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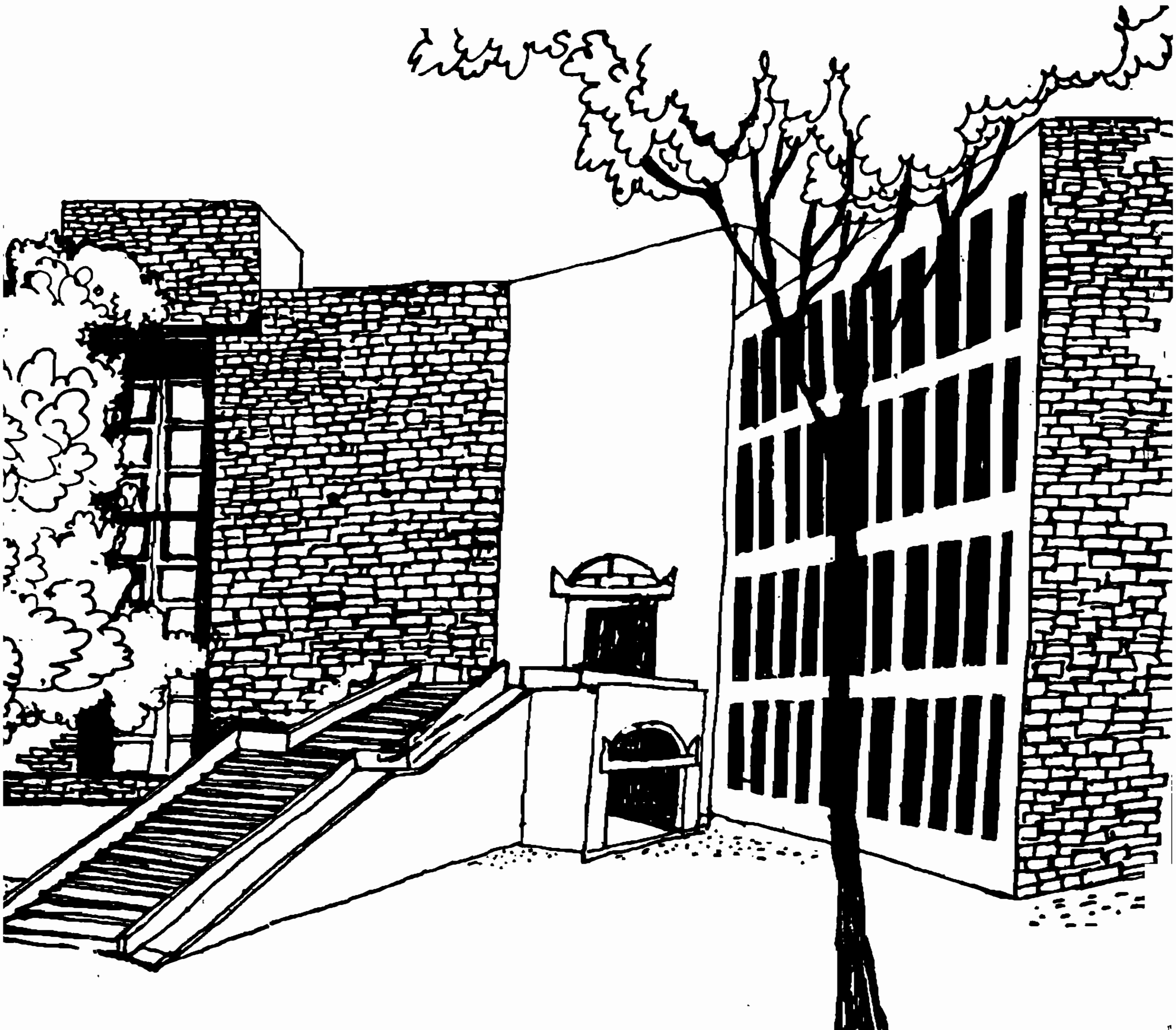

UNCOVERED CHOICE FUNCTIONS

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1. Introduction : An abiding problem in choice theory has been one of characterizing those choice functions which are obtained as a result of some kind of optimization. Specifically, the endeavour has concentrated largely on finding a binary relation (if there be any) whose best elements coincide with observed choices. An adequate survey of this line of research till the mid eighties is available in Moulin [1985]. More recently, the emphasis has focused on binary relations defined on non-empty subsets of a given set, such that the choice function coincides with the best subset corresponding to a feasible set of alternatives. This problem has been provided with a solution in Lahiri [Forthcoming], although the idea of binary relations defined on subsets is a concept which owes its analytical origins to Pattanaik and Xu [1990].

   In Dutta and Laslier [1999] one finds the device of a comparison function, which is basically a real valued function defined on all pairs of alternatives satisfying the condition that the value of an ordered pair is negative of the value of the ordered pair which is obtained by interchanging the order of the first ordered pair. Hence, in particular the value of the function along the diagonal (i.e. the set of ordered pairs with identical first and second components) is zero. A comparison function simultaneously captures the idea of preference and the intensity of preference. An alternative 'x' is preferred to another alternative 'y' if and only if the value of the comparison function at (x, y) is positive, and the value of the comparison function at (x, y) is meant to convey the intensity with which 'x' is preferred to 'y'. With the help of a comparison function they introduce the notion of 'cover': 'x' is said to cover 'y' if 'x' is preferred to 'y' (i.e. the value of the comparison function at (x, y) is positive) and for every other third element 'z' the value of the comparison function at the ordered pair (x, z) is alteast as much as the value of the comparison function at the ordered pair (y, z). Given any feasible sets, its uncovered set is the set of all elements in the feasible set which are not covered by any other element in the same set. The question that naturally arises is the following : Given a choice function , under what condition does a comparison function exist, whose uncovered sets always coincide with the choice function? This question has been discussed in Lahiri [1999], where it is observed that the binary relation 'is uncovered' is reflexive, complete and quasi-transitive and any reflexive, complete and quasi-transitive binary relation can be made to coincide with the "is uncovered" relation of some comparison function. The problem becomes much more difficult if instead of defining the covering relation globally, we considered the covering relation for each individual feasible set, by simply looking at the restriction of the comparison function to that set. In such a situation that fact that 'x' covers 'y' in a particular feasible set does not imply that 'x' covers 'y' globally. In effect, we are then concerned with what Sen [1997] calls 'menu based' relations.

In this paper we take a look at this latter problem, by considering only those comparison functions which can assume only three values : 1, 0 and -1. These comparison functions are essentially reflexive and complete binary relations. In a subsequent section we consider the problem of axiomatically characterizing those choice functions which coincide with the uncovered sets of binary relations, where 'covering' is now defined as a 'menu-based' concept.

2. Choice Functions and Uncovered Choices :- Let X be a finite, non-empty set and given any non empty subset A of X, let [A] denote the collection of all non-empty subsets of A. Thus in particular, [X] denotes the set of all non-empty subsets of X. If $A \in [X]$, then # (A) denotes the number of elements in A.

A binary relation R on X is said to be (a) reflexive if $\forall x \in X : (x, x) \in R$; (b) complete if $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$; transitive if $\forall x, y, z \in X$, $[(x, y) \in R \ \& \ (y, z) \in R$ implies $(x, z) \in R]$; anti-symmetric if $[\forall x, y \in X, (x, y) \in R \ \& \ (y, x) \in R$ implies $x = y]$. Let Π denote the set of all reflexive and complete binary relations. If $R \in \Pi$ is anti-symmetric, then R is called a tournament. Given a binary relation R, let $P(R) = \{(x, y) \in R \,/\, (y, x) \notin R\}$ and $I(R) = \{(x, y) \in R \,/\, (y, x) \in R\}$. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R.

Given $R \in \Pi$, $A \in [X]$ and $x, y \in X$, we say that x covers y via R in A if :

(i) $x, y \in A$; (ii) $(x, y) \in P(R)$; (iii) $\forall z \in A$: $[(y, z) \in R$ implies $(x, z) \in R]$; (iv) $\forall z \in A$: $[(y, z) \in P(R)$ implies $(x, z) \in P(R)]$.

Let $\hat{R}(A) = \{(x, y) \in A \times A \,/\, x$ covers y via R in A$\}$ and let $UC(A, R) = \{x \in A \,/$ if $y \in A$ then $(y, x) \notin \hat{R}(A)\}$. It is easy to see that $\forall A \in [X]$, $\hat{R}(A)$ is a transitive binary relation on A. Thus $UC(A,R) \neq \phi$ whenever $A \in [X]$. Thus for $R \in \Pi$, the function $UC(. , R) : [X] \rightarrow [X]$ is well defined and further, $UC(A, R) \subset A \ \forall A \in [X]$.

A choice function C on X is a function $C : [X] \rightarrow [X]$ such that $C(A) \subset A \ \forall A \in [X]$. Thus for $R \in \Pi$, $UC(. , R) : [X] \rightarrow [X]$ is a choice function.

A choice function C on X is said to be an uncovered choice function if there exists $R \in \Pi$ with $C(A) = UC(A, R) \ \forall A \in [X]$.

Lemma 1 :- Let $R \in \Pi$ such that $C(A) = UC(A, R) \ \forall A \in [X]$. Then $R = \{(x, y) \in X \times X / x \in C\{x, y\}\} \equiv R(C)$.

The obvious proof of this lemma is being omitted.

3. Axiomatic Characterization of Uncovered Choice Functions :- Let C be a choice function on X.
It is said to satisfy :

(1) Strong Condorcet (SC) if $\forall A \in [X]$ and $x \in A$: $[C\{x, y\}) = \{x\} \ \forall y \in A \setminus \{x\}$ implies $C(A) = \{x\}]$
(2) Converse Condorcet (CC) if $\forall A \in [X]$ and $x \in A$: $[C(x, y\}) = \{y\} \ \forall y \in A \setminus \{x\}$ implies $x \notin C(A)]$.
(3) Tie Splitting (TS) if $\forall A, B \in [X]$ with $A \cap B = \phi$: $[C(\{x, y\}) = \{x, y\} \ \forall (x, y) \in A \times B$ implies $C(A \cup B) = C(A) \cup C(B)]$;
(4) Expansion (E) if $\forall A, B \in [X]$: $C(A) \cap C(B) \subset C(A \cup B)$;
(5) Contraction (Con) if $\forall A \in [X]$ with $\#(A) \geq 4$ and $x \in C(A)$, there exists a positive integer K and $\{A_1, \ldots, A_K\} \subset ([A] \setminus \{A\})$ such that $\bigcup_{i=1}^{K} A_i = A$ and $x \in \bigcap_{i=1}^{K} C(A_i)$
(6) Strong Type 1 Property (ST1P) if $\forall x, y, z \in X$; $[\{y\} = C(\{x, y\}), \{x\} = C(\{x, z\}), z \in C(\{y, z\})]$ implies $C(\{x, y, z\}) = \{x, y, z\}$.

Proposition 1 :- Let $R \in \Pi$ and let C be a choice function on X such that $C(A) = UC(A, R) \ \forall A \in [X]$. Then C satisfies SC, CC, TS, ST1P, E and Con.

Proof :- The other properties being easy to verify let us show that C satisfies Con. Let $A \in [X]$ with $\#(A) \geq 4$ and $x \in C(A)$. Thus, $y \in A$, $y \neq x$ implies either $[(x, y) \in R]$ or [there exists $z_y \in A$ with either $((x, z_y) \in R$ and $(y, z_y) \notin R)$ or $((x, z_y) \in P(R)$ and $(y, z_y) \notin P(R))]$. Let $A_o = \{y \in A / (x, y) \in R\}$. Clearly $A_o \neq \phi$, since $x \in A_o$. Further, since there does not exist $y \in A_o$, such that y covers x via R in A_o, $x \in C(A_o)$.
Case 1:- $A_o = A$. Since $\#(A) \geq 4$, there exists $\bar{y} \in A \setminus \{x\}$ such that $A \setminus \{x, \bar{y}\} \neq \phi$. Let $A_1 = \{x, \bar{y}\}$ and $A_2 = A - \{\bar{y}\}$. Clearly $A_1 \subset\subset A$, $A_2 \subset\subset A$ and $A_1 \cup A_2 = A$. Further $x \in C(A_1) \cap C(A_2)$.

Case 2 :- $A_o \subset\subset A$. In this case, let $A_1 = A_o$ and for $y \in A \setminus A_1$, let $A_y = \{x, y, z_y\}$. Since # (A) ≥ 4, $A_y \subset\subset A$ whenever $y \in A \setminus A_1$. Further, $\forall$ $y \in A \setminus A_1$:$x \in C(A_y)$. Also, $A_1 \cup \left(\bigcup_{y \in A \setminus A_1} A_y \right) = A$. Hence C satisfies Con.

Q.E.D.

Lemma 2 :- If # (X) ≤ 3 and C is a choice function on X which satisfies SC, TS and ST1P, then C is an uncovered choice function.

Proof :- Let C and X be as in the statement of the lemma and let $R = \{(x, y) \in X \times X / x \in C(\{x, y\})\}$. If # (X) = 1, there is nothing to prove and if # (X) = 2, C(A) = UC(A, R) $\forall$ A $\in$ [X] by the definition of R. Hence suppose # (X) = 3. If A $\in$ [X] and # (A) ≤ 2, then C(A) = UC(A, R) by the way R is defined. Thus suppose $A = X = \{x, y, z\}$ with $x \neq y \neq z \neq x$. If there exists $a \in X : \{a\} = C(\{a, b\})$ $\forall$ $b \in X$, then C(X) = {a} = UC(X, R), by SC of both C and UC. Hence suppose that $\forall$ $a \in X$, there exists $b \in X \setminus \{a\} : b \in C(\{a, b\})$.

Case 1 :- C({a, b}) = {a, b} $\forall$ a, b $\in$ X. Then by TS of C and UC, C(X) = UC(X, R) = X.
Thus without loss of generality suppose, C({x, y}) = {x}. Hence, by what has been mentioned before Case 1, z $\in$ C({x, z}).

Case 2 :- {z} = C({x, z}), {y} = C({y, z}).
By ST1P, C(X) = {x, y, z} = UC(X, R).
Case 3 :- {z} = C({x, z}), {y, z} = C({y, z}).
By ST1P, C(X) = {x, y, z} = UC(X, R).
Case 4 :- {z, x} = C({x, z}) = {y} = C({y, z}).
By ST1P, C(X) = {x, y, z} = UC(X, R).
Case 5 :- {z, x} = C({x, z}), {y, z} = C({y, z}).
Thus, $\forall$ (a, b) $\in$ {z} x {x, y}:C({a, b}) = {a, b}.
By TS, C(X) = C({z}) $\cup$ C({x, y})
= {x, z} = UC(X, R).

This prooves Lemma 2.

Q.E.D.

A look at the proof of Lemma 2 reveals that we have essentially proved the following :

Lemma 3 :- Let C be a choice function on X which satisfies SC, TS and ST1P. Then $\forall$ A $\in$ [X] with # (A) ≤ 3, C(A) = UC(A, R(C)).

The above observation follows by noting that UC(A, R) depends on the restriction of R to A only.

Note :- If in Lemma 2 (: or for that matter in Lemma 3), we replace SC by CC and E we do not get the desired result as the following example reveals.

Example :- Let X = {x, y, z} with $x \neq y \neq z \neq x$. Let C(X) = {x, y}, C({x, y}) = {y}, C({y, z}) = {y}, C({x, z}) = {x}. C satisfies CC, E, TS and ST1P, the last two properties being satisfied vacuously. However, UC(X, R(C)) = {y} ≠ C(X). Note that C does not satisfy SC, since C({x, y}) = C({y, z}) = {y} and yet C(X) ≠ {x, y}.

In Dutta and Laslier [1999] we find the following property for a choice function C on X:
Type One Property (T1P) :- ∀ x, y, z ∈ X, {x} = C({x, y}), {y} =C({y, z}) and {x, z} = C({x, z}) implies C({x, y, z}) = {x, y, z}.
Clearly T1P is weaker than ST1P. However, if we replace ST1P by T1P in Lemma 2 (: or Lemma 3), we do not get the desired result as the following example reveals.
Example :- Let X = {x, y, z} with $x \neq y \neq z \neq x$. Let C(X) = {x}, C({x, y}) = {x}, C({y, z}) = {y}, C({x, z}) = {z}. Clearly C satisfies SC, TS, E, CC and T1P (: all vacuously). However, C violates ST1P which under the present situation would require C(X) = X. Further C(X) ≠ UC(X, R(C)) = X.
We are now equipped to prove the following theorem:

Theorem 1 :- A choice function C on X is an uncovered choice function if and only if C satisfies SC, TS, ST1P, E and Con.

Proof :- Proposition 1 tells us that an uncovered choice function satisfies all the properties mentioned in the theorem. Hence let C be a choice function on X satisfying SC, TS, ST1P and Con. By Lemma 3, C(A) = UC(A, R(C)) ∀ A ∈ [X] with # (A) ≤ 3. Suppose C(A) = UC(A, R(C)) ∀ A ∈ [X] with # (A) = 1,...,k, and let B ∈ [X] with #(B) = k+1. Let x ∈ C(B). Suppose k+1 ≥ 4, for otherwise there is nothing to prove. Hence by Con there exists a positive integer K and non-empty proper subsets $B_1, \ldots, B_k$ such that $B = \bigcup_{i=1}^{K} B_i$ and $x \in \bigcap_{i=1}^{K} C(B_i)$.
Clearly # $(B_i) \leq k$ whenever i ∈ {1, ..., k}.
By our induction hypothesis, $C(B_i) = UC(B_i, R(C))$ ∀ i ∈ {1, ..., k}. Thus $x \in \bigcap_{i=1}^{k} UC(B_i, R(C))$, and by E, x ∈ UC(B, R(C)). Thus, C(B) ⊂ UC(B, R(C)). By an exactly similar argument with the roles of C an UC interchanged, we get UC(B, R(C)) = C(B). By a standard induction argument, the theorem is established.

Q.E.D.

Note : The above theorem is not valid without E or Con.

Example :- Let X = {x, y,, z, w} where all of them are distinct. Let C(X) = {x}, C(A) = A if # (A) = 3, C({x, y}) = {x}, C({y, z}) = {y}, C({z, w}) = {z} and C}{w, x}) = {w}. C satisfies SC, ST1P, TS (vacuously). Further, let A_1= {x, y} and A_2 = {x, z, w}. $x \in C(X)$ and $x \in C(A_1) \cap C(A_2)$. Further $A_1 \cup A_2 = X$, with $A_1 \subset\subset X$ and $A_2 \subset\subset X$. Thus C satisfies Con. However, UC(X, R(C)) = X $\neq$ {x} = C(X). Observe that, C does not satisfy E ,since $y \in$ C({x, y, z}) $\cap$ C({y, z, x}) but $y \notin$ C(X).

Example :- Let X be as above. Let C(X) = {x, y}, C(A) = {x} if $x \in A$, C(A) = A if $x \notin A$. Clearly C satisfies SC, ST1P (:vacuously), TS and E. But C does not satisfy Con : $y \in C(X)$. If we take any finite number of non-empty proper subsets of X whose union is X, atleast one must contain 'x' and thus its choice set cannot contain 'y'.

4. Conclusion :- It is rather unfortunate that the word "tournament" has been reserved for binary relation which are reflexive, complete and anti-symmetric since most real life tournaments allow for the possibility of a "draw" (:without it actually taking place!) in addition to a win and a loss. The game of tennis is one where the possibility of a draw is ruled. Otherwise, the kind of binary relations we discuss appropriately characterize tournaments. Thus, we have provided in this paper an axiomatic characterization of choice functions which coincide with the uncovered sets of "tournaments".

References :-

1. B. Dutta and J.F. Laslier [1999] : "Comparison Functions And Choice Correspondences"' Social Choice Welfare, Vol. 16, No. 4, pp. 513-532.
2. S. Lahiri [Forthcoming] : "Path Independence And Choice Acyclicity Property"' forthcoming in Keio Economic Studies.
3. S. Lahiri [1999] : "Quasitransitive Rational Choice And Monotonic Preference For Freedom", mimeo.
4. H. Moulin [1985] : "Choice Functions Over A Finite Set : A Summary", Social Choice Welfare. Vol. 2 : 147-160.
5. P.K. Pattanaik and Y. Xu [1990] : "On Ranking Opportunity Sets In Terms Of Freedom Of Choice", Rech. Economic Louvain 56, 383-390.
6. A.K. Sen [1997]: "Maximization And The Act of Choice", Econometrica, Vol. 65, No. 4, pp. 745-780.