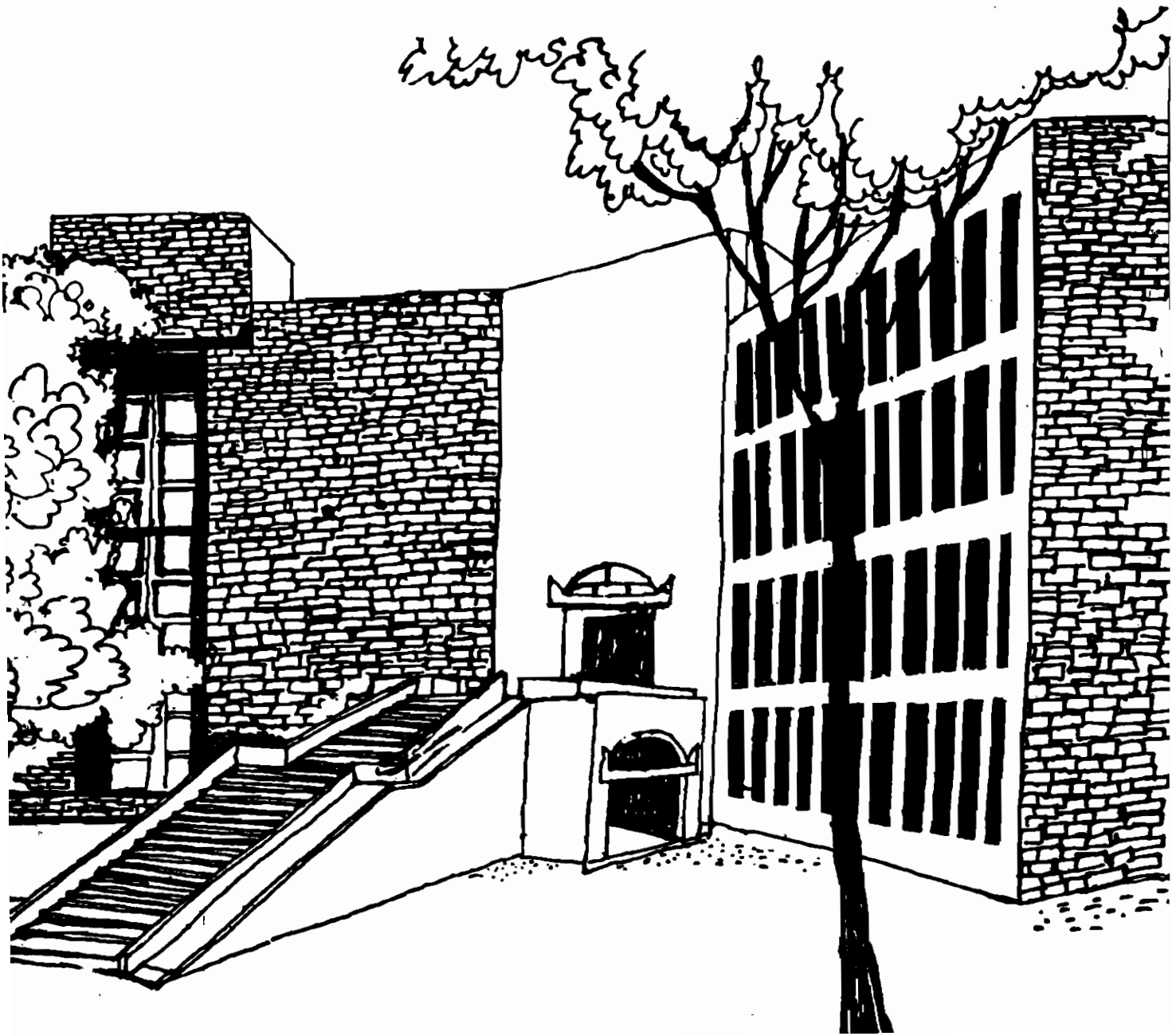




# Working Paper



**THE TOP CYCLE AND UNCOVERED  
SOLUTIONS FOR ABSTRACT GAMES:  
AXIOMATIC CHARACTERIZATIONS**

**By**

*Somdeb Lahiri*

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Abstract  
for  
**The Top Cycle and Uncovered Solutions for Abstract Games : Axiomatic  
Characterizations**  
by  
**Somdeb Lahiri**  
**Indian Institute of Management,**  
**Ahmedabad-380 015,**  
**India.**  
**e-mail:lahiri@iimahd.ernet.in**  
**November 2000.**

In this paper we consider binary relations which are reflexive and complete. Such binary relations are referred to in the literature as abstract games. Given an abstract game  $a$  (game) solution is a function which associates to each subset a non-empty collection of points of the subset. In this paper we provide axiomatic characterizations of the top cycle and uncovered solutions for abstract games. In a final section of the paper, the similarity between a game solution and a choice function of classical rational choice theory is exploited to axiomatically characterize the top cycle and uncovered choice functions.

# **The Top Cycle and Uncovered Solutions for Abstract Games : Axiomatic Characterizations**

by

**Somdeb Lahiri**

**Indian Institute of Management,**

**Ahmedabad-380 015,**

**India.**

**e-mail:lahiri@iimahd.ernet.in**

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1. **Introduction :** An abiding problem in choice theory has been one of characterizing those choice functions which are obtained as a result of some kind of optimization. Specifically, the endeavour has concentrated largely on finding a binary relation (if there be any) whose best elements coincide with observed choices. An adequate survey of this line of research till the mid eighties is available in Moulin [1985]. More recently, the emphasis has focused on binary relations defined on non-empty subsets of a given set, such that the choice function coincides with the best subset corresponding to a feasible set of alternatives. This problem has been provided with a solution in Lahiri [1999], although the idea of binary relations defined on subsets is a concept which owes its analytical origins to Pattanaik and Xu [1990]. Given a binary relation, the idea of a function which associates with each set a non-empty subset of the given set has a long history whose exact origin is very difficult to specify and in any case is unknown to the author. In Laslier [1997] can be found a very exhaustive survey of the related theory when the given binary relation is reflexive, complete and anti-symmetric. In this paper we extend the above set of binary relations to include those which are not necessarily anti-symmetric. Such binary relations which are reflexive and complete are referred to in the literature as abstract games. In a recent paper, Peris and Subiza [1999], refer to abstract games as weak tournaments. Given an abstract game, a (game)solution is a function which associates to each non-empty subset a non-empty collection of elements from the subset, on the basis of the given abstract game. Lucas [1992] has a discussion of abstract games and related solution concepts, particularly in the context of cooperative games. Much of what is discussed in Laslier [1997] and references therein carry through into this framework. An important consequence of both the frameworks is that often, a set may fail to have an element which is best with respect to the given binary relation. To circumvent this problem the concept of the top cycle set is introduced, which selects from among the feasible alternatives only those which are best with respect to the transitive closure of the given relation. The top cycle set is always non-empty and in this paper we provide an axiomatic characterization of the top-cycle solution. It is subsequently observed that the top cycle solution is the coarsest solution which satisfies two innocuous assumptions. An alternative 'x' is said to cover another alternative 'y' if and only if 'x' is preferred to 'y' and for every other third element 'z' if (a) 'y' is atleast as good as 'z', then so is

'x'; (b) if 'y' is preferred to 'z' then so is 'x'. Given any feasible set, its uncovered set is the set of all elements in the feasible set which are not covered by any other element in the same set. The question that naturally arises is the following : Given a choice function , under what condition does a binary relation exist, whose uncovered sets always coincide with the choice function? This question has been discussed in Lahiri [1999], where it is observed that the binary relation 'is uncovered' is reflexive, complete and quasi-transitive and any reflexive, complete and quasi-transitive binary relation can be made to coincide with the "is uncovered" relation of some binary relation. The problem becomes much more difficult if instead of defining the covering relation globally, we considered the covering relation for each individual feasible set, by simply looking at the restriction of the comparison function to that set. In such a situation that fact that 'x' covers 'y' in a particular feasible set does not imply that 'x' covers 'y' globally. In effect, we are then concerned with what Sen [1997] calls 'menu based' relations. In this paper we also address the problem of axiomatically characterizing the uncovered solution (where 'covering' is now defined as a 'menu-based' concept).

In Peris and Subiza [1999] an abstract game is referred to as a weak tournament and it is shown there that a considerable portion of the theory developed in the context of tournaments, carry through to weak tournaments as well. Our axiomatic characterizations are however different from the ones available Peris and Subiza [1999].

In a final selection of this paper we revert to the context of classical rational choice theory. By exploiting the close similarity between a game solution and a choice function, we discuss the necessary implications of the results established in the earlier sections of the paper, which apply to choice functions. This leads to a modest extension of the theory that has been summarized in Moulin [1985].

**Game Solutions :-** Let  $X$  be a finite, non-empty set and given any non empty subset  $A$  of  $X$ , let  $[A]$  denote the collection of all non-empty subsets of  $A$ . Thus in particular,  $[X]$  denotes the set of all non-empty subsets of  $X$ . If  $A \in [X]$ , then  $\#(A)$  denotes the number of elements in  $A$ . A binary relation  $R$  on  $X$  is said to be (a) reflexive if  $\forall x \in X : (x, x) \in R$ ; (b) complete if  $\forall x, y \in X$  with  $x \neq y$ , either  $(x, y) \in R$  or  $(y, x) \in R$ ; (c) transitive if  $\forall x, y, z \in X$ ,  $[(x, y) \in R \ \& \ (y, z) \in R \text{ implies } (x, z) \in R]$ ; (d) anti-symmetric if  $[\forall x, y \in X, (x, y) \in R \ \& \ (y, x) \in R \text{ implies } x = y]$ . Given a binary relation  $R$  on  $X$  and  $A \in [X]$ , let  $R|A = R \cap (A \times A)$ . Let  $\Pi$  denote the set of all reflexive and complete binary relations. If  $R \in \Pi$ , then  $R$  is called an abstract game. Given a binary relation  $R$ , let  $P(R) = \{(x, y) \in R / (y, x) \notin R\}$  and  $I(R) = \{(x, y) \in R / (y, x) \in R\}$ .  $P(R)$  is called the asymmetric part of  $R$  and  $I(R)$  is called the symmetric part of  $R$ . Given a binary relation  $R$  on  $X$  and  $A \in [X]$ , let  $G(A, R) = \{x \in A / \forall y \in A : (x, y) \in R\}$ . Given  $A \in [X]$ , let  $\Delta(A)$  denote the diagonal of  $A$  i.e.  $\Delta(A) = \{(x, x) / x \in A\}$ . The following example shows that given  $R \in \Pi$  and  $A \in [X]$ ,  $G(A, R)$  may be empty:  
Example 1: Let  $X = \{x, y, z\}$  and let  $R = \Delta(X) \cup \{(x, y), (y, z), (z, x)\}$ . Clearly  $G(X, R)$  is empty.

Given  $R \in \Pi$ ,  $A \in [X]$ , let  $T(R|A)$  be a binary relation on  $A$  defined as follows:  $(x, y) \in T(R|A)$  if and only if there exists a positive integer  $K$  and  $x_1, \dots, x_K$  in  $A$  with (i)  $x_1 =$

$x, x_k = y$  : (ii)  $(x_i, x_{i+1}) \in R \forall i \in \{1, \dots, k-1\}$ .  $T(R|A)$  is called the transitive hull of  $R$  in  $A$ . Clearly  $T(R|A)$  is always transitive.

Given  $R \in \Pi$ ,  $A \in [X]$ ,  $G(A, T(R|A))$  is called the top cycle set of  $R$  in  $A$ . Clearly  $G(A, T(R|A))$  is non-empty whenever  $R \in \Pi$  and  $A \in [X]$ .

Let,  $R$  belong to  $\Pi$ . An  $R$ -based game solution on  $X$  is a function  $S: [X] \rightarrow [X]$  such that:

(i)  $\forall A \in [X]: S(A) \subset A$  ;

(ii)  $\forall x, y \in X : x \in S(\{x, y\})$  if and only if  $(x, y) \in R$ .

Thus in particular,  $R = R^S \equiv \cup \{ S(\{x, y\}) \times \{x, y\} / x, y \in X \}$

If  $\forall A \in [X]$ ,  $G(A, R)$  is non-empty valued then the associated game solution is called the  $R$ - based best solution on  $X$ . In future, whenever there is no scope for confusion, an  $R$ - based game solution will be simply referred to as a solution.

The top cycle solution denoted  $TC: [X] \rightarrow [X]$  is defined as follows:  $\forall A \in [X]: TC(A) = G(A, T(R|A))$ .

Given  $R \in \Pi$ ,  $A \in [X]$  and  $x, y \in X$ , we say that  $x$  covers  $y$  via  $R$  in  $A$  if :

(i)  $x, y \in A$ ; (ii)  $(x, y) \in P(R)$ ; (iii)  $\forall z \in A: [(y, z) \in R \text{ implies } (x, z) \in R]$  ; (iv)  $\forall z \in A: [(y, z) \in P(R) \text{ implies } (x, z) \in P(R)]$ .

Given  $R \in \Pi$ , let  $\hat{R}(A) = \{(x, y) \in A \times A / x \text{ covers } y \text{ via } R \text{ in } A\}$ . Let  $UC(A) = \{x \in A / \text{if } y \in A \text{ then } (y, x) \notin \hat{R}(A)\}$ . It is easy to see that  $\forall A \in [X]$ ,  $\hat{R}(A)$  is a transitive binary relation on  $A$ . Thus  $UC(A) \neq \emptyset$  whenever  $A \in [X]$ . Thus (i)  $\forall A \in [X]: UC(A) \subset A \forall A \in [X]$ ; (ii)  $\forall x, y \in X: x \in UC(A)$  if and only if  $(x, y) \in R$ .

The solution  $UC: [X] \rightarrow [X]$  is called the uncovered solution.

Given  $A \in [X]$  and  $x \in X$  let  $s(x, A) = \#\{y \in A / (x, y) \in P(R)\} - \#\{y \in A / (y, x) \in P(R)\}$ .

The Copeland solution  $Co: [X] \rightarrow [X]$  is defined as follows:  $\forall A \in [X]:$

$Co(A) = \{x \in A / \forall y \in A: s(x, A) \geq s(y, A)\}$ .

The following proposition is available in Laslier [1997]:

Proposition 1:  $\forall A \in [X]: Co(A) \subset UC(A) \subset TC(A)$ .

Example 2: Let  $X = \{x, y, z\}$  and let  $R = \Delta(X) \cup \{(x, y), (y, z), (z, y), (x, z), (z, x)\}$ . Now  $Co(X) = \{x\} \subset \{x, z\} = UC(X) \subset X = TC(X)$ . Further  $Co(X) \subset UC(X) \subset TC(X)$ .

### 3. Axioms for the Top Cycle Solution: A solution $S$ on $X$ is said to satisfy :

Strong Condorcet (SC) if  $\forall A \in [X]: [x \in A \text{ and } [\forall y \in A \setminus \{x\}: (x, y) \in P(R)]] \text{ implies } [S(A) = \{x\}]$ ;

Expansion Independence (EI) if  $\forall A \in [X]: [x \in S(A), y \in A, (y, z) \in R] \text{ implies } [x \in S(A \cup \{z\})]$ ;

Existence of an Inessential Alternative (EIA) if  $\forall A \in [X]$  with  $\#(A) \geq 2$  and  $\forall x \in S(A)$ , there exists  $y \in A$  (possibly depending on  $A$  and  $x$ ) such that  $x \in S(A \setminus \{y\})$ .

Theorem 1: The only solution on  $X$  which satisfies SC, EI and EIA is TC.

Proof: It is clear that TC satisfies SC, EI and EIA. Hence let  $S$  be any solution that satisfies SC, EI and EIA. Let  $A \in [X]$ . If  $\#(A)$  is one or two there is nothing to prove since  $S(A) = TC(A)$  by definition. Thus suppose  $S(A) = TC(A)$  whenever  $\#(A) = 1, \dots, k$ . Let  $\#(A) = k+1$ . Let  $x \in A$ . If  $\forall y \in A \setminus \{x\}: (x, y) \in P(R)$  then  $S(A) = \{x\} = TC(A)$ . Hence suppose  $\forall x \in A$  there exists  $y \in A \setminus \{x\}$  such that  $(y, x) \in R$ .

Let  $x \in TC(A)$ . Since TC satisfies EIA, there exists  $z \in A$  such that  $x \in TC(A \setminus \{z\})$ . By the induction hypothesis  $S(A \setminus \{z\}) = TC(A \setminus \{z\})$ . If  $(x, z) \in R$  then by EI,  $x \in S(A)$ . If  $(x, z) \notin R$ , then since  $x \in TC(A) = G(A, T(R|A))$  there exists  $w \in A$  such that  $(x, w) \in T(R|A)$  and  $(w, z) \in R$ . Then by EI once again  $x \in S(A)$ . Hence  $TC(A) \subset S(A)$ . Now suppose  $x \in S(A)$  and towards a contradiction suppose  $x \notin TC(A)$ . By EIA there exists  $z \in A$  such that  $x \in S(A \setminus \{z\})$ . By the induction hypothesis  $S(A \setminus \{z\}) = TC(A \setminus \{z\})$ . If  $(x, z) \in R$  then by EI applied to TC,  $x \in TC(A)$ . Hence suppose  $(x, z) \notin R$ . Thus  $(z, x) \in P(R)$ . Let  $y \in TC(A)$ . Clearly  $y \neq x$ . Suppose  $y \neq z$ . Thus  $y \in A \setminus \{z\}$ . Thus  $(x, y) \in T(R|A)$  which combined with  $y \in TC(A)$  gives us  $x \in TC(A)$ . Hence  $y = z$ . If for some  $w \in A \setminus \{x, z\}$  we had  $(w, z) \in R$ , then since  $x \in TC(A \setminus \{z\})$  and  $w \in A \setminus \{z\}$  we would get  $x \in TC(A)$ . Thus  $\forall w \in A: (z, w) \in P(R)$ . But then by SC,  $S(A) = \{z\}$ , contradicting  $x \in S(A)$ . Thus  $x \in TC(A)$ . Hence  $S(A) \subset TC(A)$ . Thus  $S(A) = TC(A)$ . By a standard induction argument it now follows that  $\forall A \in [X]: S(A) = TC(A)$ . ♣

A solution S on X is said to satisfy:

Converse Condorcet (CC) if  $\forall A \in [X]$  and  $x \in A: [\forall y \in A \setminus \{x\}: (y, x) \in P(R)]$  implies  $[x \in S(A)]$ ;

Weak Existence of an Inessential Alternative (WEIA) if  $\forall A \in [X]$  with  $\#(A) \geq 4$  and  $\forall x \in S(A)$ , there exists  $y \in A$  (possibly depending on A and x) such that  $x \in S(A \setminus \{y\})$ .

Since TC satisfies EIA it also satisfies WEIA. Infact we can now prove the following:

Theorem 2: Let S be any solution on X which satisfies SC, CC and WEIA. Then,  $\forall A \in [X]: S(A) \subset TC(A)$ .

Proof: Step 1: Let S be any solution on X which satisfies SC and CC. Then,  $\forall A \in [X]$  with  $\#(A) \leq 3: S(A) \subset TC(A)$ .

Proof of Step 1: For  $\#(A) \leq 2$  there is nothing to prove since by the definition of a solution all of them agree on such sets. Hence suppose  $\#(A) = 3$ . Let  $A = \{x, y, z\}$  with  $x \neq y \neq z \neq x$ . Suppose without loss of generality  $x \in S(A)$ . If  $(x, y), (x, z) \in R$ , then  $x \in TC(A)$ . Thus, suppose without loss of generality that  $(y, x) \in P(R)$ . If  $(z, x) \in P(R)$  then by CC,  $x \notin S(A)$ , contradicting what we have assumed. Hence  $(x, z)$  must belong to R. If  $(z, y) \in R$ , then again  $x \in TC(A)$ . If  $(y, z) \in P(R)$ , then by SC,  $S(A) = \{y\}$ , contradicting  $x \in S(A)$ . Thus  $S(A) \subset TC(A)$ .

Step 2: Let S be any solution on X such that  $\forall A \in [X]$  with  $\#(A) \leq 3: S(A) \subset TC(A)$ . Suppose S satisfies WEIA. Then,  $\forall A \in [X]: S(A) \subset TC(A)$ .

Proof of Step 2: Suppose that  $\forall A \in [X]$  with  $3 \leq \#(A) \leq m: S(A) \subset TC(A)$ . Let  $\#(A) = m+1$ . Thus  $\#(A) \geq 4$ . Let  $x \in S(A)$ . By WEIA, there exists  $y \in A$  such that  $x \in S(A \setminus \{y\})$ . By the induction hypothesis  $S(A \setminus \{y\}) \subset TC(A \setminus \{y\})$ . Thus,  $x \in TC(A \setminus \{y\})$ . If  $(x, y) \in R$ , then clearly  $x \in TC(A)$ . Hence,  $S(A) \subset TC(A)$ . Suppose  $(y, x) \in P(R)$ . If  $\forall z \in A \setminus \{y\}: (y, z) \in P(R)$ , then by SC,  $S(A) = \{y\}$ , contradicting  $x \in S(A)$ . Hence, there exists  $z \in A \setminus \{x, y\}$  such  $(z, y) \in R$ . Since,  $x \in TC(A \setminus \{y\})$  and  $z \in A \setminus \{y\}$ ,  $(z, y) \in R$  implies  $x \in TC(A)$ . Thus  $S(A) \subset TC(A)$ .

Step 2 combined with Step 1 and a standard induction argument proves the theorem. ♣

Infact, the above proof reveals the following:



**Theorem 3:** Let  $S$  be any solution on  $X$  which satisfies SC and EIA. Then,  $\forall A \in [X]$ :  $S(A) \subset TC(A)$ .

CC is not required once we replace WEIA by EIA, since then the induction argument can begin from  $\#(A) \geq 2$ .

**The Uncovered Solution:** A solution  $S$  on  $X$  is said to satisfy Expansion (E) if  $\forall A, B \in [X]$ :  $S(A) \cap S(B) \subset S(A \cup B)$ .

It is easy to see that both TC and UC satisfy E:

(i) Let  $A, B \in [X] \times \Pi$  and suppose  $x \in UC(A) \cap UC(B)$ . Towards a contradiction suppose that  $x \notin UC(A \cup B)$ . Hence there exists  $y \in A \cup B$ , such that  $y$  covers  $x$  via  $R$  in  $A \cup B$ . Without loss of generality suppose  $y \in A$ . Since  $x \in A$ ,  $y$  covers  $x$  via  $R$  in  $A$ . This contradicts  $x \in UC(A)$ . Thus UC satisfies E.

(ii) Let  $A, B \in [X]$  and suppose  $x \in TC(A) \cap TC(B)$ . Towards a contradiction suppose that  $x \notin TC(A \cup B)$ . Hence there exists  $y \in A \cup B$ , such that  $(x, y) \notin T(R | A \cup B)$ . Without loss of generality suppose  $y \in A$ . Thus  $(x, y) \notin T(R | A \cup B)$  implies that  $(x, y) \notin T(R | A)$ . This contradicts  $x \in TC(A)$ .

Moulin [1986] has established the following:

**Proposition 2:** Let  $S$  be any solution satisfying SC and E. If  $[\forall A \in [X]$  with  $\#(A)=3$  we have  $UC(A) \subset S(A)]$  then  $[\forall A \in [X]: UC(A) \subset S(A)]$  (see Appendix for details).

A solution  $S$  on  $X$  is said to satisfy Contraction (Con) if  $\forall A \in [X]$  with  $\#(A) \geq 4$ ,  $[x \in S(A)]$  implies [there exists a positive integer  $K \geq 2$  and sets  $A_1, \dots, A_K \in [A] \setminus \{A\}$  such that (i)  $\cup \{A_k / k=1, \dots, K\} = A$ ; (ii)  $x \in \cap \{S(A_k) / k=1, \dots, K\}$ ].

Dutta and Laslier [1999] establish that UC satisfies Con. However, TC does not as the following example reveals:

**Example 3:** Let  $X = \{x, y, z, w\}$  where  $x, y, z, w$  are all distinct. Let  $R = \Delta(X) \cup \{(x, y), (z, x), (w, x), (y, z), (w, y), (z, w)\}$ . Clearly,  $x \in TC(X)$ . Let  $A \in [X] \setminus \{X\}$ , with  $\#(A) \geq 2$ . Suppose,  $y \notin A$ . Then,  $x \notin TC(A)$ . Hence,  $x \in TC(A)$  and  $\#(A) \geq 2$  implies  $y \in A$ . Suppose  $x, y \in A \cap B$  where  $A, B \in [X] \setminus \{X\}$ ,  $A \neq B$ ,  $A \not\subset B \not\subset A$ . Without loss of generality suppose  $A = \{x, y, z\}$  and  $B = \{x, y, w\}$ . Then,  $x \notin TC(B)$ . Thus, TC does not satisfy Con.

A solution  $S$  on  $X$  is said to satisfy:

**Tie Splitting (TS)** if  $\forall A, B \in [X] \times \Pi$  with  $A \cap B = \emptyset$ :  $[A \times B \subset I(R)]$  implies  $S(A \cup B) = S(A) \cup S(B)$ ;

**Strong Type 1 Property (ST1P)** if  $\forall x, y, z \in X$ ;  $[(y, x) \in P(R), (x, z) \in P(R), (z, y) \in R]$  implies  $S(\{x, y, z\}) = \{x, y, z\}$ .

**Note:** Let  $S$  be any solution satisfying E. If  $S$  satisfies ST1P then,  $[\forall A \in [X]$  with  $\#(A)=3$  we have  $UC(A) \subset S(A)$ ].

**Proposition 3 :-** Let  $S$  be a solution on  $X$  such that  $S(A) = UC(A) \forall A \in [X]$ . Then  $S$  satisfies SC, CC, TS, ST1P, E and Con.

**Proof :** We have already seen that UC satisfies E, and SC, CC, TS, ST1P being easy to verify let us show that  $S$  satisfies Con. Let  $A \in [X]$  with  $\#(A) \geq 4$  and  $x \in S(A)$ .

Thus,  $y \in A$ ,  $y \neq x$  implies either  $[(x, y) \in R]$  or [there exists  $z_y \in A$  with either  $((x, z_y) \in R$  and  $(y, z_y) \notin R)$  or  $((x, z_y) \in P(R)$  and  $(y, z_y) \notin P(R))$ ]. Let  $A_0 = \{y \in A / (x, y) \in$

R}. Clearly  $A_0 \neq \phi$ , since  $x \in A_0$ . Further, since there does not exist  $y \in A_0$ , such that  $y$  covers  $x$  via  $R$  in  $A_0$ ,  $x \in S(A_0)$ .

Case 1:-  $A_0 = A$ . Since  $\#(A) \geq 4$ , there exists  $\bar{y} \in A \setminus \{x\}$  such that  $A \setminus \{x, \bar{y}\} \neq \phi$ . Let  $A_1 = \{x, \bar{y}\}$  and  $A_2 = A - \{\bar{y}\}$ . Clearly  $A_1 \subset \subset A$ ,  $A_2 \subset \subset A$  and  $A_1 \cup A_2 = A$ . Further  $x \in S(A_1) \cap S(A_2)$ .

Case 2 :  $A_0 \subset \subset A$ . In this case, let  $A_1 = A_0$  and for  $y \in A \setminus A_1$ , let  $A_y = \{x, y, z_y\}$ . Since  $\#(A) \geq 4$ ,  $A_y \subset \subset A$  whenever  $y \in A \setminus A_1$ . Further,  $\forall y \in A \setminus A_1 : x \in S(A_y)$ . Also,  $A_1 \cup \left( \bigcup_{y \in A \setminus A_1} A_y \right) = A$ . Hence  $S$  satisfies Con. ♣

**Lemma 1 :** If  $\#(X) \leq 3$  and  $S$  is a solution on  $X$  which satisfies SC, TS and ST1P, then  $S$  is the uncovered solution.

**Proof :** Let  $S$  and  $X$  be as in the statement of the lemma. If  $\#(X) = 1$  or  $2$ , there is nothing to prove since  $S(A) = UC(A) \forall A \in [X]$  by the definition of a solution. Hence suppose  $\#(X) = 3$ . If  $A \in [X]$  and  $\#(A) \leq 2$ , then  $S(A) = UC(A)$ . Thus suppose  $A = X = \{x, y, z\}$  with  $x \neq y \neq z \neq x$ . If there exists  $a \in X : (a, b) \in P(R) \forall b \in X$ , then  $S(X) = \{a\} = UC(X)$ , by SC of both  $S$  and  $UC$ . Hence suppose that  $\forall a \in X$ , there exists  $b \in X \setminus \{a\} : (b, a) \in R$ .

Case 1 :  $I(R) = X$ . Then by TS of  $C$  and  $UC$ ,  $S(X) = UC(X) = X$ .

Thus without loss of generality suppose,  $(x, y) \in P(R)$ . Hence, by what has been mentioned before Case 1,  $(z, x) \in R$ .

Case 2 :  $(z, x), (y, z) \in P(R)$ .

By ST1P,  $S(X) = \{x, y, z\} = UC(X)$ .

Case 3 :  $(z, x) \in P(R), (y, z) \in I(R)$ .

By ST1P,  $S(X) = \{x, y, z\} = UC(X)$ .

Case 4 :  $(z, x) \in I(R), (y, z) \in P(R)$ .

By ST1P,  $S(X) = \{x, y, z\} = UC(X)$ .

Case 5 :  $(z, x) \in I(R), (y, z) \in I(R)$ .

Thus,  $\{z\} \times \{x, y\} \subset I(R)$ .

By TS,  $S(X) = S(\{z\}) \cup S(\{x, y\}) = \{x, z\} = UC(X)$ .

This proves Lemma 1. ♣

A look at the proof of Lemma 1 reveals that we have essentially proved the following:

**Lemma 2 :** Let  $S$  be a solution on  $X$  which satisfies SC, TS and ST1P. Then  $\forall A \in [X]$  with  $\#(A) \leq 3$ ,  $S(A) = UC(A)$ .

The above observation follows by noting that  $UC(A)$  depends on the restriction of  $R$  to  $A$  only.

**Note :** If in Lemma 1 (or for that matter in Lemma 2), we replace SC by CC and E we do not get the desired result as the following example reveals :

**Example 4:** Let  $X = \{x, y, z\}$  with  $x \neq y \neq z \neq x$ . Let  $S(X) = \{x, y\}$ , where  $R = \Delta(X) \cup \{(y, x), (y, z), (x, z)\}$ .  $S$  satisfies CC, E, TS and ST1P, the last two properties being satisfied vacuously. However,  $UC(X) = \{y\} \neq S(X)$ . Note that  $S$  does not satisfy SC, since  $(y, x), (y, z) \in P(R)$  and yet  $S(X) \neq \{y\}$ .

In Dutta and Laslier [1999] we find the following property for a solution  $S$  on  $X$ :  
 Type One Property (T1P) :  $\forall x, y, z \in X: [(y,x) \in P(R), (x,z) \in P(R), (z,x) \in I(R)]$   
 implies  $S(\{x, y, z\}) = \{x, y, z\}$ .

Clearly T1P is weaker than ST1P. However, if we replace ST1P by T1P in Lemma 1 (: or Lemma 2), we do not get the desired result as the following example reveals.

Example 5: Let  $X = \{x, y, z\}$  with  $x \neq y \neq z \neq x$ . Let  $S(X) = \{x\}$ , where  
 $R = \Delta(X) \cup \{(x,y), (y,z), (z,x)\}$ . Clearly  $S$  satisfies SC, TS, E, CC and T1P (: all  
 vacuously). However,  $S$  violates ST1P which under the present situation would  
 require  $S(X) = X$ . Further  $S(X) \neq UC(X) = X$ .

We are now equipped to prove the following theorem:

Theorem 4 : A solution  $S$  on  $X$  is the uncovered solution if and only if  $S$  satisfies  
 SC, TS, ST1P, E and Con.

Proof : Proposition 3 tells us that an uncovered solution satisfies all the properties  
 mentioned in the theorem. Hence let  $S$  be a solution on  $X$  satisfying SC, TS, ST1P  
 and Con. Let  $R \in \Pi$ . By Lemma 2,  $S(A) = UC(A) \forall A \in [X]$  with  $\#(A) \leq 3$ . Suppose  
 $S(A) = UC(A) \forall A \in [X]$  with  $\#(A) = 1, \dots, m$ , and let  $B \in [X]$  with  $\#(B) = m+1$ . Let  $x \in$   
 $S(B)$ . Suppose  $m+1 \geq 4$ , for otherwise there is nothing to prove. Hence by Con there  
 exists a positive integer  $K$  and non-empty proper subsets  $B_1, \dots, B_K$  such that  $B =$

$\bigcup_{i=1}^K B_i$  and  $x \in \bigcap_{i=1}^K S(B_i, R)$ . Clearly  $\#(B_i) \leq m$  whenever  $i \in \{1, \dots, K\}$ .

By our induction hypothesis,  $S(B_i) = UC(B_i) \forall i \in \{1, \dots, K\}$ . Thus  $x \in \bigcap_{i=1}^K UC(B_i)$ , and

by E,  $x \in UC(B)$ . Thus,  $S(B) \subset UC(B, R)$ . By an exactly similar argument with the  
 roles of  $S$  and  $UC$  interchanged, we get  $UC(B) \subset S(B)$ . By a standard induction  
 argument, the theorem is established. ♣

Note : The above theorem is not valid without E or Con.

Example 6: Let  $X = \{x, y, z, w\}$  where all of them are distinct. Let  $S(X) = \{x\}$ ,  $S(A) = A$   
 if  $\#(A) = 3$ , where  $R = \Delta(X) \cup \{(x,y), (y,z), (z,w), (w,x), (x,z), (z,x), (y,w), (w,y)\}$ .  $S$  satisfies  
 SC, ST1P, TS (vacuously). Further, let  $A_1 = \{x, y\}$  and  $A_2 = \{x, z, w\}$ .  $x \in S(X)$  and  $x \in$   
 $S(A_1) \cap S(A_2)$ . Further  $A_1 \cup A_2 = X$ , with  $A_1 \subset \subset X$  and  $A_2 \subset \subset X$ . Thus  $S$  satisfies  
 Con. However,  $UC(X) = X \neq \{x\} = S(X)$ . Observe that,  $S$  does not satisfy E, since  $y \in$   
 $S(\{x, y, z\}) \cap S(\{y, z, w\})$  but  $y \notin S(X)$ .

Example 7: Let  $X$  be as above. Let  $S(X) = \{x, y\}$ ,  $S(A) = \{x\}$  if  $x \in A$ ,  $S(A) = A$  if  $x \notin A$   
 where  $R = \Delta(X) \cup (\{x\} \times X) \cup (\{y, z, w\} \times \{y, z, w\})$ . Clearly  $S$  satisfies SC, ST1P (:vacuously),  
 TS and E. But  $S$  does not satisfy Con :  $y \in S(X)$ . If we take any finite number of non-  
 empty proper subsets of  $X$  whose union is  $X$ , atleast one must contain 'x' and thus  
 its choice set cannot contain 'y'.

5. Choice Functions and Extensions of Rational Choice Theory : In this section we  
 discuss the implications of the analysis reported in earlier sections of this paper, in  
 the context of classical rational choice theory.

A choice function (on  $X$ ) is a function  $C: [X] \rightarrow [X]$  such that :  $\forall A \in [X]: C(A) \subset A$ .

Given a choice function  $C$ , the binary relation revealed by  $C$  denoted  $R^C$  is defined

as follows:  $R^C \equiv \{(x,y) / x \in C(\{x,y\})\}$ . Clearly,  $R^C \in \Pi$ . A choice function  $C$  is said to be a top-cycle choice function if  $\forall A \in [X]: C(A) = G(A, T(R^C | A))$ . A choice function  $C$  is said to be an uncovered choice function if  $\forall A \in [X]: C(A) = \{x \in A / \text{if } y \in A \text{ then } (y, x) \notin \hat{R}^C(A)\}$ , where  $\hat{R}^C(A) = \{(x, y) \in A \times A / x \text{ covers } y \text{ via } R^C \text{ in } A\}$ .

A choice function  $C$  is said to satisfy:

**Strong Condorcet (SC)** if  $\forall A \in [X]: [x \in A] \text{ and } [\forall y \in A \setminus \{x\}: (x,y) \in P(R^C)]$  implies  $[C(A) = \{x\}]$ ;

**Expansion Independence (EI)** if  $\forall A \in [X]: [x \in C(A), y \in A, (y,z) \in R^C]$  implies  $[x \in C(A \cup \{z\})]$ ;

**Existence of an Inessential Alternative (EIA)** if  $\forall A \in [X]$  with  $\#(A) \geq 2$  and  $\forall x \in C(A)$ , there exists  $y \in A$  (possibly depending on  $A$  and  $x$ ) such that  $x \in C(A \setminus \{y\})$ .

As a consequence of the analysis reported earlier it follows that:

**Theorem 5:** A choice function  $C$  is a top cycle choice function if and only if  $C$  satisfies SC, EI and EIA.

A choice function  $C$  is said to satisfy:

**Converse Condorcet (CC)** if  $\forall A \in [X]$  and  $x \in A: [\forall y \in A \setminus \{x\}: (y,x) \in P(R^C)]$  implies  $[x \in C(A)]$ ;

**Weak Existence of an Inessential Alternative (WEIA)** if  $\forall A \in [X]$  with  $\#(A) \geq 4$  and  $\forall x \in S(A)$ , there exists  $y \in A$  (possibly depending on  $A$  and  $x$ ) such that  $x \in C(A \setminus \{y\})$ .

Since TC satisfies EIA it also satisfies WEIA. In fact we can now prove the following:

**Theorem 6:** Let  $C$  be any choice function which satisfies SC, CC and WEIA. Then,  $\forall A \in [X]: C(A) \subset G(A, T(R^C | A))$ .

**Theorem 7:** Let  $C$  be any choice function which satisfies SC and EIA. Then,  $\forall A \in [X]: C(A) \subset G(A, T(R^C | A))$ .

CC is not required once we replace WEIA by EIA.

A choice function  $C$  is said to satisfy **Expansion (E)** if  $\forall A, B \in [X]: C(A) \cap C(B) \subset C(A \cup B)$ .

The relevant proposition in Moulin [1986] now translates to the following:

**Proposition 4:** Let  $C$  be any choice function satisfying SC and E. If  $[\forall A \in [X]$  with  $\#(A)=3$  we have  $\{x \in A / \text{if } y \in A \text{ then } (y, x) \notin \hat{R}^C(A)\} \subset C(A)]$  then  $[\forall A \in [X]: \{x \in A / \text{if } y \in A \text{ then } (y, x) \notin \hat{R}^C(A)\} \subset C(A)]$ .

A choice function  $C$  is said to satisfy **Contraction (Con)** if  $\forall A \in [X]$  with  $\#(A) \geq 4, [x \in C(A)]$  implies  $[ \text{there exists a positive integer } K \geq 2 \text{ and sets } A_1, \dots, A_K \in [A] \setminus \{A\} \text{ such that (i) } \cup \{A_k / k=1, \dots, K\} = A; \text{(ii) } x \in \cap \{C(A_k) / k=1, \dots, K\}]$ .

A choice function  $C$  is said to satisfy:

**Tie Splitting (TS)** if  $\forall A, B \in [X] \times \Pi$  with  $A \cap B = \phi: [A \times B \subset I(R^C)]$  implies  $C(A \cup B) = C(A) \cup C(B)$ ;

**Strong Type 1 Property (ST1P)** if  $\forall x, y, z \in X: [(y,x) \in P(R^C), (x,z) \in P(R^C), (z,y) \in R^C]$  implies  $C(\{x, y, z\}) = \{x, y, z\}$ .

We now have the following theorem:

**Theorem 8 :** A choice function  $C$  is an uncovered choice function if and only if  $C$  satisfies SC, TS, ST1P, E and Con.

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Appendix

Here we provide a concise proof of Proposition 1.

**Proof of Proposition 1:** Clearly the proposition holds for  $\#(A) = 1$  or 2. Hence assume that the proposition holds for  $\#(A) = 1, \dots, K$  and now let  $\#(A) = K+1$ . Suppose  $x \in \text{Co}(A) \cup \text{UC}(A)$ . If,  $[\forall y \in A \setminus \{x\}: (x, y) \in R]$  then,  $x \in \text{TC}(A)$ . Hence, suppose that there exists  $y \in A \setminus \{x\}$  such that  $(y, x) \in P(R)$ . Clearly,  $x \in \text{Co}(A \setminus \{x\}) \cup \text{UC}(A \setminus \{x\})$ . By the induction hypothesis  $x \in \text{TC}(A \setminus \{x\}, R)$ . If  $[\forall z \in A \setminus \{y\}: (y, z) \in P(R)]$ , then  $\text{Co}(A) \cup \text{UC}(A) = \{y\}$ , contradicting  $x \in \text{Co}(A) \cup \text{UC}(A)$ . Hence there exists  $z \in A \setminus \{y\}: (z, y) \in R$ . Since  $z \in A \setminus \{y\}$  and  $x \in \text{TC}(A \setminus \{x\})$  clearly,  $(x, z) \in T(R|A)$ . Thus  $(x, y) \in T(R|A)$ . Thus,  $\text{TC}(A)$ . The proposition now follows by induction on the cardinality of A.

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