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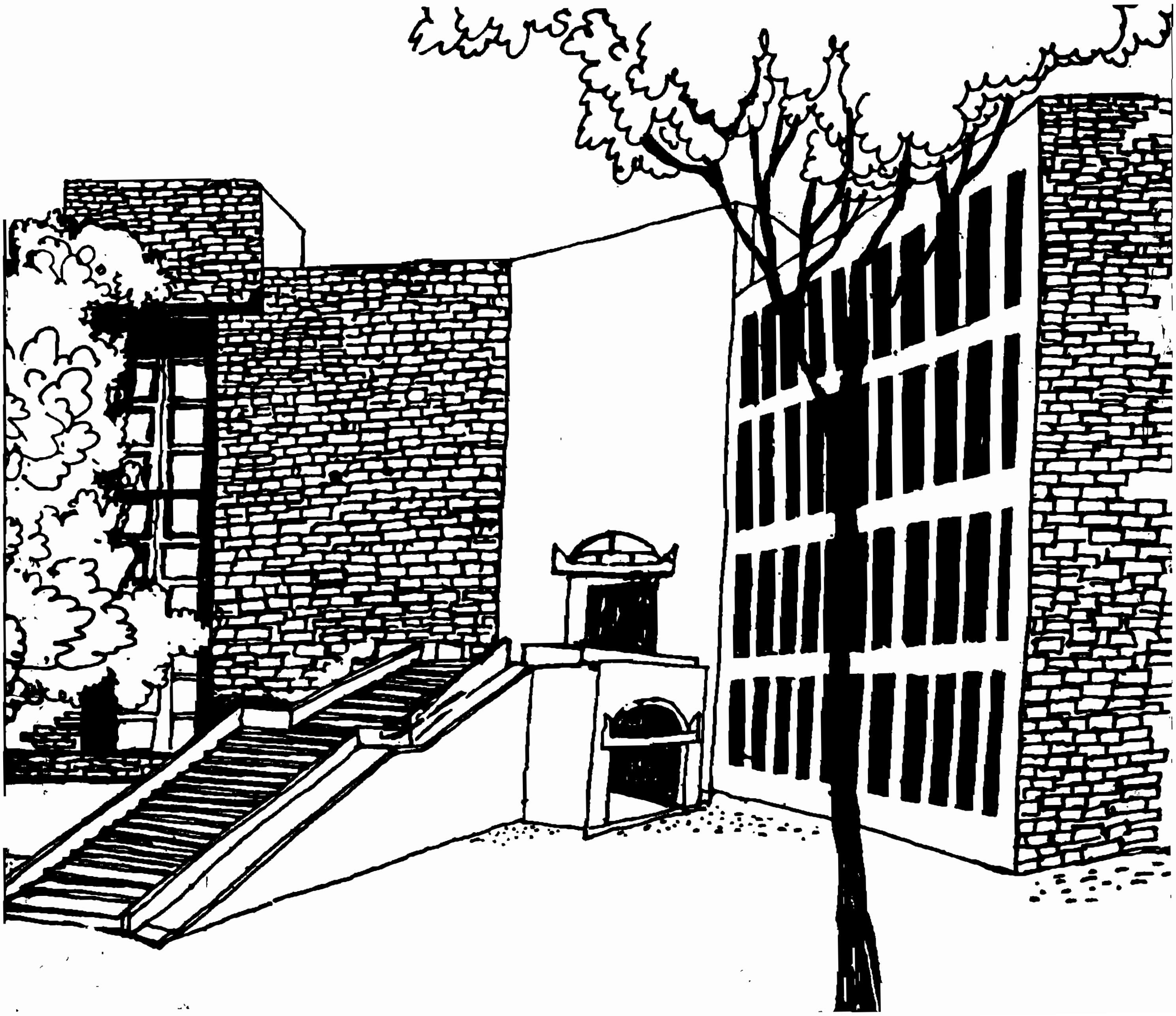

AXIOMATIC CHARACTERIZATIONS OF SOME SOLUTIONS FOR ABSTRACT GAMES

By

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W.P.No.2000-02-02
February 2000 /1580

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Axiomatic Characterizations of Some Solutions for Abstract Games

by

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1. Introduction : An abiding problem in choice theory has been one of characterizing those choice functions which are obtained as a result of some kind of optimization. Specifically, the endeavour has concentrated largely on finding a binary relation (if there be any) whose best elements coincide with observed choices. An adequate survey of this line of research till the mid eighties is available in Moulin [1985]. More recently, the emphasis has focused on binary relations defined on non-empty subsets of a given set, such that the choice function coincides with the best subset corresponding to a feasible set of alternatives. This problem has been provided with a solution in Lahiri [1999], although the idea of binary relations defined on subsets is a concept which owes its analytical origins to Pattanaik and Xu [1990].

The idea of a function which associates with each set and a binary relation a non-empty subset of the given set has a long history whose exact origin is very difficult to specify and in any case is unknown to the author.In Laslier [1997] can be found a very exhaustive survey of the related theory when binary relations are reflexive,complete and anti-symmetric.

In this paper we extend the above set of binary relations to include those which are not necessarily anti-symmetric.Such binary relations which are reflexive and complete are referred to in the literature as abstract games. An ordered pair comprising a non-empty subset of the universal set and an abstract game is referred to as a subgame.A (game)solution is a function which associates to all subgames of a given (nonempty) set of games,a nonempty subset of the set in the subgame.Lucas [1992] has a discussion of abstract games and related solution concepts, particularly in the context of cooperative games.Much of what is discussed in Laslier [1997] and references therein carry through into this framework.An important consequence of both the frameworks is that often,a set may fail to have an element which is best with respect to the given binary relation.To circumvent this problem the concept of the top cycle set is introduced,which selects from among the feasible alternatives only those which are best with respect to the transitive closure of the given relation.The top cycle set is always non-empty and in this paper we provide an axiomatic characterization of the top-cycle solution.It is subsequently observed that the top cycle solution is the coarsest solution which satisfies two innocuos assumptions.

In Dutta and Laslier [1999] one finds the device of a comparison function, which is basically a real valued function defined on all pairs of alternatives satisfying the condition that the value of an ordered pair is negative of the value of the ordered pair which is obtained by interchanging the order of the first ordered pair. Hence, in particular the value of the function along the diagonal (i.e. the set of ordered pairs with identical first and second components) is zero. A comparison function simultaneously captures the idea of preference and the intensity of preference. An alternative 'x' is preferred to another alternative 'y' if and only if the value of the comparison function at (x, y) is positive, and the value of the comparison function at (x, y) is meant to convey the intensity with which 'x' is preferred to 'y'. The device of a comparison function is a generalization of the concept of a binary relation.With the help of a comparison function they introduce the notion of 'cover': 'x' is said to cover 'y' if 'x' is preferred to 'y' (i.e. the value of the comparison function at (x, y) is positive) and for every other third element 'z' the value of the comparison function at the ordered pair (x, z) is alteast as much as the value of the comparison function at the ordered pair (y, z). Given any feasible sets, its uncovered set is the set of all elements in the feasible set which are not covered by any other element in the same set. The question that naturally arises is the following : Given a choice function , under what condition does a comparison function exist, whose uncovered sets always coincide with the choice function? This question has been discussed in Lahiri [1999], where it is observed that the binary relation 'is uncovered' is reflexive, complete and quasi-transitive and any reflexive, complete and quasi-transitive binary relation can be made to coincide with the "is uncovered" relation of some comparison function. The problem becomes much more difficult if instead of defining the covering relation globally, we considered the covering relation for each individual feasible set, by simply looking at the restriction of the comparison function to that set. In such a situation that fact that 'x' covers 'y' in a particular feasible set does not imply that 'x' covers 'y' globally. In effect, we are then concerned with what Sen [1997] calls 'menu based' relations.

In the final section of this paper we address the problem of axiomatically characterizing the uncovered solution (where 'covering' is now defined as a 'menu-based' concept), by considering only those comparison functions which can assume only three values : 1, 0 and -1. These comparison functions are nothing but abstract games.Our axiomatic characterization is different from the ones available in Dutta and Laslier [1999].

2. Game Solutions :- Let X be a finite, non-empty set and given any non empty subset A of X, let [A] denote the collection of all non-empty subsets of A.

Thus in particular, [X] denotes the set of all non-empty subsets of X. If A $\in$ [X], then # (A) denotes the number of elements in A.

A binary relation R on X is said to be (a) reflexive if $\forall$ x $\in$ X : (x, x) $\in$ R; (b) complete if $\forall$ x, y $\in$ X with x $\neq$ y, either (x, y) $\in$ R or (y, x) $\in$ R ;(c) transitive if $\forall$ x, y, z $\in$ X, [(x, y) $\in$ R & (y, z) $\in$ R implies (x, z) $\in$ R] ;(d) anti-symmetric if [$\forall$ x, y $\in$ X, (x, y) $\in$ R & (y, x) $\in$ R implies x = y]. Given a binary relation R on X and A $\in$ [X], let $R|A = R \cap (A \times A)$.
Let Π denote the set of all reflexive and complete binary relations. If R $\in \Pi$, then R is called an abstract game. An ordered pair (A,R) $\in [X]\times\Pi$ is called a subgame. Given a binary relation R, let P(R) = {(x, y) $\in$ R / (y, x) $\notin$ R} and I(R) = {(x, y) $\in$ R / (y, x) $\in$ R}. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R. Given a binary relation R on X and A $\in$ [X], let G(A,R) ={ x $\in$ A/ $\forall$y $\in$ A :(x,y) $\in$ R}. Given A $\in$ [X], let Δ (A) denote the diagonal of A i.e. Δ (A)$\equiv${(x,x)/x$\in$ A}.
The following example shows that given R $\in \Pi$ and A $\in$ [X], G(A,R) may be empty:
Example 1:Let X ={x,y,z} and let R =Δ (X)$\cup${(x,y),(y,z),(z,x)}.Clearly G(X,R) is empty.
Given R $\in \Pi$, A $\in$ [X] ,let T(R|A) be a binary relation on A defined as follows: (x, y) $\in$ T(R|A) if and only if there exists a positive integer K and $x_1,\ldots,x_K$ in A with (i) x_1 = x, x_K = y : (ii) $(x_i, x_{i+1}) \in R$ $\forall$ i $\in$ {1,...K-1}. T(R|A) is called the transitive hull of R in A.Clearly T(R|A) is always transitive.
Given R $\in \Pi$, A $\in$ [X], G(A,T(R|A)) is called the top cycle set of R in A.Clearly G(A,T(R|A)) is non-empty whenever R $\in \Pi$ and A $\in$ [X].
Let,Λ be a non-empty subset of Π.
A (game) solution on (X, Λ) is a function S: $[X]\times\Lambda \to [X]$ such that:
(i) $\forall$(A,R)$\in [X]\times\Lambda$:S(A,R) $\subset$ A ;
(ii) $\forall$(A,R), (A,Q)$\in [X]\times\Lambda$:R|A=Q|A implies S(A,R)=S(A,Q);
(iii) $\forall$x,y$\in$ X and R$\in \Lambda$:x$\in$S(A,R) if and only if (x,y) $\in$R.
If $\forall$(A,R)$\in [X]\times\Lambda$, G(A,R) is non-empty valued then the associated solution is called the best solution on (X, Λ).
Rational choice theory till the mid eighties has concentrated on the following problem:given a non-empty subset Λ of Π, specify a finite minimal set of (reasonable axioms) on solutions on (X, Λ) such that whenever a solution on (X, Λ) satisfies those properties it coincides with the best solution on (X, Λ).This theory has been adequately surveyed in Moulin [1985].In what follows we will assume that $\Lambda = \Pi$,and refer to solutions on (X, Π) merely as solutions on X.
The top cycle solution denoted TC: $[X]\times\Pi \to [X]$ is defined as follows:
$\forall$(A,R)$\in [X]\times\Pi$:TC(A,R) = G(A,T(R|A)).
Given R $\in \Pi$, A $\in$ [X] and x, y $\in$ X, we say that x covers y via R in A if :
(i) x, y $\in$ A; (ii) (x, y) $\in$ P(R); (iii) $\forall$ z $\in$ A: [(y, z) $\in$ R implies (x, z) $\in$ R] ; (iv) $\forall$ z $\in$ A: [(y, z) $\in$ P(R) implies (x, z) $\in$ P(R)].

Let $\hat{R}(A) = \{(x, y) \in A\times A / x$ covers y via R in A$\}$ and let $UC(A, R) = \{x \in A /$ if $y \in A$ then $(y, x) \notin \hat{R}(A)\}$. It is easy to see that $\forall A \in [X]$, $\hat{R}(A)$ is a transitive binary relation on A. Thus $UC(A,R) \neq \phi$ whenever $A \in [X]$. Thus (i)$\forall (A,R) \in [X]\times \Pi$: $UC(A, R) \subset A \ \forall A \in [X]$;(ii) $\forall(A,R), (A,Q)\in [X]\times\Pi$:$R|A=Q|A$ implies $UC(A,R)=UC(A,Q)$;(iii) $\forall x,y\in X$ and $R\in \Pi$:$x\in UC(A,R)$ if and only if $(x,y) \in R$.
The solution UC: $[X]\times\Pi\to[X]$ is called the uncovered choice function.
Given $R \in \Pi$, $A \in [X]$ and $x \in X$ let $s(x,A,R)=\#\{y\in A/(x,y) \in P(R|A)\}$.
The Copeland solution Co: $[X]\times\Pi\to[X]$ is defined as follows:
$\forall (A,R) \in[X]\times \Pi:Co(A,R)=\{x\in A/\forall y\in A: s(x,A,R) \geq s(y,A,R)\}$.

The following proposition is available in Laslier [1997]:
Proposition 1: $\forall (A,R) \in[X]\times \Pi:Co(A,R)\cup UC(A,R)\subset TC(A,R)$.
Example 2:Let $X =\{x,y,z\}$ and let $R =\Delta (X)\cup\{(x,y),(y,z),(z,y),(x,z),(z,x)\}$.Now $Co(X,R)=\{x\}\subset\subset \{x,z\} =UC(X,R) \subset\subset X =TC(X,R)$.Further $Co(X,R) \cup UC(X,R) \subset\subset TC(X,R)$.

3. Axioms for the Top Cycle Solution:A solution S on X is said to satisfy :
Strong Condorcet (SC) if $\forall (A,R) \in[X]\times \Pi$: $[x\in A]$ and $[\forall y\in A\setminus\{x\}:(x,y) \in P(R)]$ implies $[S(A,R) =\{x\}]$;
Expansion Independence (EI) if $\forall (A,R) \in[X]\times \Pi:[x\in S(A,R),y\in A,(y,z) \in R]$ implies $[x\in S(A\cup\{z\},R)]$;
Existence of an Inessential Alternative (EIA) if $\forall (A,R) \in[X]\times \Pi$ with $\#(A) \geq 2$ and $\forall x\in S(A,R)$, there exists $y\in A$ (possibly depending on A,R and x) such that $x\in S(A\setminus\{y\},R)$.

Theorem 1:The only solution on X which satisfies SC,EI and EIA is TC.

Proof:It is clear that TC satisfies SC,EI and EIA.Hence let S be any solution that satisfies SC,EI and EIA.Let $(A,R)\in[X]\times\Pi$.If $\#(A)$ is one or two there is nothing to prove since $S(A,R)=TC(A,R)$ by definition.Thus suppose $S(A,R)=TC(A,R)$ whenever $\#(A)=1,...k$.Let $\#(A)=k+1$.Let $x\in A$.If $\forall y\in A\setminus\{x\}:(x,y) \in P(R)]$ then $S(A,R) =\{x\}=TC(A,R)$.Hence suppose $\forall x\in A$ there exists $y \in A\setminus\{x\}$ such that $(y,x) \in R$.
Let $x \in TC(A,R)$.Since TC satisfies EIA,there exists $z \in A$ such that $x \in TC(A\setminus\{z\},R)$.By the induction hypothesis $S(A\setminus\{z\},R)= TC(A\setminus\{z\},R)$.If $(x,z) \in R$ then by EI , $x \in S(A,R)$.If $(x,z)\notin R$,then since $x \in TC(A,R)= G(A,T(R|A))$ there exists $w\in A$ such that $(x,w) \in T(R|A)$ and $(w,z) \in R$.Then by EI once again $x \in S(A,R)$. Hence $TC(A,R)\subset S(A,R)$.
Now suppose $x \in S(A,R)$ and towards a contradiction suppose $x \notin TC(A,R)$.By EIA there exists $z \in A$ such that $x \in S(A\setminus\{z\},R)$.By the induction hypothesis $S(A\setminus\{z\},R)= TC(A\setminus\{z\},R)$. If $(x,z) \in R$ then by EI applied to TC, $x \in TC(A,R)$. Hence suppose $(x,z)\notin R$.Thus $(z,x)\in P(R)$. Let $y \in TC(A,R)$. Clearly $y\neq x$.Suppose $y\neq z$.Thus $y\in A\setminus\{z\}$.Thus $(x,y)\in T(R|A)$ which combined with $y \in TC(A,R)$ gives us $x \in TC(A,R)$.Hence $y=z$.If for some $w \in A\setminus\{x,z\}$ we had

$(w,z)\in R$, then since $x \in TC(A\backslash\{z\},R)$ and $w \in A\backslash\{z\}$ we would get $x \in TC(A,R)$.Thus $\forall w\in A:(z,w)\in P(R)$.But then by SC,$S(A,R)=\{z\}$, contradicting $x \in S(A,R)$.Thus $x \in TC(A,R)$. Hence $S(A,R)\subset TC(A,R)$. Thus $S(A,R)=TC(A,R)$.
By a standard induction argument it now follows that $\forall A\in[X]$: $S(A,R)=TC(A,R)$.This being true whenever $R\in\Pi$ the theorem follows.

Q.E.D.

A solution S on X is said to satisfy:
Converse Condorcet (CC) if $\forall$ $(A,R)\in[X]\times\Pi$ and $x\in A$:$[\forall y\in A\backslash\{x\}:(y,x)\in P(R)]$ implies $[x\notin S(A,R)]$;
Weak Existence of an Inessential Alternative (WEIA) if $\forall$ $(A,R) \in[X]\times\Pi$ with $\#(A)\geq 4$ and $\forall$ $x\in S(A,R)$, there exists $y\in A$ (possibly depending on A,R and x) such that $x\in S(A\backslash\{y\},R)$.
Since TC satisfies EIA it also saisfies WIEA. Infact we can now prove the following:

Theorem 2:Let S be any solution on X which satisfies SC,CC and WEIA.Then, $\forall$ $(A,R) \in[X]\times\Pi$: $S(A,R) \subset TC(A,R)$.

Proof:
Step 1: Let S be any solution on X which satisfies SC and CC.Then, $\forall$ $(A,R) \in[X]\times\Pi$ with $\#(A)\leq 3$: $S(A,R) \subset TC(A,R)$.
Proof of Step 1:For $\#(A)\leq 2$ there is nothing to prove since by the definition of a solution all of them agree on such sets.Hence suppose $\#(A)=3$.Let $A=\{x,y,z\}$ with $x\neq y\neq z\neq x$.Suppose without loss of generality $x\in S(A,R)$.If $(x,y),(x,z) \in R$,then $x \in TC(A,R)$.Thus,suppose without loss of generality that $(y,x) \in P(R)$.If $(z,x) \in P(R)$ then by CC, $x\notin S(A,R)$,contradicting what we have assumed.Hence (x,z) must belong to R.If $(z,y) \in R$,then again $x \in TC(A,R)$.If $(y,z) \in P(R)$,then by SC,$S(A,R)=\{y\}$,contradicting $x\in S(A,R)$.Thus $S(A,R) \subset TC(A,R)$.
Step 2: Let S be any solution on X such that $\forall$ $(A,R) \in[X]\times\Pi$ with $\#(A)\leq 3$: $S(A,R) \subset TC(A,R)$.Suppose S satisfies WEIA.Then, $\forall$ $(A,R) \in[X]\times\Pi$: $S(A,R) \subset TC(A,R)$.
Proof of Step 2: Suppose that $\forall$ $(A,R) \in[X]\times\Pi$ with $3\leq\#(A)\leq m$: $S(A,R) \subset TC(A,R)$.Let $\#(A)=m+1$.Thus $\#(A)\geq 4$.Let $x\in S(A,R)$.By WEIA,there exists $y\in A$ such that $x\in S(A\backslash\{y\},R)$.By the induction hypothesis $S(A\backslash\{y\},R) \subset TC(A\backslash\{y\},R)$.Thus, $x\in TC(A\backslash\{y\},R)$.If $(x,y) \in R$, then clearly $x\in TC(A,R)$.Hence, $S(A,R) \subset TC(A,R)$.Suppose $(y,x) \in P(R)$.If $\forall z\in A\backslash\{y\}:(y,z)\in P(R)$,then by SC,$S(A,R)=\{y\}$,contradicting $x\in S(A,R)$.Hence ,there exists $z \in A\backslash\{x,y\}$ such $(z,y) \in R$.Since, $x\in TC(A\backslash\{y\},R)$ and $z \in A\backslash\{y\}$, $(z,y) \in R$ implies $x\in TC(A,R)$.Thus $S(A,R) \subset TC(A,R)$.
Step 2 combined with Step 1 and a standard induction argument proves the theorem.

Q.E.D.

Infact,the above proof reveals the following:
Theorem 3:Let S be any solution on X which satisfies SC and EIA.Then, $\forall$ $(A,R) \in [X] \times \Pi$: $S(A,R) \subset TC(A,R)$.
CC is not required once we replace WEIA by EIA,since then the induction argument can begin from $\#(A) \geq 2$.

The Uncovered Solution: A solution S on X is said to satisfy Expansion (E) if $\forall$ $(A,R),(B,R) \in [X] \times \Pi$: $S(A,R) \cap S(B,R) \subset S(A \cup B,R)$.

It is easy to see that both TC and UC satisfy E:
(i)Let $(A,R),(B,R) \in [X] \times \Pi$ and suppose $x \in UC(A,R) \cap UC(B,R)$.Towards a contradiction suppose that $x \notin UC\ (A \cup B,R)$.Hence there exists $y \in A \cup B$,such that y covers x via R in $A \cup B$.Without loss of generality suppose $y \in A$.Since $x \in A$, y covers x via R in A.This contradicts $x \in UC(A,R)$.Thus UC satisfies E.
(ii) Let $(A,R),(B,R) \in [X] \times \Pi$ and suppose $x \in TC(A,R) \cap TC(B,R)$.Towards a contradiction suppose that $x \notin TC\ (A \cup B,R)$.Hence there exists $y \in A \cup B$,such that $(x,y) \notin T(R|A \cup B)$. Without loss of generality suppose $y \in A$.Thus $(x,y) \notin T(R|A \cup B)$ implies that $(x,y) \notin T(R|A)$.This contradicts $x \in TC(A,R)$.

Moulin [1986] has established the following:

Proposition 2:Let S be any solution satisfying SC and E.Then $\forall$ $(A,R) \in [X] \times \Pi$: $UC(A,R) \subset S(A,R)$.

A solution S on X is said to satisfy Contraction (Con) if $\forall$ $(A,R) \in [X] \times \Pi$ with $\#(A) \geq 4$,$[x \in S(A,R)]$ implies [there exists a positive integer $K \geq 2$ and sets $A_1,\ldots,A_K \in [A] \setminus \{A\}$ such that (i)$\cup\{A_k/k=1,\ldots K\} = A$;(ii) $x \in \cap\{S(A_k,R)/k=1,\ldots K\}$].

Dutta and Laslier [1999] establish that UC satisfies Con.However,TC does not as the following example reveals:

Example 3:Let $X = \{x,y,z,w\}$ where x,y,z,w are all distinct. Let,$R = \Delta(X) \cup \{(x,y),(z,x),(w,x),(y,z),(w,y),(z,w)\}$.Clearly,$x \in TC(X,R)$.Let $A \in [X] \setminus \{X\}$, with $\#(A) \geq 2$. Suppose,$y \notin A$.Then,$x \notin TC(A,R)$.Hence, $x \in TC(A,R)$ and $\#(A) \geq 2$ implies $y \in A$.Suppose $x,y \in A \cap B$ where $A,B \in [X] \setminus \{X\}$, $A \neq B$, $A \not\subset B \not\subset A$.Without loss of generality suppose $A=\{x,y,z\}$ and $B=\{x,y,w\}$.Then, $x \notin TC(B,R)$.Thus,TC does not satisfy Con.
A solution S on X is said to satisfy:
Tie Splitting (TS) if $\forall$ (A,R), $(B,R) \in [X] \times \Pi$ with $A \cap B = \phi$: $[A \times B \subset I(R)$ implies $S(A \cup B,R) = S(A,R) \cup S(B,R)]$;
Strong Type 1 Property (ST1P) if $\forall$ $x, y, z \in X$; $[(y,x) \in P(R), (x,z) \in P(R), (z,x) \in R]$ implies $S(\{x, y, z\},R) = \{x, y, z\}$.

Proposition 3 :- Let $R \in \Pi$ and let S be a solution on X such that S(A,R) = UC(A, R) $\forall$ A $\in$ [X]. Then S satisfies SC, CC, TS, ST1P, E and Con.

Proof : We have already seen that UC satisfies E, and SC,CC,TS,ST1P being easy to verify let us show that S satisfies Con. Let $(A,R) \in [X] \times \Pi$ with # (A) $\geq$ 4 and $x \in S(A,R)$. Thus, $y \in A$, $y \neq x$ implies either $[(x, y) \in R]$ or [there exists $z_y \in A$ with either $((x, z_y) \in R$ and $(y, z_y) \notin R)$ or $((x, z_y) \in P(R)$ and $(y, z_y) \notin P(R))$]. Let $A_o = \{y \in A / (x, y) \in R\}$. Clearly $A_o \neq \phi$, since $x \in A_o$. Further, since there does not exist $y \in A_o$, such that y covers x via R in A_o, x $\in$
$S(A_o,R)$.
Case 1:- $A_o = A$. Since # (A) $\geq$ 4, there exists $\bar{y} \in A \setminus \{x\}$ such that $A \setminus \{x, \bar{y}\} \neq \phi$. Let $A_1 = \{x, \bar{y}\}$ and $A_2 = A - \{\bar{y}\}$. Clearly $A_1 \subset\subset A$, $A_2 \subset\subset A$ and $A_1 \cup A_2 = A$. Further $x \in S(A_1,R) \cap S(A_2,R)$.

Case 2 : $A_o \subset\subset A$. In this case, let $A_1 = A_o$ and for $y \in A \setminus A_1$, let $A_y = \{x, y, z_y\}$. Since # (A) $\geq$ 4, $A_y \subset\subset A$ whenever $y \in A \setminus A_1$. Further, $\forall\, y \in A \setminus A_1 : x \in S(A_y, R)$. Also, $A_1 \cup \left(\bigcup_{y \in A \setminus A_1} A_y \right) = A$. Hence S satisfies Con.

Q.E.D.

Lemma 1 : If # (X) $\leq$ 3 and C is a choice function on X which satisfies SC, TS and ST1P, then S is the uncovered solution.

Proof : Let S and X be as in the statement of the lemma and let $R \in \Pi$. If # (X) = 1 or 2, there is nothing to prove since S(A,R) = UC(A, R) $\forall$ A $\in$ [X] by the definition of a solution. Hence suppose # (X) = 3. If A $\in$ [X] and # (A) $\leq$ 2, then S(A,R) = UC(A, R). Thus suppose A = X = {x, y, z} with $x \neq y \neq z \neq x$. If there exists $a \in X : (a,b) \in P(R)\ \forall\, b \in X$, then S(X,R) = {a} = UC(X, R), by SC of both S and UC. Hence suppose that $\forall\, a \in X$, there exists $b \in X \setminus \{a\} : (b,a) \in R$.

Case 1 : I(R) = X. Then by TS of C and UC, S(X,R) = UC(X, R) = X.
Thus without loss of generality suppose, $(x, y) \in P(R)$. Hence, by what has been mentioned before Case 1, $(z,x) \in R$.

Case 2 : $(z,x), (y,z) \in P(R)$.
By ST1P, S(X,R) = {x, y, z} = UC(X, R).
Case 3 : $(z,x) \in P(R)$, $(y,z) \in I(R)$.
By ST1P, S(X,R) = {x, y, z} = UC(X, R).
Case 4 : $(z,x) \in I(R)$, $(y,z) \in P(R)$.
By ST1P, C(X) = {x, y, z} = UC(X, R).
Case 5 : $(z,x) \in I(R)$, $(y,z) \in I(R)$.
Thus, $\{z\} \times \{x, y\} \subset I(R)$.
By TS, S(X,R) = S({z},R) $\cup$ S({x, y},R)

$= \{x, z\} = UC(X, R)$.

This proves Lemma 1.

Q.E.D.

A look at the proof of Lemma 1 reveals that we have essentially proved the following :

Lemma 2 : Let S be a solution on X which satisfies SC, TS and ST1P. Then $\forall (A, R) \in [X] \times \Pi$ with $\# (A) \leq 3$, $S(A,R) = UC(A, R)$.

The above observation follows by noting that UC(A, R) depends on the restriction of R to A only.

Note : If in Lemma 1 (: or for that matter in Lemma 2), we replace SC by CC and E we do not get the desired result as the following example reveals :

Example 4: Let $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$. Let $S(X,R) = \{x, y\}$, where $R=\Delta(X) \cup \{(y,x),(y,z),(x,z)\}$, and let S=UC otherwise. S satisfies CC, E, TS and ST1P, the last two properties being satisfied vacuously. However, $UC(X, R) = \{y\} \neq S(X,R)$. Note that S does not satisfy SC, since $(y,x),(y,z) \in P(R)$ and yet $S(X,R) \neq \{y\}$.

In Dutta and Laslier [1999] we find the following property for a solution S on X:

Type One Property (T1P) : $\forall x, y, z \in X$; $[(y,x) \in P(R), (x,z) \in P(R), (z,x) \in I(R)]$ implies $S(\{x, y, z\},R) = \{x, y, z\}$..

Clearly T1P is weaker than ST1P. However, if we replace ST1P by T1P in Lemma 1 (: or Lemma 2), we do not get the desired result as the following example reveals.

Example 5: Let $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$. Let $S(X,R) = \{x\}$, where $R=\Delta(X) \cup \{(x,y),(y,z),(z,x)\}$, and let S=UC otherwise. Clearly S satisfies SC, TS, E, CC and T1P (: all vacuously). However, S violates ST1P which under the present situation would require $S(X,R) = X$. Further $S(X,R) \neq UC(X, R) = X$.

We are now equipped to prove the following theorem:

Theorem 4 : A solution S on X is the uncovered solution if and only if S satisfies SC, TS, ST1P, E and Con.

Proof : Proposition 3 tells us that an uncovered solution satisfies all the properties mentioned in the theorem. Hence let S be a solution on X satisfying SC, TS, ST1P and Con. Let $R \in \Pi$. By Lemma 2, $S(A,R) = UC(A, R)$ $\forall (A,R) \in [X] \times \Pi$ with $\# (A) \leq 3$. Suppose $S(A) = UC(A, R)$ $\forall A \in [X]$ with

(A) = 1,...,m, and let $B \in [X]$ with #(B) = m+1. Let $x \in S(B,R)$. Suppose $m+1 \geq 4$, for otherwise there is nothing to prove. Hence by Con there exists a positive integer K and non-empty proper subsets $B_1, \ldots, B_K$ such that $B = \bigcup_{i=1}^{K} B_i$ and $x \in \bigcap_{i=1}^{K} S(B_i, R)$. Clearly # $(B_i) \leq m$ whenever $i \in \{1, \ldots, K\}$.
By our induction hypothesis, $S(B_i, R) = UC(B_i, R)$ $\forall$ $i \in \{1, \ldots, K\}$. Thus $x \in \bigcap_{i=1}^{K} UC(B_i, R)$, and by E, $x \in UC(B, R)$. Thus, $S(B,R) \subset UC(B, R)$. By an exactly similar argument with the roles of S an UC interchanged, we get $UC(B, R) \subset S(B,R)$. By a standard induction argument, the theorem is established.

Q.E.D.

Note : The above theorem is not valid without E or Con.
Example 6: Let X = {x, y, z, w} where all of them are distinct. Let S(X,R) = {x}, S(A,R)=A if # (A) = 3, where $R=\Delta(X)\cup\{(x,y),(y,z),(z,w),(w,x),(x,z),(z,x),(y,w),(w,y)\}$,and let S=UC otherwise. S satisfies SC, ST1P, TS (vacuously). Further, let A_1= {x, y} and A_2 = {x, z, w}. $x \in S(X,R)$ and $x \in S(A_1, R) \cap S(A_2, R)$. Further $A_1 \cup A_2 = X$, with $A_1 \subset\subset X$ and $A_2 \subset\subset X$. Thus S satisfies Con. However, $UC(X, R) = X \neq \{x\} = S(X,R)$. Observe that, S does not satisfy E ,since $y \in S(\{x, y, z\},R) \cap S(\{y, z, w\},R)$ but $y \notin S(X,R)$.

Example 7: Let X be as above. Let S(X,R) = {x, y}, S(A,R) = {x} if $x \in A$, S(A,R) = A if $x \notin A$ where $R=\Delta(X)\cup(\{x\}\times X)\cup(\{y,z,w\}\times\{y,z,w\})$,and let S=UC otherwise. Clearly S satisfies SC, ST1P (:vacuously), TS and E. But S does not satisfy Con : $y \in S(X,R)$. If we take any finite number of non-empty proper subsets of X whose union is X, atleast one must contain 'x' and thus its choice set cannot contain 'y'.

Acknowledgement::I would like to thank Prof.Dipak Banerji for useful discussions relating to this paper.

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