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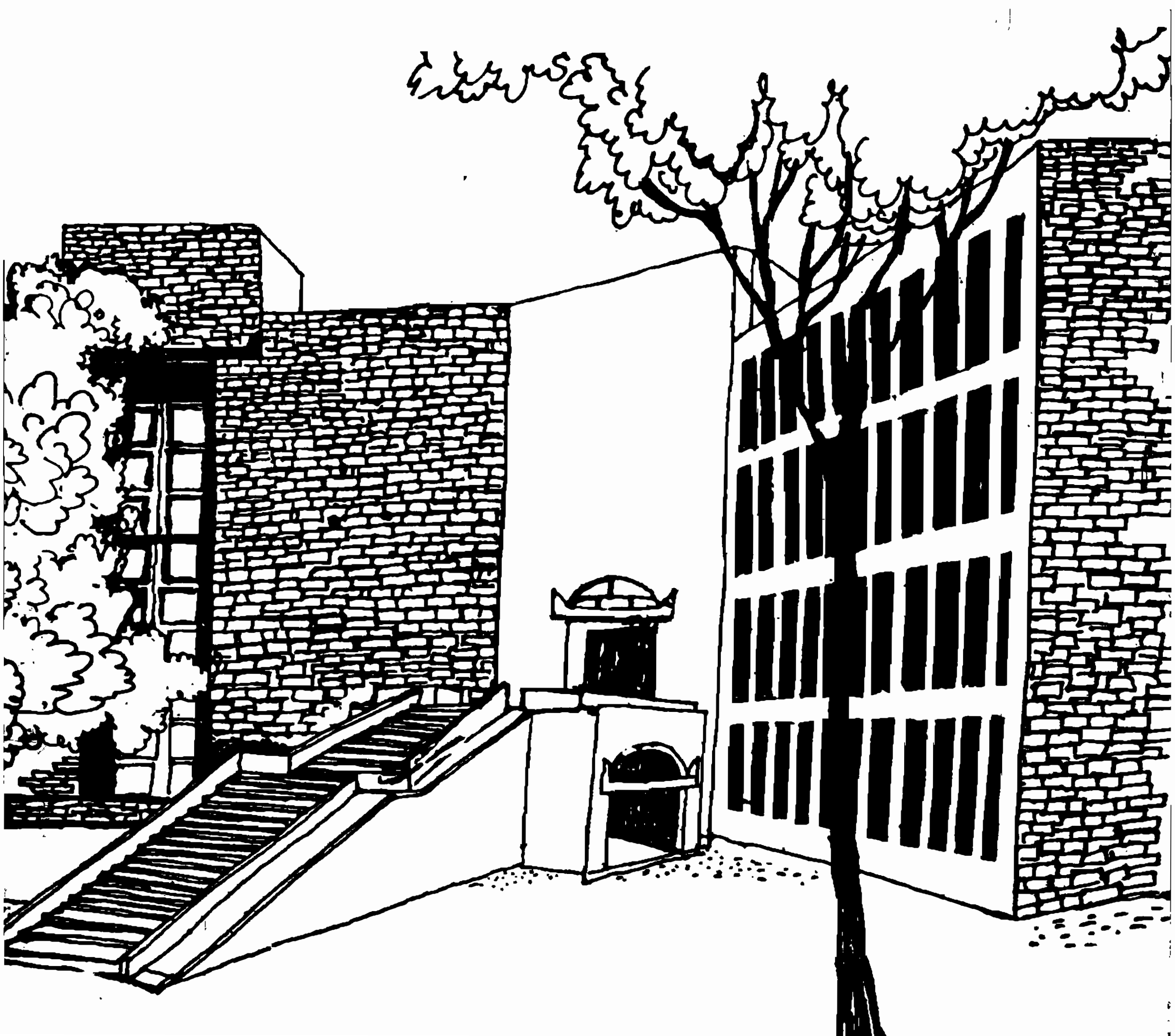

LOCAL SOCIAL DECISION FUNCTIONS: A SURVEY

By

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Abstract

Since the publication of Arrow's [1951] impossibility theorem, much effort has been spent on the analysis and rationalizability of committee decision making. The traditional approach to this problem considers rules which aggregate individual binary relations to a binary relation for society. In this survey we call such rules, which are assumed to be defined on profiles of individual rankings, by the name *social decision functions.*
In Aleskerov (1999) can be found a property that social decision functions are required to satisfy.This property is called locality.In Arrow's original work it was called independence of irrelevant alternatives.Social Decision functions which satisfy locality are called *local social decision functions.*Aleskerov (1999) not only contains a state of the art survey of local social decision functions,but several original contributions to the literature as well.However,Aleskerov does not restrict the domain of social decision functions to be profiles of individual rankings.In different characterization theorems,different domains are considered.All these domains contain the set of profiles of individual rankings as a subset and usually as a strict subset. It is well known in the theory of axiomatic choice theory that a characterization valid on a given domain may fail to hold on a subdomain.Our purpose in this survey is to show that such is not the case with local social decision functions.
It is necessary to justify the domain we have chosen for our survey. Social sciences in general and economic theory in particular, has never confronted any major problem while representing individual preferences by a strict ranking.It is only the issue concerning social preferences by a strict ranking which has been at the centre of the debate concerning aggregation of preferences in social choice theory.Thus the domain comprising profiles of individual rankings is consistent with the demands of economic theory and yet highlights the problems that arise very naturally in social aggregation procedures.

Local Social Decision Functions : A Survey

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October 2000

1. Introduction :- Since the publication of Arrow's [1951] impossibility theorem, much effort has been spent on the analysis and rationalizability of committee decision making. The traditional approach to this problem considers rules which aggregate individual binary relations to a binary relation for society. In this survey we call such rules, which are assumed to be defined on profiles of individual rankings, by the name *social decision functions.*
In Aleskerov (1999) can be found a property that social decision functions are required to satisfy.This property is called locality.In Arrow's original work it was called independence of irrelevant alternatives.Social Decision functions which satisfy locality are called *local social decision functions.*Aleskerov (1999) not only contains a state of the art survey of local social decision functions,but several original contributions to the literature as well.However,Aleskerov does not restrict the domain of social decision functions to be profiles of individual rankings.In different characterization theorems,different domains are considered.All these domains contain the set of profiles of individual rankings as a subset and usually as a strict subset. It is well known in the theory of axiomatic choice theory that a characterization valid on a given domain may fail to hold on a subdomain.Our purpose in this survey is to show that such is not the case with local social decision functions.
It is necessary to justify the domain we have chosen for our survey. Social sciences in general and economic theory in particular, has never confronted any major problem while representing individual preferences by a strict ranking.It is only the issue concerning social preferences by a strict ranking which has been at the centre of the debate concerning aggregation of preferences in social choice theory.Thus the domain comprising profiles of individual rankings is consistent with the demands of economic theory and yet highlights the problems that arise very naturally in social aggregation procedures.

2. The Model :- Let X be a non-empty finite set, containing at least three distinct elements. Let $\Delta(X)=\{(x,x)/x\in X\}$. $\Delta(X)$ is called the diagonal of X.
A binary relation R on X is said to be reflexive if $\Delta(X)\subset R$ and complete if $X\backslash\Delta(X)\subset R\cup R^{-1}$, where $R^{-1}=\{(x,y)/(y,x)\in R\}$ is the inverse of R. A binary relation R on X is said to be an abstract game if R is reflexive and complete.

Let B(X) denote the set of abstract games. An abstract game R is said to be a tournament if $\forall$ x,y$\in$X with x$\neq$y, R$\cap${(x,y),(y,x)} is a singleton.

A binary relation R on X is said to be transitive if $\forall$ x,y,z$\in$X : [{(x,y),(y,z)}$\subset$R] implies [(x,z)$\in$R]. Let Tr.(X) = {R$\in$B(X)/R is transitive}.

A tournament R on X is said to be a linear order if R$\in$Tr.(X). Let L(X) denote the set of linear orders on X.

Given a binary relation R on X, its asymmetric part P(R)={(x,y)$\in$R/(y,x)$\notin$R} and its symmetric part I(R)={(x,y)$\in$R/(y,x)$\in$R}. A binary relation R on X is said to be <u>quasi-transitive</u> if P(R) is transitive. Let QT(X)={R$\in$B(X)/R is quasi-transitive}.

Given a binary relation R on X, let T(R)={(x,y)$\in$XxX / there does not exist a positive integer 's' and elements $x_1,\ldots,x_s$ in X such that $x=x_1$, $y=x_s$ and $(x_i,x_{i+1})\in P(R)\,\forall i\in\{1,\ldots,s-1\}$}. A binary relation R on X is said to be acyclic if T(P(R))$\cap\Delta$(X)=ϕ. Let A(X) denote the set {R$\in$B(X)/R is acyclic}.

A binary relation R in B(X) is said to be weakly sourced if there exists x$\in$X such that for no w$\in$X~{x} is it the the case that (w,x)$\in$T(R)~T(I(R)). A binary relation R in B(X) is said to be single sourced if there exists x$\in$X such that : (i) for no y$\in$X~{x} is it the case that (y,x)$\in$T(R)~T(IR)); (ii) $\forall$y$\in$X~{x} : that (x,y)$\in$T(P(R)). Let WS(X) denote the set of all weakly sourced relations on X and SS(X) denote the set of all single sourced binary relations on X. Clearly, SS(X)$\subset$WS(X).

A binary relation R in B(X) is said to be an interval order if $\forall$ x,y,z,v$\in$X : [{(x,y),(z,v)}$\subset$P(R)] implies [{(x,v),(z,y)}$\cap$P(R)$\neq\phi$]. Let IO(X) denote the set of interval orders on X.

Then L(X) $\subset$Tr.(X)$\subset$IO(X)$\subset$QT(X)$\subset$ A(X).

Let n be a positive integer greater than or equal to two and let N={1,2,...,n}. N denotes the agent set i.e. the set of agents. Let L^N(X) denote the set of all functions from N to L(X) and if f$\in L^N$(X), then for i$\in$N,f(i) (denoted f_i) represents the preferences of agent i over the alternatives in X. A social decision function is a function F:L^N(X)$\rightarrow$B(X), such that $\forall$ f$\in L^N$(X) $\bigcap_{i\in N} f_i \subset F(f)\subset \bigcup_{i\in N} f_i$. Hence if (x,y)$\in$ $\bigcap_{i\in N} f_i$, then since f$\in L^N$(X), we have (y,x)$\notin$ $\bigcup_{i\in N} f_i$. Thus (x,y)$\in$P(F(f)). This property of a social decision function is called <u>unanimity</u>. Given a non-empty subset Y of X and R$\in$B(X), let R/Y $\equiv$R$\cap$(YxY). Given f$\in L^N$(X), let f/Y be the function on N such that $\forall$ i$\in$N : (f/Y)(i)=f_i/Y.

A social decision function F:L^N(X)$\rightarrow$B(X) is said to be local (: or satisfy locality) if $\forall$ x,y,$\in$X and f, g$\in L^N$(X) : [f/{x,y}=g/{x,y}] implies [F(f)/{x,y}=F(g)/{x,y}].

Given F$\in L^N$(X) and x,y$\in$X with x$\neq$y, let V(x,y;f)={i$\in$N/(x,y)$\in f_i$}. Clearly V(y,x;f)=N~V(x,y;f) $\forall$ x,y$\in$X with x$\neq$y and f$\in L^N$(X). Given a local social decision function (LSDF) F, and (x,y)$\in$(XxX)~Δ(X), let

$W(x,y;F)=\{w/[w=V(x,y;f)\in L^N(X)]$ implies $[(x,y)\in P(F(f)]\}$. Clearly $W(x,y;F)\neq\phi$, since $N\in W(x,y;F)$.

A LSDF, F is said to be monotonic if $\forall(x,y)\in(XxX)\sim\Delta(X);[\ \overline{w}\in W(x,y;F)$ and $\overline{w}\subset w']$ implies $w'\in W(x,y;F)$. It is said to be neutral if there exists a set W(F) such that $\forall\ (x,y)\in(XxX)\sim\Delta(X):W(x,y;F)=W(F)$.

3. Characterization of monotonic and neutral LSDF's :-

Theorem 1 :- Let F be a monotonic and neutral LSDF. Then there exists a non-empty set $\overline{W}(F)$ of non-empty subsets of N such that $\forall f\in L^N(X)$ and $x,y\in X:(x,y)\in F(f)\leftrightarrow(y,x)\notin \bigcup_{w\in\overline{W}(F)}\left[\bigcap_{i\in w}P(f_i)\right]$.

Proof :- Let $\overline{W}(F)$ be a non-empty set of non-empty subsets of N such that $\forall f\in L^N(X)$ and $x,y\in X;(x,y)\in F(f)\leftrightarrow (y,x)\notin \bigcup_{w\in\overline{W}(F)}\left[\bigcap_{i\in w}P(f_i)\right]$.

Let $\overline{w}\in W(x,y;F)$ and let $\overline{w}\subset w'$. Let $f\in L^N(X); w'=V(x,y,;f)$.

Let $g\in L^N(X)$: $\overline{w}=V(x,y;g)$. Since $w\in W(x,y;F)$, $(x,y)\in P(F(g))$. Thus $(y,x)\notin F(g)$.

Hence $(x,y)\in \bigcup_{w\in\overline{W}(F)}\left[\bigcap_{i\in w}P(g_i)\right]$. Hence there exists $w\in\overline{W}(F):(x,y)\in\bigcap_{i\in w}P(g_i)$.

Thus $w\subset\overline{w}\subset w'$. Thus $(x,y)\in\bigcap_{i\in w}P(f_i)$. Hence $(x,y)\in P(F(f))$. Thus $w'\in W(x,y;F)$.

Hence F is monotonic. Let $\overline{w}\in W(x,y;F)$ and let $(z,v)\in(XxX)\sim\Delta(X)$. Hence there exists (by a similar argument as above) $w\in\overline{W}(F):w\subset\overline{w}$.

Let $f\in L^N(X)$. Then $(z,w)\in\bigcap_{i\in w}P(f_i)$ implies $(z,v)\in P(F(f))$. $\overline{w}\in W(z,v;F)$. Now suppose F is a LSDF which is monotonic and neutral. Hence there exists a set W(F) such that $\forall(x,y)\in(XxX)\sim\Delta(X):W(F)=W(x,y;F)$. Let $\overline{W}(F)=\{w\in W(F):[w'\in W(F),w'\subset w]$ implies $w'=w\}$.

Let $f\in L^N(X)$ and $(x,y)\in\bigcup_{w\in\overline{W}(F)}\left[\bigcap_{i\in w}P(f_i)\right]$. Hence there exists $w\in\overline{W}(F)$:

$(x,y)\in\bigcap_{i\in w}P(f_i)$. $\therefore w\subset V(x,y;f)$. By monotonicity, $V(x,y;f)\in W(F)$. Thus

$(x,y)\in P(F(f))$. Hence, $\bigcup_{w\in\overline{W}(F)}\left[\bigcap_{i\in w}P(f_i)\right]\subset P(F(f))$..

Now, let $(x,y)\in P(F(f))$. Since F is an LSDF, $V(x,y;f)\in W(x,y;F4{:}29\ PM)=W(F)$. Thus there exists $\overline{w}\in\overline{W}(F):\overline{w}\subset=V(x,y;f)$. Hence $(x,y)\in\bigcap_{i\in w}P(f_i)$..

Thus $(x,y)\in\bigcup_{w\in W(F)}\left[\bigcap_{i\in w}P(f_i)\right]$.

$\therefore P(F(f)) = \bigcup_{w \in \overline{W}(F)} \left[\bigcap_{i \in w} P(f_i) \right]$.

This proves the theorem.

Q.E.D.

Let F be a LSDF such that $\forall f \in L^N(X)$ and $x,y \in X:(x,y) \in F(f)$ if and only if $(y,x) \notin \bigcup_{w \in \overline{W}(F)} \left[\bigcap_{i \in w} P(f_i) \right]$, for some non-empty collection $\overline{W}(F)$ of non-empty subsets of N. If $\overline{W}(F)$ is a singleton, then F is callled an oligarchy. If there exists a positive integer 'k' such that $\overline{W}(F)=\{w \subset N /$ cardinality of $w=k\}$, the F is called a k-votes rule. On the other hand if F is an oligarchy for which $\overline{W}(F)=\{w\}$ where w is a singleton, then F is called a dictatorial rule.

Let F be an LSDF such that $\forall\, f \in L^N(X)$ and $x,y \in X:(x,y) \in F(f)$ if and only if $(y,x) \notin \bigcup_{w \in \overline{W}(F)} \left[\bigcap_{i \in w} P(f_i) \right]$ for some non-empty collection of non-empty subsets of N. If $\bigcap_{w \in \overline{W}(F)} w \neq \phi$, then f is called a veto-rule. This is because, unless every agent in w prefers x to y, it is not posssible for the social decision function to exhibit a preference for x over y. A veto rule F is clled a <u>collegium rule</u> if whenever $\eta: N \to N$ is a one-to-one mapping with $\eta\left(\bigcap_{w \in \overline{W}(F)} w \right) = \bigcap_{w \in \overline{W}(F)} w$, then $\forall f,g \in L^N(X)$ with $f_i = g_{\eta(i)}\ \forall i \in N$, $F(f)=F(g)$.

4. <u>Acyclic LSDF's</u>:

<u>Lemma 1</u> :- Let F be an LSDF. Then range $(F) \subset A(X)$ if and only if given any positive integer s, less than or equal to m (i.e. the cardinality of X) and $x_1,\ldots,x_s \in X:[w_i \in W(x_i,x_{i+1};F)$ for $i \in \{1,\ldots,s-1\}$ and $w_s \in W(x_s,x_1;F)]$ implies [either $\bigcap_{i=1}^{s} w_i$ or $\bigcup_{i=1}^{s} w_i \subset\subset N$].

<u>Proof</u> :- Let F be an LSDF and suppose range $(F) \not\subset A(X)$. Let $f \in L^N(X)$ and $x_1,\ldots,x_s \in X$ such that $(x_i,x_{i+1}) \in P(F(f))$ for $i \in \{1,\ldots,s-1\}$ and $(x_s,x_1) \in P(F(f)))$. Thus $V(x_i,x_{i+1};f) \in W(x_i,x_{i+1};f)$ for $i \in \{1,\ldots,s-1\}$ and $V(x_s,x_1;f) \in W(x_s,x_1;f)$. If $\left[\bigcap_{i=1}^{s-1} V(x_i,x_{i+1};f) \right] \cap V(x_s,x_i;f) \in P(f_j)$ then there exists $j \in N:(x_i,x_{i+1}) \in P(f_j) \forall i \in \{1,\ldots,s-1\}$ and $(x_s,x_1) \in P(f_j)$. This contradicts $f_j \in L(X)$. Hence, $\left[\bigcap_{i=1}^{s-1} V(x_i,x_{i+1};f) \right] \cap V(x_s,x_1;f)=\phi$. If $\left[\bigcap_{i=1}^{s-1} V(x_i,x_{i+1};f) \right] \cup V(x_s,x_1;f) \subset\subset N$, there exists

$j \in N:(x_{i+1},x_i) \in P(f_j) \forall i \in \{1,...,s-1\}$ and $(x_1,x_s) \in P(f_j)$. This contradicts $f \in L(X)$. Hence if $w_i \in W(x_i,x_{i+1};f)$ for $i \in \{1,...,s-1\}$ amd $w_s \in w(x_s,x_1;f)$ implies either $\bigcap_{i=1}^{s} w_i \neq \phi$ or

$\bigcup_{i=1}^{s} w_i \subset\subset N$, then range(F)$\subset$A(X).

Now suppose range (F)$\subset$A(X) and let $x_1,...,x_s \in X$ with $w_i \in W(x_i,x_{i+1};F)$ for $i \in \{1,...,s-1\}$ and $w_s \in W(x_s,x_1;F)$. Towards a contradiction suppose $\bigcap_{i=1}^{s} w_i = \phi$

and $\bigcup_{i=1}^{s} w_i = N$. Then there exists $f \in L^N(X)$: $w_i = V(x_i,x_{i+1};f)$ for $i \in \{1,..,s-1\}$ and $w_s = V(x_s,x_1;f)$. Hence $(x_i,x_{i+1}) \in P(F(f))$ for $i \in \{1,...,s-1\}$ and $(x_s,x_1) \in P(F(f))$, contradicting range (F)$\subset$A(X).
This proves the lemma.

Q.E.D.

Let F be an LSDF satisfying neutrality. Then, clearly there exists a non-empty collection W(F) of non-empty subsets of N such that $\forall x,y \in X$ with $x \neq y$; $W(x,y;F)=W(F)$. This observation was used in the proof of Theorem 1. Given an LSDF F satisfying neutrality, the Nakamura number of F, $\nu(F)$ is a natural number, such that W(F) contains at least one collection of $\nu(F)$ sets whose intersection is empty and intersection of fewer sets is always non-empty. If $\bigcap_{w \in W(F)} w \neq \phi$, then $\nu(F)$ is set equal to $+\infty$.

Theorem 2:- Let F be an LSDF satisfying monotonicity and neutrality. Then range(F)$\subset$A(X) if and only if $\nu(F)>m$, where m is equal to the cardinality of X.

Proof :- As in Lemma 1, there exists a non-empty collection $\overline{W}(F)$ of non-empty subsets of N such that $\forall f \in L^N(X)$ and $x,y \in X$:

$(x,y) \in F(f)$ if and only if $(y,x) \notin \bigcup_{w \in \overline{W}(F)} \left[\bigcap_{i \in w} P(f_i) \right]$. Further, $\overline{W}(F)$ =

$\{w \in W(F): [w' \in W(F), w' \subset w]$ implies $w' = w\}$.

Suppose range (F)$\subset$A(X) and towards a contradiction suppose $\nu(F) \leq m$.

Hence there exists $w_1,...,w_{\nu(F)} \in W(F)$ such that $\bigcap_{i=1}^{m} w_i = \phi$ and $\bigcap_{i \in j} w_i \neq \phi$ whenever J is a non-empty proper subset of $\{1,...,\nu(F)\}$. Let $x_1,...,x_{\nu(F)} \in X$ and let $f \in L^N(X)$:
$V(x_i,x_{i+1};f)=w_i$ for $i \in \{1,...,\nu(F)-1\}$

$V(x_{\nu(F)},x_1;f)=w_{\nu(F)} \cup \left[N \sim \left\{ \bigcup_{i=1}^{\nu(F)-1} w_i \right\} \right]$.

By monotonicity, $V(x_{\nu(F)},x_1;f)\in W(F)$. Thus, $(x_i,x_{i+1})\in P(F(f))$ $\forall i\in\{1,\ldots,\nu(F)-1\}$ and $(x_{\nu(F)},x_1)\in P(F(f))$ contradicting range $(F)\subset A(X)$. Thus, $\nu(F)>m$.

Conversely suppose, $\nu(F)>m$. Then if s is any positive interger less than or equal to m and $w_1,\ldots,w_s\in W(F)$, then $\bigcap_{i=1}^{s} w_i \neq \phi$. By Lemma 1, range$(F)\subset A(X)$. This proves the theorem.

Q.E.D.

Theorem 3 :- Let F be a veto rule. Then range $(F)\subset A(X)$.
Proof :- If F is veto rule, then $\bigcap_{w\in W(F)} w \neq \phi$. Hence $\nu(F)=+\infty>m$.
By Theorem 2, range$(F)\subset A(X)$.

Q.E.D.

Theorem 4 :- Let$m\geq n$ where m is the cardinality of $\overline{X}$. Let F be an LSDF satisfying monotonicity and neutrality. Then range$(F)\subset A(X)$ implies F is a veto rule.

Proof :- Suppose F is as above except that it is not a veto rule. Hence $\bigcap_{w\in \overline{W}(F)} w = \phi$. Given $i\in N$, let $W_{-i}=\{w\in \overline{W}(F)/i\notin w\}$. For each $i\in N$, choose $w_i\in W_{-i}$.
Then $\bigcap_{i\in N} w_i = \phi$. Hence, $\nu(F)\leq n\leq m$. By Theorem 2, range$(F)\not\subset A(X)$ and a contradiction.This proves the theorem.

Q.E.D.

Let F be an LSDF. It is said to satisfy anonymity if $\forall i,j\in N$ and $f,g\in L^N(X)$ if $[f_h=g_h \forall h\in N\sim\{i,j\}, f_i=g_i$ and $f_j=g_j]$, then $[F(f)=F(g)]$.

Theorem 5:- Let F be an LSDF satisfying neutrality, monotonicity and Anonymity. Then F is a k-votes rule.

Proof :- By Theorem 1,
$P(F(f))=\bigcup_{w\in\overline{W}(F)}\left[\bigcap_{i\in w} P(f_i)\right] \forall f\in L^N(X)$, where $\overline{W}(F)$ is a non-empty collection of non-empty subsets of N. By anonymity, $w\in \overline{W}(F)$, $\phi\neq w'\subset N$ and cardinality of w = cardinality of w', implies $w'\in \overline{W}(F)$. This proves the theorem.

Q.E.D.

Theorem 6:- Let F be a k-votes rule. Then [range(F)$\subset$A(X)]$\leftrightarrow m<\left[\frac{n}{n-k}\right]$

where $\left[\frac{n}{n-k}\right]$ is the smallest integer greater than or equal to $\left[\frac{n}{n-k}\right]$.

Proof :- Let F be a k-votes rule and let $\nu(F)$ be its Nakamura number. Hence there exists a collection of $\nu(F)$ sets in $\overline{W}(F)$ each of size 'k' whose intersection is empty. The union of the complement of each such set is thus N. The complement of each set in $\overline{W}(F)$ is of size (n-k). Hence $\nu(F)$, $(n-k)\geq n$.

If $\left[\frac{n}{n-k}\right]>m$, then $\nu(F)>m$. By Theorem 2, range(F)$\subset$A(X).

Now suppose F is a k-votes rule with range(F)$\subset$A(X). Thus $\nu(F)>m$.

Let $\{S_1,\ldots,S_p\}$ ne a partition of N:

(i) S_i=n-ki for i=1,...,p-1

(ii) $S_p\leq n$-k.

$\therefore (p-1)(n-k)+$cardinality of $S_p = n$

$\therefore p(n-k)\geq n\geq(p-1)n-k$

$\therefore p\geq\left[\frac{n}{n-k}\right]\geq p-1$

$\therefore p=\left[\frac{n}{n-k}\right]$.

Now $S_i^c\in W(F)\ \forall i\in\{1,\ldots,p\}$ and $\bigcap_{i=1}^{p} S_i^c=\phi$

$\therefore \nu(F)\leq p=\left[\frac{n}{n-k}\right]$

$\therefore m<\left[\frac{n}{n-k}\right]$.

Q.E.D.

Let $B^o(X) = B(X)\cap A(X)$. $B^o(X)$ is the set of tournaments on X.

Proposition 1:- Suppose 'n' is odd and F is an LSDF. Then F satisfies monotonicity, neutrality, anonymity and range(F)$\subset B^o(X)$ if and if F is an $\left[\frac{n+1}{2}\right]$ votes rule. Such an F is called the majority SDF.

<u>Proof</u> :- If F is the majority SDF, then F clearly satisfies the desired properties.Hence suppose F is an LSDF satisfying monotonicity, neutrality, anonymity and range(F)$\subset B^o(X)$. By theorem 5, there exists a positive integer 'k' such that $\forall f \in L^N(X)$:

$$P(F(f)) = \bigcup_{w \in \overline{W}(F)} \left[\bigcap_{i \in w} P(f_i) \right]$$

where $\overline{W}(F) = \{w \subset N / \text{cardinality of } w = k\}$.

Suppose $k < \left[\frac{n+1}{2}\right]$. Thus $2k \leq n$.

Let w and w' be two disjoints sets in $\overline{W}(F)$ and let $x,y \in X$ with $x \neq y$. Let $f \in L^N(X)$:$V(x,y;f)=w$. Thus $w' \subset V(y,x;f)$. Since $w \in \overline{W}(F)$, $(x,y) \in P(F(f))$. If $w' \in \overline{W}(F)$ and $w' \subset V(y,x;f)$, by monotonicity, $(y,x) \in P(F(f))$, which contradicts $(x,y) \in P(F(f))$. Now Suppose $k > \left[\frac{n+1}{2}\right]$. Thus $2k > n+1 > n$.

Let $f \in L^N(X)$ and $x,y \in X$ with $x \neq y$ and let, $V(x,y;f) = \{1,2,\ldots\ldots,\frac{n+1}{2}\}$

$V(y,x;f) = \{\frac{n+3}{2}, \ldots\ldots, n\}$.

Neither $V(x,y;f)$ nor $V(y,x;f)$ belongs to W(F). Hence $(x,y) \notin P(F(f))$ and $(y,x) \notin P(F(f))$. Thus $(x,y) \in I(F(f))$, contradicting range(F)$\subset B^o(X)$. Hence $k = \frac{n+1}{2}$.

<u>Q.E.D.</u>

5. <u>Quasitransitive SDF's</u>

<u>Theorem 7</u> :- Let F be a k-votes rule. Then range(F)$\subset$QT(X) if and only if k=n.

<u>Proof</u> :- If k=n, then $P(F(f)) = \bigcap_{i \in N} P(f_i)\ \forall f \in L^N(X)$. Hence range(F)$\subset$QT(X). Now suppose range(F)$\subset$QT(X) and towards a contradiction suppose k<n. Hence $\overline{W}(F)$ has atleast two non-empty subsets of N. Let w, $w' \in \overline{W}(F)$ with $w \neq w'$. Let $\overline{w} = w' \cup (N \sim w)$. Clearly $\overline{w} \in W(F)$. $\overline{W}(F) = \{\{i\}/i \in N\}$. Let $f \in L^N(X)$ with $V(x,y;f) = \{1\}$ for some $x,y \in X$. Since $\{1\} \in \overline{W}(F)$, $(x,y) \in P(F(f))$. Since $N \sim \{1\} \in W(F)$, $(y,x) \in P(F(f))$ and a contradiction.

Let $x,y,z \in X$ with $x \neq y \neq z \neq x$ and let $f \in L^N(X)$ with

(i) $V(x,y;f) = w$

(ii) $V(y,z;f) = \overline{w}$

(iii) $V(x,z;f) = w \cap w'$.

This construction is possible since $w \cap \overline{w} = w \cap w'$. Clearly $w \cap w' \notin W(F)$. Hence $\{(x,y),(y,z)\} \subset P(F(f))$, but $(x,z) \notin P(F(f))$. Thus $range(F) \not\subset QT(X)$ and a contradiction. This proves the theorem.

Q.E.D.

Lemma 2 :- Let F be a LSDF with $[range(F) \subset QT(X)]$. Then, $[\forall x,y,z \in X: w_1 \in W(x,y;F),\ w_2 \in W(y,z;F)$ and $w_1 \cap w_2 \subset w \subset W_1 \cup W_2]$ implies $[W \in W(x,z;F)]$.

Proof :- Let w_1, w_2 and w be as above and let $f \in L^N(X)$:

(i) $V(x,y;F)=w1$

(ii) $V(y,z;F)=w2$

(iii) $V(x,z;F)=w$

This is possible since $w_1 \cap w_2 \subset w \subset w_1 \cup w_2$. Thus $(x,y) \in P(F(f))$, $(y,z) \in P(F(f))$.Since $range(F) \subset QT(X)$, $(x,z) \in P(F(f))$. thus $w \in W(x,z;F)$.

Q.E.D.

Lemma 3:- Let F be a LSDF with $[range(F) \subset QT(X)]$. Then F is neutral.

Proof :- Let $w \in W(x,y;F)$ and $w' \in W(y,z;F)$. By Lemma 2, since $w \cap w' \subset w \subset w \cup w'$, we get $w \in W(x,z:F)$. Let $v \in X \sim \{x,z\}$ and let $w'' \in W(v,x;F)$. Then $w'' \cap w \subset w \subset w \cup w''$ and Lemma 2, implies $w \in W(v,z;F)$. Thus $w \in W(v,z;F)\ \forall (v,z) \in (Xx(X \sim \{x\})) \sim \Delta(X)$. Let $\tilde{w} \in W(Y,x;F)$ and $v \in X \sim \{y,x\}$. Since $w \in W(v,y;F)$ and $w \cap \tilde{w} \subset w \subset w \cup \tilde{w}$, we get $w \in W(v,x:F) \forall v \in X \sim \{y,x\}$ by Lemma 2. Since $w \in W(y,v;F)$ and $w \in W(v,x;F)$, Lemma 2 implies that $w \in W(y,x;F)$. Thus $w \in W(v,z:F) \forall (v,z) \in (XxX) \sim \Delta(X)$. Thus there exists a non-empty collection W(F) of non-empty subsets of N, such that $W(F)=W(v,z:F) \forall (v,z) \in (XxX) \sim \Delta(X)$.

Q.E.D.

Lemma 4 :- Let F be a LSDF with $[range(F) \subset QT(X)]$. Then F is monotonic.

Proof :- By Lemma 3, there exists W(F) such that $\forall (x,y) \in (XxX) \sim \Delta(X)$. $W(F)=W(x,y;F)$. Let $w \in W(F)$ and $w \subset w'$. Let $x,y,z \in X$ with $x \neq y \neq z \neq x$ and let $f \in L^N(X)$:

(i) $V(x,y;f)=w$

(ii) $V(y,z;f)=N$

(iii) $V(x,z;f)=w'$

This is possible since $w \subset w' \subset N$.

Since $w \in W(F)$ we get $(x,y) \in P(F(f))$. By definition of an SDF, $(y,z) \in P(F(f))$. Since $\text{range}(F) \subset QT(X)'$, $(x,z) \in P(F(f))$. Thus $w' \in W(F)$.

Q.E.D.

Lemma 5 :- Let F be an LSDF with $\text{range}(F) \subset QT(X)$. Then F is an oligarchy.
Proof :- By Lemmas 3,4 and Theorem 1, there exists a non-emtpy collection $\overline{W}(F)$ of non-empty subsets of N such that $\forall f \in L^N(X)$:

$$P(F(f)) = \bigcup_{w \in \overline{W}(F)} \left[\bigcap_{i \in w} P(f_i) \right]$$

Since $\text{range}(F) \subset QT(X)$, by Lemmas 2 and 3, $\bigcap_{w \in \overline{W}(F)} w \in \overline{W}(F)$. Since

$\overline{W}(F) = \{w \in W(F) / \ w' \in W(F),\ w' \subset w \text{ implies } W' = w\}$, $\overline{W}(F) = \left\{ \bigcap_{w \in \overline{W}(F)} w \right\}$. Hence

$\overline{W}(F)$ is a singleton. Thus F is an oligarchy.

Q.E.D.

Theorem 8 :- Let F be an LSDF. Then $[\text{range}(F) \subset QT(X)]$ if and only if [F is an oligarchy].

Proof :- If F is an oligarchy, then clearly range $(F) \subset QT(X)$. This coupled with Lemma 5, proves Theorem 8.

Q.E.D.

Note :- If cardinality of X is three, then $QT(X)=WS(X)$. To see this, let $X=\{x,y,z\}$ and let $R \in WS(X)$. Let (x,y), $(y,z) \in P(R)$. Towards a contradiction suppose $(z,x) \in R$. Then $(y,x) \in T(R) \sim T(I(R))$. This contradicts the assumption that $R \in WS(X)$. Hence $(x,z) \in P(R)$. Thus $R \in QT(X)$.

Lemma 6 :- Let F be an LSDF, with $[\text{range}(F) \subset WS(X)$. Then F is an oligarchy.
Proof :- If cardinality of X is three, then by the note above and Theorem 8, there exists a non-empty subset W of N such that $\forall f \in L^N(X)$.
$P(F(f)) = \bigcap_{i \in w} P(f_i)$ i.e. F is an oligarchy. Thus F satisfies monotonicity and neutrality. Hence suppose, cardinality of X is equal to m, which is greater than 3. Suppose the lemma is true for cardinality of X less than m. Choose $x \in X$ and let $I(x)$ be a subset of $L^N(X)$ such that $f \in I(x)$ if and only if $\forall i \in N$ and $y \in X \sim \{x\}$ it is the case that $(y,x) \in f_i$. Now if $f \in h(x)$, since $F(f) \in WS(X)$, it must be the case that there exists $y \in X \sim \{x\}$ such that for no $v \in X \sim \{y\}$ is it the case that $(v,y) \in T(F(f)) \sim T(I(F(f)))$.

Let G_x:$L^N(X)\sim\{x\})\to B(X\sim\{x\})$ be defined as follows : $\forall g\in L^N(X\sim\{x\})$, let $G_x(g)=F(f)\,|\,X\sim\{x\}$ where $f\in h(x)$ and $f\,|\,X\sim\{x\}=g$. Clearly G_x is a well defined LSDF, and range$(G_x)\subset WS(X\sim\{x\})$. Hence by the induction hypothesis, there exists a non-empty subset W_x of N such that $\forall g\in L^N(X\sim\{x\})$:$P(G_x(g))= \bigcap_{i\in w_x} P(g_i)$. Hence $\forall f\in I(x)$:$P(F(f)\,|\,X\sim\{x\})= \bigcap_{i\in w_x} P(f_i|X\sim\{x\})$. Since F satisfies locality $\forall f\in L^n(X)$: $P(F(f)\,|\,X\sim\{x\}) = \bigcap_{i\in w_x} P(f_i|X\sim\{x\})$. Let x,y,z,v be distinct elements in X. Then $(z,v)\in P(F(f)\,|\,X\sim\{x\}) \leftrightarrow (z,v)\in \bigcap_{i\in w_x} P(f_i|X\sim\{x\})$ and $(z,v)\in P(F(f)\,|\,X\sim\{y\})\leftrightarrow(z,v)\in \bigcap_{i\in w_y} P(f_i|X\sim\{y\})$. Thus $w_x=w_y$. Hence there exists a non-empty subset w of N such that $\forall x\in X$ and $\forall f\in L^N(X)$: $P(F(f)\,|\,X\sim\{x\})$ $= \bigcap_{i\in w_x} P(f_i|X\sim\{x\})$. Hence $P(F(f))= \bigcap_{i\in w_x} P(f_i|X\sim\{x\})$. Hence $P(F(f))= \bigcap_{i\in W} P(f_i)$.
Hence F is an oligarchy. By a standard induction argument the lemma stands proved.

Q.E.D.

6. Dictatorial Social Decision Functions :-

Theorem 9 : Let F be an LSDF. Then range(F)$\subset$SS(X) if and only if F is dictatorial.
Proof :- If F is dictatorial then range(F) is clearly a subset of SS(X). Hence suppose range (F)$\subset$SS(X)$\subset$WS(X). By Lemma 6, there exists a non-empty subset w of N such that $\forall f\in L^N(X)$:$P(F(f))\subset \bigcap_{i\in W} P(f_i)$. Towards a contradiction suppose w is not a singleton. Let $j\in w$. Hence $W\sim j\neq\phi$. Let $X=\{x_1,\ldots,x_m\}$ and let $f\in L^N(X)$:
(i) $\forall i\in\{1,\ldots,m-1\}$:$(x_i,x_{i+1})\in f$.
(ii)$\forall h\in N\sim\{j\}$ and $\forall i\in\{1,\ldots,m-1\}$: $(x_{i+1},x_i)\in f_h$.
Thus, $P(F(f))=\phi$, since $\{j\}\subset\subset w$ and $w\not\subset N\sim\{j\}$. Now range(F)$\subset$SS(X) implies that there exists $x\in X$:$\forall y\in X\sim\{x\}$, $(x,y)\in T(P(F(f)))$, which is contradicted by $P(F(f))=\phi$(:since $P(F(f))=\phi$ implies $T(P(F(f)))=\phi$). Thus w is a singleton. Hence F is dictatorial.

Q.E.D.

Theorem 10 : Let cardinality of X be atleast four. Let F be an LSDF. Then range(F)$\subset$IO(X) if and only if F is dictatorial.
Proof :- If F is dictatorial, range(F)$\subset$IO(X). Since IO(X)$\subset$QT(X), by theorem 8, there exists a non-empty subset w of N such that $\forall f\in L^N(X)$: $P(F(f))= \bigcap_{i\in W} P(f_i) =$.

Towards contradiction suppose w is not a singleton. Let $j \in w$. Hence $w \sim \{j\} \neq \phi$. Let $x,y,z,v \in X$ with all four being distinct. Let $f \in L^N(X)$ such that
(i) (x,y), (y,z), $(z,v) \in f_j$.
(ii) $(z,v),(v,x),(x,y) \in f_h$ whenever $h \in W \sim \{j\}$
(iii) $(y,v),(v,z),(z,x) \in f_h$ whenever $h \in N \sim w$.

Now $(x,y) \in \bigcap_{i \in W} P(f_i)$. Hence $(x,y) \in P(F(f))$, $(z,v) \in \bigcap_{i \in W} P(f_i)$. Hence $(z,v) \in P(F(f))$. Since $F(f) \in IO(X)$, either $(z,y) \in P(F(f))$ or $(x,v) \in P(F(f))$. However, $V(z,y:f)=w \sim \{j\} \subset\subset w$ and $V(x,v;f)=\{j\} \subset\subset w$, contradicting the definition of F. Hence w must be a singleton. Thus F is dictatorial.

Q.E.D.

Note :- The clause, "Cardinality of X is at least four" is necessary since when cardinality of X is three $IO(X)=QT(X)$ and F need not be dictatorial. For instance for $X=\{x,y,z\}$, F defined by $P(F(f))= \bigcap_{i \in W} P(f_i)$, has range in $IO(X)$.

However F is not dictatorial.

Corollary of Theorem 10 :- Let cardinality of X be atleast four and let F be an LSDF. Then range $(F) \subset Tr.(X)$ if and only if F is dictatorial.
Proof :- This follows immediately from the fact that $IO(X) \subset T(X)$.

Q.E.D.

Theorem 11 :- Let F be an LSDF. Then F is dictatorial if and only if range$(F) \subset Tr.(X)$.
Proof :- If F is dictatorial, then clearly range$(F) \subset Tr.(X)$. By Theorem 8, there exists a non-empty subset w of N such that $\forall f \in L^N(X) : P(F(f)) = \bigcap_{i \in W} P(f_i)$.

Towards a contradiction suppose, w is not a singleton. Let $j \in w$. Hence $w \sim \{j\} \neq \phi$. Let $x,y,z \in X$ with $x \neq y \neq z \neq x$ and let $f \in L^N(X)$:
(i) $V(x,z;f)=w \sim \{j\}$
(ii) $V(z,y;f)=\{j\}$
(iii) $V(x,y;f)=w$
Let $i \in w \sim \{j\}$. Then $(x,y),(y,z)$, $(x,z) \in f_i$. Let $i=j$. Then $(x,y),(z,y),(z,x) \in f_i$. Let $i \in N \sim w$. Then $(y,x),(y,z),(z,x) \in f_i$. Hence there is no inconsistency in the definition of f.
Then $V(x,y;f)=w$, we get $(x,y) \in P(F(f))$. Since $V(z,y;f)=\{j\} \subset\subset w$, $(z,y) \notin P(F(f))$. Since $w \not\subset V(y,z;f)=N \sim \{j\}$, $(y,z) \notin P(F(f))$. $\therefore (y,z) \in I(F(f))$. Since $F(f) \in Tr.(X)$, $(x,z) \in P(F(f))$. But $V(x,z;f)=w \sim \{j\}$ and $w \not\subset w \sim \{j\}$. This contradiction proves the theorem.

Q.E.D.

Note :- Theorem 11 is the celebrated theorem due to Arrow and in the literature on social choice theory it is referred to as Arrow's Impossibility Theorem.

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