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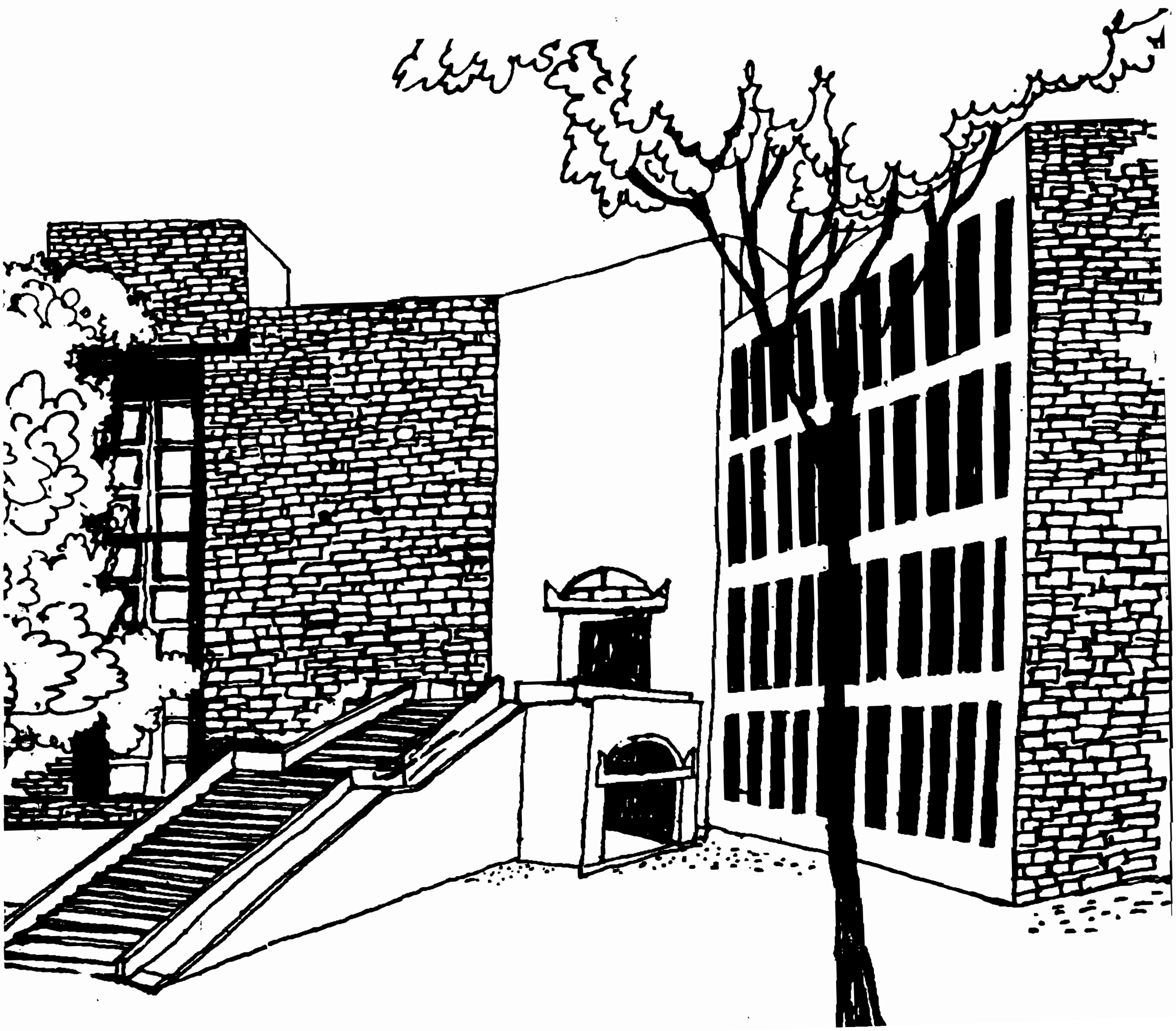

A NOTE ON CHARACTERISING THE MEDIAN

By

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A Note On Characterising the Median

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1.Introduction: Consider the situation where one has to choose one among three differently priced birthday cakes,to give to a friend.It is very likely, that in the absence of strong personal reasons,one would select the cake whose price lies between the two extremes.A similar emphasis on the middle path is found in the teachings of Buddha as also in Confucian philosophy.That the choice of an alternative from a finite set of alternatives,need not conform to some optimising behavior,is a possibility that has been discussed in Baigent and Gaertner (1996).In a sense this is a position on human behavior which is contrary to the received view of a decision maker as an optimizer of some objective function that is favored for instance by Sen (1993).That the median does not satisfy the requirements of underlying optimising behavior has been noted by Kolm (1994) and Gaertner and Xu (1999).However, the median is a reasonable compromise,in practical decision making.

In Gaertner and Xu (1999) can be found a first axiomatic characterisation of the choice rule which selects the median from a finite set of alternatives.The axiomatic characterisation is valid for a universal set containing at least four alternatives as Example 1 in our paper points out.For universal sets containing three alternatives the above mentioned axiomatic characterisation is no longer valid.Hoever,decision theory as opposed to decision algorithms,has overriding importance only when the set of alternatives is sufficiently small.For large sets the computational complexity of a solution may substantially offset its decision theoretic virtues.For a set containing a small number of alternatives we may ignore computational issues and concentrate only on decision theoretic properties.Hence, it is our view in this paper, that the real test of a theory of decision making takes place only when the universal set of alternatives is relatively small.

In this paper we provide two theorems which characterizes the median choice function when the universal set has atleast three alternatives.Several examples are provided to highlight the relationship between the axioms emphasised in this paper.It is also noted here that our second axiomatic characterisation breaks down if the universal set contains precisely two elements.

2. <u>The Model</u>: Let N denote the set of positive integers and let $X=\{i \in N / i \leq n\}$ (:the set of first n positive integers) for some $n \in N$ with $n \geq 3$.Let [X] denote the set of all non-empty subsets of X. Given $A \in [X]$, let #(A) denote the cardinality of A. A choice function on X is a function $C:[X] \rightarrow [X]$,such that $C(A) \subset A\ \forall A \in [X]$.The <u>median choice function</u> $M:[X] \rightarrow [X]$ is defined as follows: $\forall A \in [X]$,

$M(A)=\{k\}$ if (a) $\#\{i \in A/i<k\}=\#\{i \in A/i>k\}$,and (b) #(A) is an odd number;
$=\{j,k\}$ if (a)$j<k$,(b) $\#\{i \in A/i<j\}=\#\{i \in A/i>k\}$,and (c) #(A) is an even number.

The following axioms are due to Gaertner and Xu (1999):

<u>Axiom 1</u>: $\forall i,j \in X, C(\{i,j\})=\{i,j\}$.

<u>Axiom 2</u>: $\forall i,j,k \in X$,with $i \neq j \neq k \neq i$, $C(\{i,j,k\}) \neq \{i,j,k\}$.

<u>Axiom 3</u>: $\forall A \in [X]$ and $i \in X \backslash A$,if $B=\{i\} \cup (A \backslash C(A))$ then $C(A \cup B)= C(C(A) \cup C(B))$.

<u>Axiom 4</u>: $\forall A \in [X], C(A)=\{i,j\}$ with $i \neq j, C(A \backslash \{i\})=\{j\}$.

<u>Axiom 5</u>: If $i,j,k,m \in X$,where all of them are distinct,then $C(\{i,j,k,m\})=\{i,j\}$ implies there exists $a \in \{k,m\}$ such that $i \in C(\{i,j,a\})$.

<u>Example 1</u>:Let n=3.Let $C(\{1,2,3\})=\{1\}$ and C(A)=A,otherwise.Clearly C satisfies all the five axioms given above.

<u>Example 2</u>: Let $C(A)=\{i \in A/i \geq j\ \forall j \in A\}$, whenever $A \in [X]$.Then C satisfies all the above axioms except for Axiom 1.

<u>Example 3</u>:Let $C(A)=A\ \forall A \in [X]$.Then C satisfies all the above axioms except for Axiom 2.

<u>Example 4</u>:Let C(A)=A if $A \in [X]$ and #(A)=2, and let $C(A)=\{i \in A/i \geq j\ \forall j \in A\}$, otherwise.Then C satisfies all the axioms above,if n=3 and all except Axiom 4,if $n \geq 4$.For, let $A=\{1,2,3\}$ and $i=4$.Then $\{1,2,4\}=\{4\} \cup (A \backslash C(A))$ since $C(A)=\{3\}$.Let $B=\{1,2,4\}$.Thus,$A \cup B=\{1,2,3,4\}$ and $C(A \cup B)=\{4\}$. However $C(C(A) \cup C(B))=\{3,4\} \neq \{4\}=C(A \cup B)$,contradicting Axiom 3.

<u>Example 5</u>: Let $C(A)=\{i \in A/i \geq j\ \forall j \in A \text{ or } i \leq j\ \forall j \in A\}$, $\forall A \in [X]$. Then C satisfies all the above axioms except for Axiom 4.For let $A=\{1,2,3\}$. Then, $C(A)=\{1,3\}$.However,$C(\{1,2\})=\{1,2\}$ and $C(\{2,3\})=\{2,3\}$, contradicting Axiom 4.

Example 5: Let $n = 4$.Let $C(A)=A$ if $\#(A)$ is an even number,and let $C(A)=\{i\in A/i \geq j\ \forall j\in A\}$, otherwise.C satisfies all the above axioms above except for Axiom 5.For,$1,2 \in C(\{1,2,3,4\})$, but $1\notin C(\{1,2,3\})$ and $1\notin C(\{1,2,4\})$,contradicting Axiom 5.

The following axiom is implied by Axiom 1:
Binary Injective Invariance (BII): $\forall$ i,j $\in$ X with $i \neq j$, if $f: \{i,j\}\rightarrow X$ is one to one and order preserving (:in the sense that $f(i) > f(j)$ if and only if $i>j$),then $C(\{f(i),f(j)\})= \{f(k)/\ k \in C(\{i,j\})\}$.

That Axiom 1 implies BII is an easy observation.

The following axiom is crucial for what follows:
Invriance with respect to Best and Worst outcomes (IBW): $\forall A\in[X]$ and for all $i,j \in X\backslash A$, $[i < k\ \forall k\in A]\ \&[j>k\ \forall k\in A]$ implies $C(A \cup \{i,j\})=C(A)$.

Observe that the choice function in example 1 does not satisfy IBW.

Example 7: Let $C(A)= \{i\in M(A)/i \geq j\ \forall j\in M(A)\}$.Clearly $C \neq M$, since $C(\{1,2\}) = \{2\}\neq \{1,2\}=M(A)$ and yet C satisfies BII and IBW.However C does not satisfy Axiom 1.
Example 7: Let $C(A)= \{i\in M(A)/i \geq j\ \forall j\in M(A)\}$ if $1 \notin A$ and $C(A)= \{i\in M(A)/i \leq j\ \forall j\in M(A)\}$,otherwise. $C(\{1,2\})=\{1\}\neq \{3\}=C(\{2,3\})$.Let $f: \{1,2\}\rightarrow X$ be defined by $f(i) = i+1$ for $i \in \{1,2\}$. f is order preseving. However, $C(\{f(1),f(2)\}) \neq\{f(k)/\ k \in C(\{1,2\})\}$ contradicting BII.However, C satisfies IBW.
The choice function defined in Example 2 above satisfies BII.However it does not satisfy IBW.

3. Some Results:

Proposition 1:Let C be a choice function satisfying IBW.Then:
(i) $C(A) = C(M(A))\ \forall A\in[X]$;
(ii) $C(A) = M(A)\ \forall A\in[X]$ with $\#(A)$ being an odd number.

Proof:Given $A\in[X]$, either $A=M(A)$ or, there exists $k \in N$ and $\{i_j \in X\backslash M(A)/\ j \in\{1,\ldots,2k\}\}$ such that (a) $i_j > i_{j-1}$, $\forall j \in \{2,\ldots,2k\}$; (b) $i_j < a < i_{j+k}$, $\forall j \in \{1,\ldots,k\}$; (c) $A= M(A) \cup \{i_j \in X\backslash M(A)/\ j \in\{1,\ldots,2k\}\}$.

If A=M(A), then C(A)=C(M(A)).Otherwise, by IBW, C(M(A)) = C(M(A) $\cup$ { i_k , i_{k+1}}).By IBW, C(M(A) $\cup$ { i_{k-j} ,..., i_{k+j+1} })= C(M(A) $\cup$ { i_{k-j-1},..., i_{k+j+2} }). Thus C(M(A)) = C(A).
If #(A) is an odd number,then M(A) is a singleton,whence C(M(A)) =M(A). This proves the proposition. ♥

<u>Theorem 1</u>: The only choice funtion on X which satisfies Axiom 1 and IBW is M.

Proof: M clearly satisfies Axiom 1 and IBW. Hence let C be any choice function on X which satisfies Axiom 1 and IBW.By Proposition 1,C(A) = C(M(A)) $\forall A \in [X]$,and in particular C(A) = M(A) whenever #(A) is an odd number.However if #(A) is an even number then M(A) is a set consisting two distinct elements,whence by Axiom 1, C(M(A))=M(A).Hence, C(A)=M(A) $\forall A \in [X]$. ♥

4.<u>Another Axiomatic Characterisation</u>: A final property we invoke is the following:

<u>Partial Fidelity (PF)</u>: $\forall A \in [X]$ with $\#(A) \geq 2$ and $\forall a \in X \backslash A$, if [<u>either</u> ($a < i$ $\forall i \in A$), <u>or</u> ($a > i$ $\forall i \in A$)], then $C(A \cup \{a\}) \cap C(A) \neq \phi$ (: the empty set).

<u>Proposition 2</u>:Let C be a choice function satisfying IBW and PF and let $A \subset X$ with $M(A) \subset X \backslash \{1,n\}$.Then, C(A) = M(A).

Proof: By Proposition 1,C(A) = C(M(A)) $\forall A \in [X]$,and in particular C(A) = M(A) whenever #(A) is an odd number.Since #(A) is an odd number if and only if M(A) is a singleton, we need only consider the case where #(M(A)) =2.Thus let M(A)= {i,j} with $i < j$.
<u>Case 1</u>: C(M(A)) = {i}. Clearly $j < n$ and M(A)$\cup$ {n} = {i,j,n}.By Proposition 1, C(M(A)$\cup${n}) = {j}.This contradicts PF,since then C(M(A) $\cup$ {n}) $\cap$ C(M(A)) = ϕ .
<u>Case 2</u>: C(M(A)) = {j}. Clearly $1 < i$ and M(A)$\cup$ {1} = {1,i,j}.By Proposition 1, C(M(A)$\cup${1}) = {i}.This contradicts PF,since then C(M(A) $\cup$ {1}) $\cap$ C(M(A)) = ϕ .
Thus since C(M(A)) $\neq \phi$, we must have C(M(A))= M(A). ♥

Note : The choice function in Example 8,satisfies PF as well.Thus it satisfies IBW and PF but not BII. The choice function in Example 7,satisfies BII and IBW but not PF.

Example 9:Let C(A)=A $\forall A \in [X]$.Then C satisfies BII and IBW but not PF.

Theorem 2: The only choice funtion on X which satisfies BII,IBW and PF is M.

Proof: M clearly satisfies BII ,IBW and PF. Hence let C be any choice function on X which satisfies BII ,IBW and BF.By Proposition 1,C(A) = C(M(A)) $\forall A \in [X]$,and in particular C(A) = M(A) whenever #(A) is an odd number.However #(A) is an even number if and only if M(A) is a set consisting two distinct elements.By Propsition 2,if $M(A) \subset X \setminus \{1,n\}$,then C(A) = M(A).Hence let us assume that #(M(A)) =2 and $M(A) \cap \{1,n\} \neq \phi$. Let us first show that for all $i \in X$, with $1 < i < n$, $i \in C(\{1,i\}) \cap C(\{i,n\})$. Towards a contradiction suppose $i \notin C(\{1,i\})$.Thus $C(\{1,i\}) = \{1\}$.However $C(\{1,i.n\}) = \{i\}$, and this contradicts PF,since we get $C(\{1,i\}) \cap C(\{1,i.n\}) = \phi$.Thus suppose $i \notin C(\{i,n\})$.Thus $C(\{i,n\}) = \{n\}$.However $C(\{1,i.n\}) = \{i\}$, and this contradicts PF,since we get $C(\{i,n\}) \cap C(\{1,i,n\}) = \phi$.Hence $i \in C(\{1,i\}) \cap C(\{i,n\})$.
Let $f : \{1,i\} \rightarrow X$ be defined by f (1) = i and f (i) = n . f is order preserving.Since $i \in C(\{1,i\})$, by BII,$n \in C(\{i,n\})$.Hence, $C(\{i,n\}) = \{i,n\}$.
Now let $g : \{i,n\} \rightarrow X$ be defined by g (i) = 1 and g (n) = i . g is order preserving.Since $i \in C(\{i,n\})$, by BII,$1 \in C(\{1,i\})$.Hence, $C(\{1,i\}) = \{1,i\}$.
Now let $h : \{2,n\} \rightarrow X$ be defined by h (2) = 1 and h (n) = n . h is order preserving.Since $C(\{2,n\}) = \{2,n\}$, by BII, $C(\{1,n\}) = \{1,n\}$.
Thus C(M(A)) = M(A) $\forall A \in [X]$.This in conjunction with Proposition 1, proves the theorem. ♥

Example 10:Let C(A)=A if $\#(A) \neq 2$ and let $C(\{i,j\}) = \{i\}$ if $A = \{i,j\}$ with $i < j$.Then C satisfies PF and BII.However C does not satisfy Axiom 1.
Example 11:Suppose $n \geq 4$. Let C(A)=A if #(A)= 1 or 2 and let $C(A) = \{i \in M(A) / i \geq j \ \forall j \in M(A)\}$ if $\#(A) \geq 3$.Then C satisfies Axiom 1 but not PF: let $A = \{1,2,3\}$ and let a = 4.Then $C(A)=\{2\}$ and $C(A \cup \{4\})=\{3\}$ violating PF.

Remark 1: If we had not insisted on $\#(A) \geq 2$ in the definition of PF, then the modified axiom would imply Axiom 1.This is because for $i<j$, $C(\{i,j\}) = \{j\}$