sigma^M$, $M \in$ *P*, suppose

(a) $u_i(V) = u_j(V) \ \forall \ i, j \in M$.

Then F(V) = E(V).

Proof : Let V satisfy the hypothesis of the lemma.

Suppose $\{x \in V / y > x \rightarrow y \notin V\} = \{x \in V / y >> x \rightarrow y \in V\}$. We will show that F(V) has all coordinates equal , so that by WPO it must agree with E(V).

Let $P = \{i, j\} \subset M$ and let $S = V_P$. Let $k \in I \setminus \{i, j\}$ and let $Q = \{i, j, k\}$. Construct $T \in \Sigma^Q$ as follows : Let $S^0 = S$,

$S^1 = \{(y_j, y_k) \in \Re_+^{\{j,k\}} / y_j = x_i, y_k = x_j, \text{ for some } (x_i, x_j) \in S\}$

$S^2 = \{(y_k, y_i) \in \Re_+^{\{k,i\}} / y_k = x_i, y_i = x_j, \text{ for some } (x_i, x_j) \in S\}$.

Let T be the smallest comprehensive convex set containing S^0, S^1, S^2. Clearly, $T_P = S$ and by WPO and AN, F (T) = E (T). By WRGP, $F(T_P^{F(T)}) = F_P(T) = E_P(T)$; by HOM, $F(T_P^{F(T)})$

$= F(\lambda(T_P, F_P(T))T_P) = \lambda(T_P, F_P(T))F(T_P)$.

Therefore, $F(S) = F(T_P) = \frac{1}{\lambda(T_P, F_P(T))} E_P(T)$.

Therefore, $F(V_P)$ has both coordinates equal, whenever $P = \{i, j\} \subset M$.

By WRGP,

$F_P(V) = F(V_P^{F(V)}) = F(\lambda(V_P, F_P(V))\, V_P)$

$= \lambda(V_P, F_P(V))\, F(V_P)$ (by HOM).

Thus, $F_P(V)$ has both coordinates equal whenever $P = \{i, j\} \subset M$.

Thus, F(V) has all its coordinates equal.

By WPO, F (V) = E (V).

Q.E.D.

Proof of Theorem 2 Continued : The argument now is identical to the argument in the proof of Theorem 1 appearing after the proof of Lemma 1.

Note : The above theorem and proof would be equally valid if instead of assuming $I = \aleph$, we had assumed $|I| \geq 3$.

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