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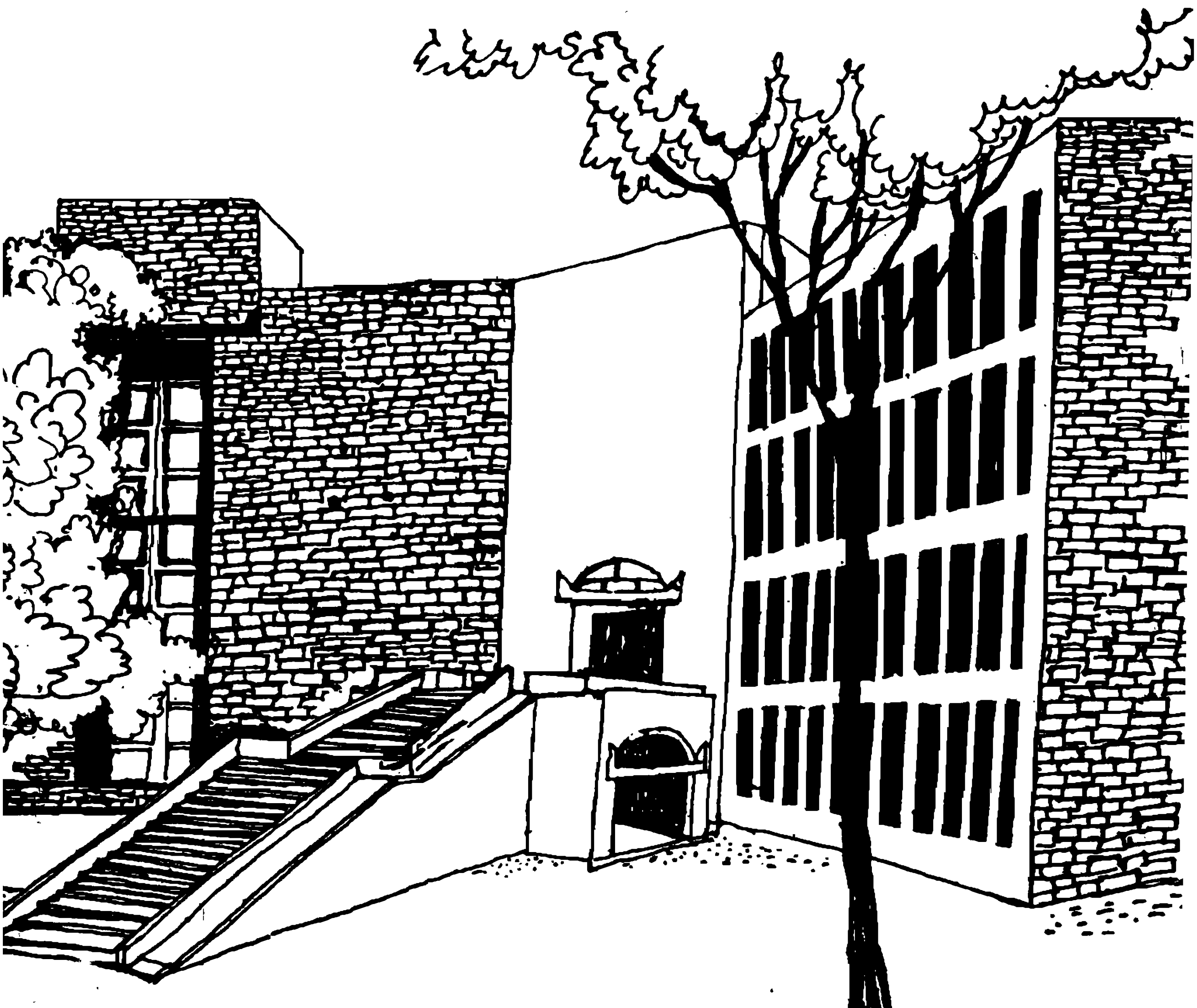

THRESHOLD AND MEDIAN RANK SOLUTIONS FOR TRANSITIVE ABSTRACT GAMES

By

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Threshold and Median Rank Solutions
for
Transitive Abstract Games

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Introduction

The idea of a function which associates with each set and a binary relation a non-empty subset of the given set has a long history whose exact origin is very difficult to specify and in any case is unknown to the author.In Laslier [1997] can be found a very exhaustive survey of the related theory when binary relations are reflexive,complete and anti-symmetric.

In a related paper (Lahiri [2000b])we extend the above set of binary relations to include those which are not necessarily anti-symmetric.Such binary relations which are reflexive and complete are referred to in the literature as abstract games. An ordered pair comprising a non-empty subset of the universal set and an abstract game is referred to as a subgame.A (game)solution is a function which associates to all subgames of a given (nonempty) set of games,a nonempty subset of the set in the subgame.Lucas [1992] has a discussion of abstract games and related solution concepts, particularly in the context of cooperative games.Moulin [1986],is really the rigorous starting point of the axiomatic analysis of game solutions defined on tournaments,i.e.anti-symmetric abstract games.Much of what is discussed in Laslier [1997] and references therein carry through into this framework.In Lahiri [2000 c],we obtain necessary and sufficient conditions that an abstract game needs to satisfy so that every subgame has atleast one von Neumann-Morgenstern stable set.

In this paper we consider solutions defined on the class of transitive games.A solution is said to be a threshold solution, if for every subgame there exists an alternative such that the solution set for the subgame coincides with the set of feasible alternatives which are no worse than the assigned alternative.Such solutions are closely related to the threshold choice functions of Aizerman and Aleskerov [1995].We provide an axiomatic characterization of such such solutions using three properties. The first property says that if one alternative is strictly superior to another,then given a choice between the two,the inferior alternative is never chosen.The second property is functional acyclicity due to Aizerman and Aleskerov [1995].The third property requires that if two feasible alternatives are indifferent to each other,then either they are both chosen or they are both rejected.In order to make the presentation self contained we also provide a simple proof of an extension theorem due to Suzumura [1983],which is used to prove the above mentioned axiomatic characterization.

charaterization of the median choice function.Neither of these two choice functions satisfy the axiom due to Nash which was used in characterizing the greatest and least choice functions.

Abstract games arise very naturally in the theory of elections as a consequence of majority voting.Laslier [1997] shows that every tournament corresponds to the preference of a majority where each individual preference is summarized by a transitive tournament.In a final section we show that every abstract game corresponds to the preference of a majority where each individual preference is summarized by a transitive tournament.

Game Solutions

Let X be a finite, non-empty set and given any non empty subset A of X, let [A] denote the collection of all non-empty subsets of A. Thus in particular, [X] denotes the set of all non-empty subsets of X. If $A \in [X]$, then # (A) denotes the number of elements in A.

A binary relation R on X is said to be (a) reflexive if $\forall x \in X : (x, x) \in R$; (b) complete if $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$;(c) transitive if $\forall x, y, z \in X$, $[(x, y) \in R$ & $(y, z) \in R$ implies $(x, z) \in R]$;(d) anti-symmetric if $[\forall x, y \in X, (x, y) \in R$ & $(y, x) \in R$ implies $x = y]$. Given a binary relation R on X and $A \in [X]$, let $R|A = R \cap (A \times A)$.

Let Π denote the set of all reflexive and complete binary relations. If $R \in \Pi$, then R is called an abstract game. An ordered pair $(A,R) \in [X] \times \Pi$ is called a subgame. Given a binary relation R, let $P(R) = \{(x, y) \in R / (y, x) \notin R\}$ and $I(R) = \{(x, y) \in R / (y, x) \in R\}$. P(R) is called the asymmetric part of R and I(R) is called the symmetric part of R.

Given a binary relation R on X define a binary relation T(R) on X as follows : $(x, y) \in T(R)$ if and only if there exists a positive integer K and $x_1,\ldots,x_K$ in X with (i) $x_1 = x$, $x_K = y$: (ii) $(x_i, x_{i+1}) \in R \ \forall i \in \{1,\ldots K-1\}$. T(R) is called the transitive hull of R .Clearly T(R) is always transitive. Further $T(I(R)) \subset I(T(R))$.

A binary relation R on X is said to be acyclic if T(P(R)) is asymmetric. It is said to be consistent if there does not exist any x in X such that $(x,x) \in T(R) \setminus T(I(R))$.

Given a binary relation R on X a binary relation Q on X is said to extend (be an extension of) R if $R \subset Q$ and $P(R) \subset P(Q)$.

Given $A \in [X]$, let $\Delta(A)$ denote the diagonal of A i.e. $\Delta(A)=\{(x,x)/x \in A\}$.

Let Π^0 be the set of all anti-symmetric abstract games.An element of Π^0 is called a tournament.Let Π^T denote the set of transitive abstract games.An element of $\Pi^0 \cap \Pi^T$ is by analogy called a transitive tournament.

The following theorem is due to Suzumura [1983]:

Suzumura's Extension Theorem : If R is a reflexive binary relation on X then it has an extension Q which is a transitive abstract game, if and only if R is consistent.

A Simple Proof of Suzumura's Extension Theorem

Given a binary relation R on X and given any non-empty subset S of X, let M(S,R) denote $\{x \in S / (y, x) \in P(R)$ implies $y \notin S\}$.

The following well known theorem,for which we provide a simple proof, is due to Szpilrajn [1930]:

Subsequently we focus our attention on solutions defined on transitive tournaments (i.e.transitive and anti-symmetric abstract games).Such solutions are essentially rank solutions i.e. solutions which depend on the ranks of the alternatives and not on any other physical characteristic.Consider the situation where one has to choose one among three differently priced birthday cakes,to give to a friend.It is very likely, that in the absence of strong personal reasons, one would select the cake whose price lies between the two extremes.A similar emphasis on the middle path is found in the teachings of Buddha as also in Confucian philosophy.That the choice of an alternative from a finite set of alternatives,need not result in choosing the alternative with the highest rank,is a possibility that has been discussed in Baigent and Gaertner (1996).In a sense this is a position on human behavior which is contrary to the received view of a decision maker as an optimizer of some objective function that is favored for instance by Sen (1993).That the median does not satisfy the requirements of underlying optimising behavior has been noted by Kolm (1994) and Gaertner and Xu (1999).However, the median is a reasonable compromise,in practical decision making. In Gaertner and Xu (1999) can be found a first axiomatic characterisation of the choice rule which selects the median from a finite set of alternatives.The axiomatic characterisation is valid for a universal set containing at least four alternatives as Example 1 in our paper points out.For universal sets containing three alternatives the above mentioned axiomatic characterisation is no longer valid.However,decision theory as opposed to decision algorithms,has overriding importance only when the set of alternatives is sufficiently small.For large sets the computational complexity of a solution may substantially offset its decision theoretic virtues.For a set containing a small number of alternatives we may ignore computational issues and concentrate only on decision theoretic properties.Hence, it is our view in this paper, that the real test of a theory of decision making takes place only when the universal set of alternatives is relatively small. In this paper we provide two theorems which characterizes the median choice function when the universal set has atleast three alternatives.Several examples are provided to highlight the relationship between the axioms emphasised in this paper.It is also noted here that our second axiomatic characterisation breaks down if the universal set contains precisely two elements.
Following our discussion of the median rank solution, we provide two more axiomatic characterizations.The first is a simultaneous axiomatic characterization of two solutions: one being that which always chooses the element with the highest rank from a set and the other being that which always selects the element with the lowest rank from a set.This is accomplished by using two axioms one of which is very well known in the literature on choice theory and is due to Nash (1950).The other is a property we invoke for the charaterization of the median rank solution.We provide this axiomatic characterization to emphasise the generality of the property which is common in the characterization of the median solution as well as in this theorem and to highlight the distinguishing features of those properties which are not common to both.The second is also a simultaneous axiomatic characterization of two solutions: one being that which always chooses the greatest element from the median choice set of a set and the other being that which always selects the least element from the median choice set of a set. This is accomplished by using three axioms two of which are invoked for the

Szpilrain's Extension Theorem: If R is a reflexive and transitive binary relation on X then it has an extension Q which is an abstract game.

Proof : Since R is transitive, it is clearly acyclic.Thus whenever A is a non-empty subset of X, M(A, R) is non-empty.

Let A_1 = M(X, R) and having defined A_n, let $A_{n+1} = M(X \setminus \bigcup_{i=1}^{n} A_i, R)$. Since

X is finite, there exists a positive integer r such that $A_r \neq \phi$ and $X = \bigcup_{i=1}^{r} A_i$. Further if $i \neq j$, then $A_i \cap A_j = \phi$. Define $f : X \to \Re$ (the set of real numbers) as follows : f(x) = r - i +1 if $x \in A_i$. Suppose $(x, y) \in P(R)$. Then $x \in A_i$, $y \in A_j$ implies by our method of construction that i < j. Thus f(x) > f(y). Now suppose $(x, y) \in R$ and towards a contradiction suppose that f(y) > f(x). Hence if $y \in A_j$ and $x \in A_i$, clearly j < i. Thus, $A_j = M(X \setminus \bigcup_{k=1}^{j-1} A_k, R)$, $X \setminus \bigcup_{k=1}^{j-1} A_j$ is finite and R is transitive implies that there exists $z \in A_j$ such that $(z, x) \in P(R)$ (: since $x \in (X \setminus \bigcup_{k=1}^{j-1} A_k) \setminus Aj$). By transitivity if R, $(z, y) \in P(R)$, contradicting $y \in A_j$. Thus, $f(x) \geq f(y)$.

Let $Q = \{(x,y) \in X \times X / f(x) \geq f(y)\}$. Thus,Q is an abstract game which extends R.

♥

Proof of Suzumura's Extension Theorem :If R has an extension Q which is an abstract game then $(x,x) \in T(R) \setminus T(I(R))$ implies $(x,x) \in T(Q) \setminus T(I(Q)) \subset P(Q)$, since Q is transitive.However this is not possible since P(Q) is the asymmetric part of Q.

Now suppose that R is consistent and reflexive.Since T(R) is reflexive and transitive T(R) has an extension Q which is an abstract game.Thus,$(x,y) \in R \subset T(R)$ implies $(x,y) \in Q$. Let $(x,y) \in P(R)$.Thus $(x,y) \in R \subset T(R)$. If $(y,x) \in T(R)$ then $(x,x) \in T(R) \setminus T(I(R))$,since $(x,y) \in P(R)$.This contradicts that R is consistent. Thus, $(x,y) \in P(R)$ implies $(x,y) \in P(T(R)) \subset P(Q)$.Thus the abstract game Q extends R.

♥

Lahiri [2000 a] provides a useful summary of related results.

Game Solutions

Let Λ be a non-empty subset of Π.

A (game) solution on Λ is a function S: $[X] \times \Lambda \to [X]$ such that:

(i) $\forall (A,R) \in [X] \times \Lambda : S(A,R) \subset A$;

(ii) $\forall (A,R), (A,Q) \in [X] \times \Lambda : R|A = Q|A$ implies S(A,R)=S(A,Q);

(iii) $\forall x,y \in X$ and $R \in \Lambda$: if $(x,y) \in R$ then $x \in S(\{x,y\},R)$.

In the sequel we will be concerned solely with solutions defined on Π^T and $\Pi^0 \cap \Pi^T$.

Let Λ be a non-empty subset of Π^T. A solution S on Λ is said to be a threshold solution if : (i) $\forall x,y \in X$ and $R \in \Lambda$: $x \in S(\{x,y\},R)$ if and only if $(x,y) \in R$;(ii) $\forall (A,R) \in [X] \times \Lambda$,there exists $V(A,R) \in A$: $S(A,R) = \{x \in A: (x,V(A,R)) \in R\}$.

Let Λ be a non-empty subset of Π^T.The best solution on Λ denoted B is defined as follows: $\forall (A,R) \in [X] \times \Lambda$: $B(A,R) = \{ x \in A: \forall y \in A, (x,y) \in R\}$.

<u>Axiomatic Characterization of Threshold Solutions</u>

Let Λ be a non-empty subset of Π. A solution S on Λ is said to satisfy:
(a) Binary Property (BP) if $\forall x,y \in X$ and $R \in \Lambda$: if $x \in S(\{x,y\},R)$, then $(x,y) \in R$;
(b) Functional Acyclicity (FA) if there does not exist a positive integer K sets $A_1,\ldots,A_K \in [X]$ and $R \in \Lambda$ such that : (i) $\forall$ i $\in \{1,\ldots,K-1\}$:$S(A_i,R) \cap (A_{i+1} \setminus S(A_{i+1},R)) \neq \phi$; and (ii) $S(A_K,R) \cap (A_1 \setminus S(A_1,R)) \neq \phi$;
(c) Indifference Property (IP) if $\forall (A,R) \in [X] \times \Lambda$ and $x,y \in A$: if $[(x,y) \in I(R)]$ then $[x \in S(A,R) \leftrightarrow y \in S(A,R)]$.

<u>Theorem 1</u>: A solution S on Π^T is a threshold solution if and only if S satisfies BP,FA and IP.
<u>Proof</u>: Let S be a threshold solution on Π^T.Thus, there exists a function V : $[X] \times \Pi^T \to X$ such that : (i) $\forall$ $(A,R) \in [X] \times \Pi^T$: $V(A,R) \in A$; (ii)$S(A,R) = \{ x \in A / (x, V(A,R)) \in R\}$.Towards a contradiction suppose that there exists a positive integer K, sets $A_1,\ldots,A_K \in [X]$ and $R \in \Pi^T$ such that : (i) $\forall$ i $\in \{1,\ldots,K-1\}$:$S(A_i,R) \cap (A_{i+1} \setminus S(A_{i+1},R)) \neq \phi$; and (ii) $S(A_K,R) \cap (A_1 \setminus S(A_1,R)) \neq \phi$. Let $x_t \in S(A_t,R) \cap (A_{t+1} \setminus S(A_{t+1},R))$, for t = 1, ..., K-1 and let $x_K \in S(A_K,R) \cap (A_1 \setminus S(A_1,R)) \neq \phi$. Thus $(x_t, V(A_t,R)) \in R$, for t = 1, ..., K, $(V(A_{t+1},R), x_t) \in P(R)$ for t = 1, ..., K-1, and $(V(A_1,R), x_K) \in P(R)$.Since R is transitive we get $(V(A_1,R), V(A_1,R)) \in P(R)$,contradicting the reflexivity of R. This contradiction implies that S must satisfy FA.The fact that S satisfies BP and IP are self evident.
Now suppose that S satisfies FA and IP. Let $R \in \Pi^T$ and let C(A) denote S(A,R) $\forall$ A $\in$ R. Let $R_C^* = \{C(S) x (S \setminus C(S) / S \in [X]\}$ and let $Q = \Delta_X \cup R_C^*$. Q is reflexive and by Functional Acyclicity Q is consistent. Further, $P(Q) = R_C^*$ .Thus by Suzumura's Extension Theorem there exists an abstract game Q' which extends Q. Given $A \in [X]$, let $V(A,R) \in \{x \in C(A) / \forall y \in C(A):(y,x) \in Q'\}$. Clearly,if $x \in C(A)$ then $(x,V(A,R)) \in Q'$. Now, suppose $x \in A$ and $(x,V(S)) \in Q$ and towards a contradiction suppose $x \notin C(A)$. Thus, $(V(S), x) \in R_C^*$. Thus by the above $(V(A,R), x) \in P(Q')$which contradicts$(x,V(A,R)) \in Q'$. Thus $x \in A$, $(x,V(A,R)) \in Q'$ implies $x \in C(A)$.Hence, $S(A,R) = C(A) = \{x \in A: (x,V(A,R)) \in Q'\}$.
Now suppose $A \in [X]$ and let $x \in A$. Let, $(x,V(A,R)) \in P(R)$. By BP, $S(\{x,V(A,R)\} = \{x\}$. Thus, $(x,V(A,R)) \in P(Q')$.If $(x,V(A,R)) \in I(R)$ then by IP, $x \in S(A,R)$.Hence, $(x,V(A,R)) \in Q'$. Thus, $\{x \in A: (x,V(A,R)) \in R\} \subset \{x \in A: (x,V(A,R)) \in Q'\} = S(A,R)$.
Now suppose $A \in [X]$ and let $x \in A$ with $(x,V(A,R)) \in Q'$.Towards a contradiction suppose $(V(A,R),x) \in P(R)$. By BP, $S(\{x,V(A,R)\} = \{V(A,R)\}$.This contradicts $(x,V(A,R)) \in Q'$. Thus, $S(A,R) = \{x \in A: (x,V(A,R)) \in Q'\} \subset \{x \in A: (x,V(A,R)) \in R\}$. Thus, $S(A,R) = \{x \in A: (x,V(A,R)) \in R\}$.
♥
A close study of the proof of Theorem 1,reveals that we have essentially proved the following:

<u>Theorem 2</u>: A solution S on $\Pi^T \cap \Pi^0$ is a threshold solution if and only if S satisfiesBP and FA .
<u>Proof</u>: On $\Pi^T \cap \Pi^0$, IP follows from the reflexivity of a tournament.
♥

The three assumptions BP, FA and IP that have been invoked in Theorem 1,are logically independent:

Theorem 3: (a) There exists a solution S on Π^T which satisfies BP and FA but does not satisfy IP.Further,this solution is not a threshold solution.
(b) There exists a solution S on Π^T which satisfies BP and IP but does not satisfy FA.Further,this solution is not a threshold solution.
(c) There exists a solution S on Π^T which satisfies FA and IP but does not satisfy BP.Further,this solution is not a threshold solution.
Proof: Let X ={x,y,z} with $x \neq y \neq z \neq x$.
(a)Let, Q = $\{(x,y),(y,x),(y,z),(x,z)\} \cup \Delta(X)$. Let S(X,Q) = {x} and let S(A,R) = B(A,R) for $(A,R) \in ([X] \times \Pi^T) \setminus \{(X,Q)\}$. Clearly S satisfiesBP and FA but does not satisfy IP: $x \in S(X,Q)$, $y \in X$ and $(x,y) \in I(Q)$;yet $y \notin S(X,Q)$.Further S is not a threshold solution.
(b) Let, Q = $\{(x,y),(y,z),(x,z)\} \cup \Delta(X)$. Let S(A,R) = B(A,R) for $(A,R) \in ([X] \times \Pi^T) \setminus \{(X,Q)\}$ and let S(X,Q) = {z}.Clearly S satisfies BP and IP but not FA: $x \in (X \setminus S(X,R)) \cap S(\{x,z\},Q)$ and $z \in (\{x,z\} \setminus S(\{x,z\},R)) \cap S(X,Q)$ contradicting FA. Further S is not a threshold solution.
(c) Let, Q = $\{(x,y),(y,z),(x,z)\} \cup \Delta(X)$. Let S(A,Q) = A $\forall A \in [X] \setminus \{X\}$,S(X,Q) ={z} and let S(A,R) = B(A,R) for $(A,R) \in ([X] \times \Pi^T) \setminus \{(A,Q)/ A \in [X]\}$. Clearly S satisfiesFA and IP but does not satisfy BP: $(x,y) \in P(Q)$ and yet $y \in S(\{x,y\},Q)$.Further S is not a threshold solution.
♥

Rank Solutions:

Let N denote the set of positive integers and let X ={i ∈N/ i ≤ n} (:the set of first n positive integers) for some n∈N with n ≥ 3.
A rank solution is a function S :$[X] \times (\Pi^T \cap \Pi^0) \rightarrow [X]$ such that (i) $\forall (A,R) \in [X] \times \Lambda$:S(A,R) ⊂ A ;(ii) $\forall R,Q \in (\Pi^T \cap \Pi^0)$ if Q ={(f(x),f(y))/(x,y)∈R}where f:X→X is a bijection, then ∀A ∈ [X] : S(B,Q) = {f(x)/x∈S(A,R)}, whenever B ={f(x)/x∈A}.
Hence in the study of rank solutions it is enough to focus our attention on $R^* = \{(i,j) \in N \times N / i \leq j\}$,since given any transitive tournament Q we can always find a bijection f:X→X such that Q ={(f(x),f(y))/(x,y)∈R^*}.
Let S be a rank solution.The choice function on X corresponding to S is a function C:[X]→[X]such that C(A)= S(A, R^*) ∀A∈[X].In the sequel when we talk about a choice function we will have precisely this interpretation in mind.
The median choice function M:[X]→[X] is defined as follows: ∀A∈[X],
M(A)={k} if (a) #{i ∈A/i<k}=#{i ∈A/i>k},and (b) #(A) is an odd number;
=$\{j,k\}$ if (a)j<k,(b) #{i ∈A/i<j}=#{i ∈A/i>k},(c) #(A) is an even number,and (d) A= {i ∈A/i<j}∪ {i ∈A/i>k}∪ {j,k}.

Let G: [X]→[X] be defined by G(A) = {i∈A/i ≥ j,∀j∈A} and L: [X]→[X] be defined by L(A) = {i∈A/j ≥ i,∀j∈A} whenever A ∈[X].G and L are known as the greatest and least choice functions respectively.Clearly G and L are single valued choice functions.Let G(A) = {g(A)} and L(A) = {h(A)} whenever A ∈[X]. Let GM: [X]→[X] be defined by G(M(A)) and LM: [X]→[X] be defined by L(M(A)) whenever A ∈[X].GM and LM are known as the greatest and lowest median choice functions respectively.Thus G(M(A))= {g(M(A))} and L(A) = {h(M(A))} ∀ A ∈[X].

The following axioms are due to Gaertner and Xu (1999):

Axiom 1: ∀i,j∈X,C ({i,j})={i,j}.

Axiom 2: ∀i,j,k∈X,with i≠j≠k≠i, C ({i,j,k})≠{i,j,k}.

Axiom 3: ∀A∈[X] and i ∈X\A,if B={i}∪(A\C(A)) then C(A∪B)= C(C(A)∪C(B)).

Axiom 4: ∀A∈[X],C(A)={i,j} with i ≠j,C(A\{i})={j}.

Axiom 5: If i,j,k,m ∈X,where all of them are distinct,then C({i,j,k,m})={i,j} implies there exists a ∈{k,m} such that i∈C({i,j,a}).

Example 1:Let n=3.Let C({1,2,3})={1} and C(A)=A,otherwise.Clearly C satisfies all the five axioms given above.

Example 2: Let C(A)=G(A), whenever A∈[X].Then C satisfies all the above axioms except for Axiom 1.

Example 3:Let C(A)=A ∀A∈[X].Then C satisfies all the above axioms except for Axiom 2.

Example 4:Let C(A)=A if A∈[X] and #(A)=2, and let C(A)=G(A), otherwise.Then C satisfies all the axioms above,if n=3 and all except Axiom 4,if n ≥ 4.For, let A={1,2,3} and i = 4.Then {1,2,4}={4}∪ (A\C(A)) since C(A)={3}.Let B={1,2,4}.Thus,A∪B={1,2,3,4} and C(A∪B)={4}. However C(C(A) ∪C(B))={3,4}≠{4}=C(A∪B),contradicting Axiom 3.

Example 5: Let C(A)=G(A)∪L(A), ∀A∈[X]. Then C satisfies all the above axioms except for Axiom 4.For let A={1,2,3}. Then, C(A)={1,3}.However,C({1,2})={1,2} and C({2,3})={2,3}, contradicting Axiom 4.

Example 6: Let n = 4.Let C(A)=A if #(A) is an even number,and let C(A)=G(A), otherwise.C satisfies all the above axioms above except for Axiom 5.For,1,2 ∈ C({1,2,3,4}), but 1∉C({1,2,3}) and 1∉C({1,2,4}),contradicting Axiom 5.

The following axiom is implied by Axiom 1:

Binary Injective Invariance (BII): ∀ i,j ∈ X with i ≠ j, if f : {i,j }→ X is one to one and order preserving (:in the sense that f (i) > f (j) if and only if i>j),then C({f (i),f (j)})= {f (k)/ k ∈ C({i,j})}.

That Axiom 1 implies BII is an easy observation.

The following axiom is crucial for what follows:

Invriance with respect to Best and Worst outcomes (IBW): ∀A∈[X] and for all i,j ∈ X [i <h(A)] &[j>g(A)] implies C(A ∪ {i,j})=C(A).

Observe that the choice function in example 1 does not satisfy IBW.

Example 7: Let C(A)=GM(A).Clearly C ≠ M, since C({1,2}) = {2}≠ {1,2}=M(A) and yet C satisfies BII and IBW.However C does not satisfy Axiom 1.

Example 8: Let C(A)= GM(A) if 1 ∈M(A) and C(A) =LM(A),otherwise. C({1,2})={2}=C({2,3}).Let f : {1,2}→ X be defined by f (i) = i+1 for i ∈ {1,2}. f is order preseving. However, C({f (1),f (2)}) ≠{f (k)/ k ∈ C({1,2})} contradicting BII.However, C satisfies IBW.

The choice function defined in Example 2 above satisfies BII.However it does not satisfy IBW.

Proposition 1:Let C be a choice function satisfying IBW.Then:

(i) C(A) = C(M(A)) ∀A∈[X];

(ii) C(A) = M(A) ∀A∈[X] with #(A) being an odd number.

Proof:Given A∈[X], either A=M(A) or, there exists k ∈N and {i_j ∈X\M(A)/ j ∈{1,...,2k}} such that (a) i_j > i_{j-1} , ∀j ∈ {2,...,2k}; (b) i_j < a < i_{j+k} ,∀j ∈ {1,...,k}; (c) A= M(A) ∪ {i_j ∈X\M(A)/ j ∈{1,...,2k}}.

If A=M(A), then C(A)=C(M(A)).Otherwise, by IBW, C(M(A)) = C(M(A) ∪ { i_k , i_{k+1}}).By IBW, C(M(A) ∪ { i_{k-j} ,...,i_{k+j+1} })= C(M(A) ∪ { i_{k-j-1},...,i_{k+j+2} }). Thus C(M(A)) = C(A).

If #(A) is an odd number,then M(A) is a singleton,whence C(M(A)) =M(A). This proves the proposition.

♥

Theorem 4 : The only choice function on X which satisfies Axiom 1 and IBW is M.

Proof: M clearly satisfies Axiom 1 and IBW. Hence let C be any choice function on X which satisfies Axiom 1 and IBW.By Proposition 1,C(A) = C(M(A)) ∀A∈[X],and in particular C(A) = M(A) whenever #(A) is an odd number.However if #(A) is an even number then M(A) is a set consisting two distinct elements,whence by Axiom 1, C(M(A))=M(A).Hence, C(A)=M(A) ∀A∈[X].

♥

A property we invoke now is the following:

Partial Fidelity (PF): ∀A∈[X] with #(A) ≥ 2 and ∀a∈X\A, if [either (a < h(A)), or (a > g(A))], then C(A∪ {a}) ∩ C(A)≠ϕ (: the empty set).

Proposition 2:Let C be a choice function satisfying IBW and PF and let A ⊂ X with M(A) ⊂ X\ {1,n}.Then, C(A) = M(A).

Proof: By Proposition 1,C(A) = C(M(A)) ∀A∈ [X],and in particular C(A) = M(A) whenever #(A) is an odd number.Since #(A) is an odd number if and only if M(A) is a singleton, we need only consider the case where #(M(A)) =2.Thus let M(A)= {i,j} with i < j.

Case 1: $C(M(A)) = \{i\}$. Clearly $j < n$ and $M(A) \cup \{n\} = \{i,j,n\}$.By Proposition 1, $C(M(A) \cup \{n\}) = \{j\}$.This contradicts PF,since then $C(M(A) \cup \{n\}) \cap C(M(A)) = \phi$.
Case 2: $C(M(A)) = \{j\}$. Clearly $1 < i$ and $M(A) \cup \{1\} = \{1,i,j\}$.By Proposition 1, $C(M(A) \cup \{1\}) = \{i\}$.This contradicts PF,since then $C(M(A) \cup \{1\}) \cap C(M(A)) = \phi$.
Thus since $C(M(A)) \neq \phi$, we must have $C(M(A)) = M(A)$.
♥

Note : The choice function in Example 8,satisfies PF as well.Thus it satisfies IBW and PF but not BII. The choice function in Example 7,satisfies BII and IBW but not PF.

Example 9:Let $C(A)=A$ $\forall A \in [X]$.Then C satisfies BII and PF but not IBW.

Theorem 5: The only choice function on X which satisfies BII,IBW and PF is M.

Proof: M clearly satisfies BII ,IBW and PF. Hence let C be any choice function on X which satisfies BII ,IBW and BF.By Proposition 1,$C(A) = C(M(A))$ $\forall A \in [X]$,and in particular $C(A) = M(A)$ whenever $\#(A)$ is an odd number.However $\#(A)$ is an even number if and only if M(A) is a set consisting two distinct elements.By Propsition 2,if $M(A) \subset X \setminus \{1,n\}$,then $C(A) = M(A)$.Hence let us assume that $\#(M(A)) =2$ and $M(A) \cap \{1,n\} \neq \phi$.
Let us first show that for all $i \in X$, with $1 < i < n$, $i \in C(\{1,i\}) \cap C(\{i,n\})$.
Towards a contradiction suppose $i \notin C(\{1,i\})$.Thus $C(\{1,i\}) = \{1\}$.However $C(\{1,i.n\}) = \{i\}$, and this contradicts PF,since we get $C(\{1,i\}) \cap C(\{1,i.n\}) = \phi$.Thus suppose $i \notin C(\{i,n\})$.Thus $C(\{i,n\}) = \{n\}$.However $C(\{1,i.n\}) = \{i\}$, and this contradicts PF,since we get $C(\{i,n\}) \cap C(\{1,i,n\}) = \phi$.Hence $i \in C(\{1,i\}) \cap C(\{i,n\})$.
Let $f : \{1,i\} \rightarrow X$ be defined by $f(1) = i$ and $f(i) = n$. f is order preserving.Since $i \in C(\{1,i\})$, by BII,$n \in C(\{i,n\})$.Hence, $C(\{i,n\}) = \{i,n\}$. Now let $g : \{i,n\} \rightarrow X$ be defined by $g(i) = 1$ and $g(n) = i$. g is order preserving.Since $i \in C(\{i,n\})$, by BII,$1 \in C(\{1,i\})$.Hence, $C(\{1,i\}) = \{1,i\}$. Now let $h : \{2,n\} \rightarrow X$ be defined by $h(2) = 1$ and $h(n) = n$. h is order preserving.Since $C(\{2,n\}) = \{2,n\}$, by BII, $C(\{1,n\}) = \{1,n\}$.
Thus $C(M(A)) = M(A)$ $\forall A \in [X]$.This in conjunction with Proposition 1, proves the theorem.
♥
Example 10:Let $C(A)=A$ if $\#(A) \neq 2$ and let $C(\{i,j\})= \{i\}$ if $A = \{i,j\}$ with $i < j$.Then C satisfies PF and BII.However C does not satisfy Axiom 1.
Example 11:Suppose $n \geq 4$. Let $C(A)=A$ if $\#(A)= 1$ or 2 and let $C(A)= GM(A)$ if $\#(A) \geq 3$.Then C satisfies Axiom 1 but not PF: let $A = \{1,2,3\}$ and let $a = 4$.Then $C(A)=\{2\}$ and $C(A \cup \{4\})=\{3\}$ violating PF.

Remark 1: If we had not insisted on $\#(A) \geq 2$ in the definition of PF, then the modified axiom would imply Axiom 1.This is because for $i<j$, $C(\{i,j\}) = \{j\}$ and $C(\{i\}) = \{i\}$ would violate the non-empty intersection requirement in the definition of PF as would $C(\{i,j\}) = \{i\}$ and $C(\{j\}) = \{j\}$.

Remark 2: The assumption that $n \geq 3$ is crucial for Theorem 5.If $X = \{1,2\}$, then $C(\{i\})=\{i\}$ for all $i \in \{1,2\}$ and $C(X)=\{1\}$ satisfies all the properties mentioned in Theorem 5.However, $C \neq M$.

It is worth noting that both G and L satisfy the following property due to Nash (1950):

Nash's Independence of Irrelevant Alternatives (NIIA) : (a) $\forall$ $A \in [X]$, $\# C(A) = 1$; (b) $\forall$ $A, B \in [X]$, with $A \subset B$, [$C(B) \subset C(A)$ implies $C(B) = C(A)$].

Theorem 6: The only two choice function on X which satisfy BII and NIIA are G and L.

Proof: Let $A = \{1,2\}$ and let $B = \{i,j\}$ with $i<j$.Clearly, the function $f: A \to X$, where $f(1) = i$ and $f(2) = j$, is order preserving.Thus by BII, $C(A)=G(A)$ implies $C(B) = G(B)$ $\forall$ $B \in [X]$ with $\# B = 2$ and $C(A)=L(A)$ implies $C(B) = L(B)$ $\forall$ $B \in [X]$ with $\# B = 2$.
Without loss of generality suppose, $C(B) = G(B)$ $\forall$ $B \in [X]$ with $\# B = 2$. Let $D \in [X]$ and towards a contradiction suppose $C(D) \neq G(D)$.By (a) of NIIA,
$\# C(D) = 1$. Let $C(D) = \{i\}$ and let $G(D) = \{j\}$.Clearly, $j>i$.However, $\# \{i,j\}=2$ implies that $C(\{i,j\}) = G(\{i,j\}) = \{j\}$.Since, $\{i,j\} \subset D$ and $C(D) \subset \{i,j\}$, (b) of NIIA implies $\{i\} = C(D) = C(\{i,j\}) = G(\{i,j\}) = \{j\}$ which is not possible since $i \neq j$.Thus, $C(B) = G(B)$ $\forall$ $B \in [X]$ with $\# B = 2$ implies $C(B) = G(B)$ $\forall$ $B \in [X]$.Similarly, $C(B) = L(B)$ $\forall$ $B \in [X]$ with $\# B = 2$ implies $C(B) = L(B)$ $\forall$ $B \in [X]$.
♥
Remark: It is by now a standard result in choice theory that the satisfaction of NIIA by a choice function C is equivalent to the existence of a function $u: X \to \Re$ (the set of real numbers) such that $\forall$ $A \in [X]$:$C(A) = \{x \in A / \forall y \in A: u(x) \geq u(y)\}$(see Aizerman and Aleskerov (1995) Theorem 2.10,for instance).However NIIA does not imply that the choice function satisfies BII. Thus there are choice functions which satisfy NIIA and yet do not coincide with either G or L.The following example illustrates this fact:
Example 12: Define a function $u: X \to \Re$ as follows:
$u(k) = 2k-n$,if $k>g(M(X))$;
$= n-(2k-1)$, if $k< h(M(X))$;
$= 0$, if $k= g(M(X))$
$= -1$, if $k= h(M(X))<g(M(X))$.
Let $C(A) = \{x \in A / \forall y \in A: u(x) \geq u(y) \}$.Clearly C satisfies NIIA.However $C(\{1,n\})= \{n\}$ and $C(\{1,n-1\})=\{1\}$ contradicting BII.

Example 13: Let $C(A) = LM(A)$ $\forall$ $A \in [X]$. Clearly, C satisfies BII but not NIIA: $C(\{1,2,3\})=\{2\} \subset \{1,2\}$ and yet $C(\{1,2\})=\{1\}$.

Observe that neither GM nor LM satisfy NIIA.However both satisfy the following property:
Single Value (SV): $\forall$ $A \in [X]$, $\# C(A) = 1$.
SV is simply the first part of NIIA.The following provides a dual axiomatic characterization of GM and LM:

Theorem 7: The only two choice function on X which satisfy BII,IBW and SV are GM and LM.

Proof: GM and LM clearly satisfy BII ,IBW and SV. Hence let C be any choice function on X which satisfies BII ,IBW and SV.By Proposition 1,C(A) = C(M(A)) ∀A∈[X],and in particular C(A) = M(A) whenever #(A) is an odd number.By IBW, if C({1,2})={1}, then C(A)=LM(A) ∀A∈[X] and if C({1,2})={1}, then C(A)=LM(A) ∀A∈[X].This proves the theorem.

♥

It is worth noting that the results reported in this section would remain valid if we replaced BII by the following stronger axiom:

Injective Invariance (II): ∀ A ∈ [X] if f : A → X is one to one and order preserving (:in the sense that f (i) > f (j) if and only if i>j),then C(f (A))=
f (C(A)).

Although II implies BII the converse is not true.

Example 14: Let n=4 and let C(A) =G(A) if 1∈A or # A =2;C(A)= L(A) otherwise. Clearly, C satisfies BII but not II: C({1,2,3})={3} but C({2,3,4}) ={2} even though f:{1,2,3}→X defined by f(i) =i+1 is one to one and order preserving and f({1,2,3})={2,3,4}.

Abstract Games and Majority Voting

Let I be a non-empty finite set.A preference profile on I is a function $F: I \to \Pi^T \cap \Pi^0$

Given a preference profile F on I, let M(F) = {(x,y)/#{i∈I / (x,y)∈F(i)} ≥ #{i∈I / (y,x)∈F(i)}}. M(F) is called the majority rule generated by F. It is easy to see that whenever F is a preference profile on I, M(F) belongs to Π.

Theorem 8 : Let R ∈Π. Then there exists a finite set I, and a prefence profile $F : I \to \Pi^T \cap \Pi^0$,such that R=M(F).

Proof : Let X ={x_1,..., x_n} and let, I = R×{1,2}.For (x,y) ∈ R let, (w,z)∈F(((x,y),1)) if either (a)w = x;or (b) w = y and z ≠x; or (c) w = x_j , z = x_k , j ≤ k and {w,z} ∩ {x,y} = ϕ. For (x,y) ∈ R let, (w,z)∈F(((x,y),2)) if either (a)z = y;or (b) z = x and w ≠y; or (c) w = x_j , z = x_k , j ≥ k and {w,z} ∩ {x,y} = ϕ. Straightforward verification now shows that (x,y) ∈ R if and only if (x,y) ∈ M(F).

♥

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