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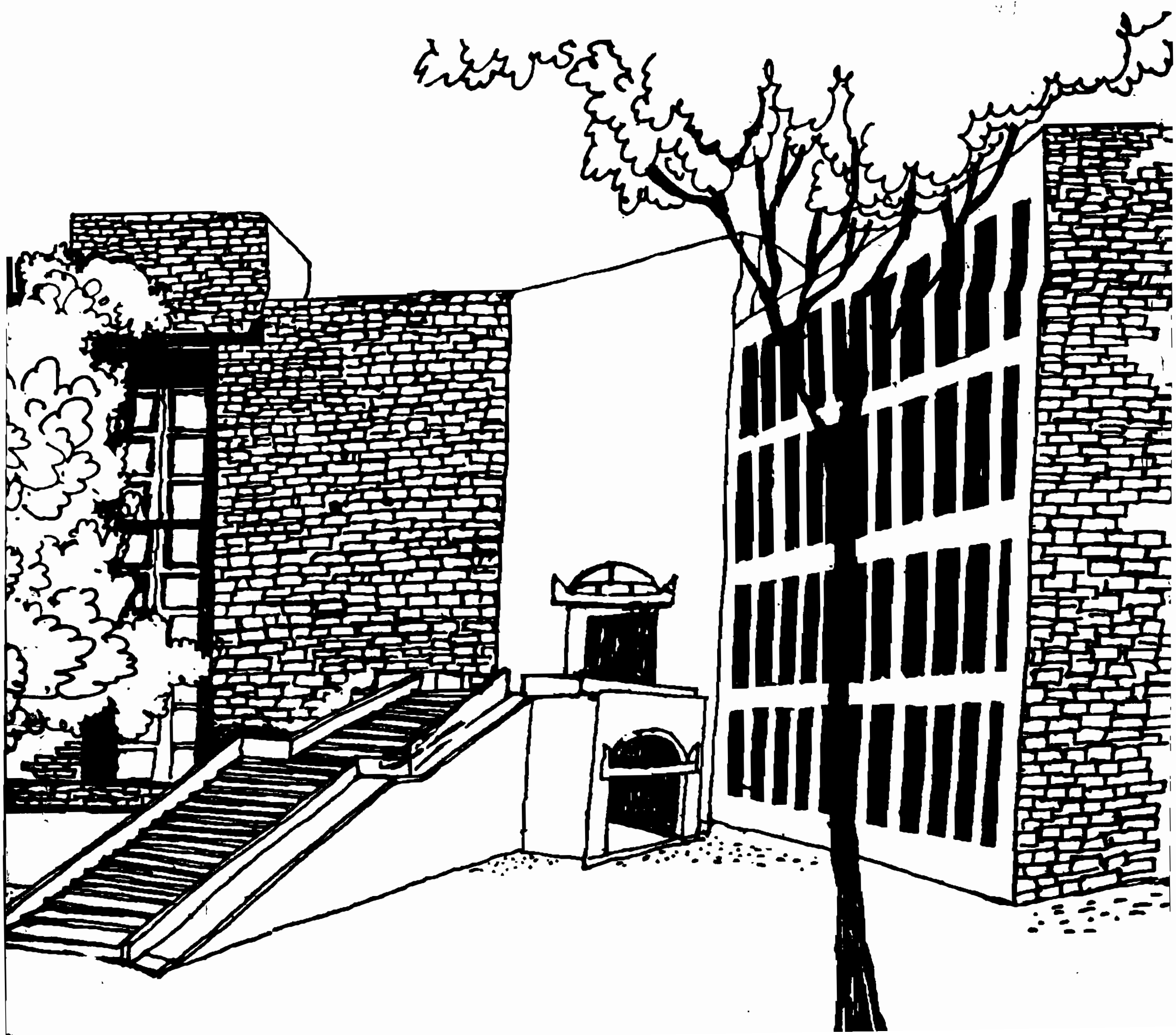

VECTOR OPTIMIZATION WITH MULTIPLE CONSTRAINTS

By

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Abstract

In this paper we show that a vector optimization problem with convex constraint functions which satisfy a constraint qualification can be reduced to a vector optimization problem with a single constraint, if the objective function satisfy a certain generalization of quasi-concavity.

Vector Optimization With Multiple Constraints

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1. Introduction :- Microeconomic theory is largely concerned with solving optimization problems related to resource allocation. Thus the theory of optimization has found applications in microeconomics like nowhere else.

   In solving optimization problems, one is lead to setting up the Lagrangean for the problem and then setting the partial derivatives of the Lagrangean with respect to the variables and the Lagrange multipliers equal to zero. The first order necessary conditions for constrained optimization are usually obtained by applying the Implicit Function Theorem. However, if there is just one binding constraint, it is easily established that to establish the first order necessary conditions, one does not require the use of the Implicit Function Theorem (see Lahiri [2000]). The proof of the Implicit Function Theorem when compared to other theorems is advanced calculus, is far more complicated. Thus, it would enhance ones understanding of the proof the necessary conditions for constrained optimization, if the use of the Implicit Function Theorem could be avoided. Hence, it would be welcome if an optimization problem with multiple constraints could be reduced to an optimization problem with a single constraint. Luenberger [1968] shows that such is indeed possible if the objective function is quasi-concave, the constraints are convex and satisfy a standard constraint qualification.

   In this paper we extend this result to vector optimization problems. Such problems naturally arise when we seek Pareto efficient allocations of resources in welfare economics. The optimal solutions are usually found by considering non-negative linear combinations of the objective functions of the agents among whom resources are meant to be distributed. One could alternatively reduce a vector optimization problem with multiple constraints to a vector optimization problem with a single constraint and then proceed to form the associated Lagrangean function of the reduced vector optimization problem. This is precisely the motivation behind this paper.

2. The Mathematical Theory : Let X be a non-empty convex subset of $\mathbb{R}^n$, $u : X \to \mathbb{R}^k$ and $g : X \to \mathbb{R}^m$ be continuous functions, where n, k and m are positive

integers. Let C be a non-empty open subset of $\mathbb{R}^k$ satisfying the following properties :

(i) $x \in cl(C)$, $y \in C$ implies $x+y \in C$;

(ii) $x \in C$, $y \in C$ implies $z \in C$ where $\forall i \in \{1,\ldots,k\}$: $z_i = \min\{x_i , y_i\}$.

Let $\mathbb{R}_{++} = \{x \in \mathbb{R} / x>0\}$, $\mathbb{R}^k_{++} = \{x \in \mathbb{R}^k / x_i>0 \forall i \in \{1,\ldots,k\}\}$ and $\mathbb{R}^k_{+} = \{x \in \mathbb{R}^k / x_i \geq 0 \ \forall i \in \{1,\ldots,k\}\}$.

Assumption : (1) u is continuous and C-quasi-concave i.e. $\forall$ x,y$\in$X and t$\in$[0,1] : $u(tx+(1-t)y) - (\min\{u_1(x), u_1(y)\}, \ldots, \min\{u_k(x), u_k(y)\}) \in cl(C)$;

(2) g is continuous and convex i.e. $\forall$x,y$\in$X and t$\in$[0,1]:$g_i(tx+(1-t)y \leq tg_i(x)+(1-t)g_i(y) \forall i \in \{1,\ldots,m\}$;

(3) g satisfies the following constraint qualification : there exists $\bar{x} \in X$ such that $g_i(\bar{x})<0 \forall i \in \{1,\ldots,m\}$.

The triplet [u,g,C] is called a vector optimization problem. Say that $x^* \in X$ solves the vector optimization problem [u,g,C] if : (a) $g(x^*) \leq 0 \ \forall i \in \{1,\ldots,m\}$; (b) there does not exist $x \in X$ with $g_i(x) \leq 0 \forall i \in \{1,\ldots,m\}$ and $u(x) - u(x^*) \in C$.

Theorem :- Suppose x* solves the vector optimization [u,g,C]. Then there exists non-negative real numbers $\lambda_1,\ldots,\lambda_m$ not all zero such that :

(a) $\sum_{i=1}^{m} \lambda_i g_i(x^*) \leq 0$;

(b) there does not exist $x \in X$ with $\sum_{i=1}^{m} \lambda_i g_i(x) \leq 0$ and $u(x)-u(x^*) \in C$.

Proof :- Let $A = \{z \in \mathbb{R}^m / z_i \geq g_i(x) \forall i \in \{1,\ldots,m\}$ and $x \in X\}$ and $B = \{z \in \mathbb{R}^m / z_i \geq g_i(x) \forall i \in \{1,\ldots,m\}$, $x \in X$ and $u(x)-u(x^*) \in C\}$.

If $B=\phi$, then there does not exist $x \in C$ with $u(x)-u(x^*) \in C$. Hence by taking $\lambda_i=1$, $i \in \{1,\ldots,m\}$ we manage to obtain a proof of the theorem.

Hence suppose $B \neq \phi$. Since $0 \in A$, $A \neq \phi$. Since each g_i is convex, so is A. Let $z,z' \in B$ and $t \in [0,1]$. Thus there exists $x,x' \in X$ with (a) $z_i \geq g_i(x) \forall i \in \{1,\ldots,m\}$; (b) $z'_i \geq g_i(x') \forall i \in \{1,\ldots,m\}$; (c) $u(x)-u(x^*) \in C$; (d) $u(x')-u(x^*) \in C$. Thus $tz_i + (1-t)z_i' \geq tg_i(x) + (1-t)g_i(x') \geq g_i(tx+(1-t)x') \forall i \in \{1,\ldots,m\}$. Further, $u(tx+(1-t)x') - (\min\{u_1(x), u_1(x')\}, \ldots, \min\{u_k(x), u_k(x')\}) \in cl(C)$,by C-quasi concavity of u. Since by hypothesis, $u(x)-u(x^*) \in C$ and $u(x')-u(x^*) \in C$, property (ii) of C gives,$(\min\{u_1(x)-u_1(x^*), u_1(x')-u_1(x^*)\}, \ldots ,\min\{u_k(x)-u_k(x^*),u_k(x')-u_k(x^*)\}) \in C$. Hence, $(\min\{u_1(x),u_1(x')\}, \ldots ,\min\{u_k(x), u_k(x')\})-u(x^*) \in C$. Hence by property (I) of C,$u(tx + (1-t)x')-u(x^*) \in C$. Thus $tz+(1-t)z' \in B$. Thus B is convex. By hypothesis $A \cap B=\phi$. Hence by the separating hyperplane theorem due to Minkowski (see de la Fuente [2000]), there exists $\lambda \in \mathbb{R}^m \backslash \{0\}$: $\forall z \in B$ and $z' \in A$: $\lambda z \geq \lambda z'$. Since $z \in B$ and $z' \in \mathbb{R}^m$ with $z_i' \geq z_i \ \forall \ i \in \{1,\ldots, m\}$ implies $z' \in B$, we must have $\lambda_i \geq 0 \ \forall i \in \{1,\ldots,m\}$. Since $0 \in A$: $\lambda z \geq 0 \ \forall \ z \in B$. Since $g_i(x^*) \leq 0 \ \forall i \in 1,..,m$: $\sum_{i=1}^{m} \lambda_i g_i(x^*) \leq 0$. Towards a contradiction suppose there exists $x \in X$ with $\sum_{i=1}^{m} \lambda_i g_i(x) \leq 0$ and $u(x)-u(x^*) \in C$.

Let $x^t = x + t(\bar{x}-x)$ where $g_i(\bar{x}) < 0\ \forall i \in \{1,...,m\}$ and let $t\in(0,1)$. Since X is convex, $x^t \in X$. Further $\sum_{i=1}^{m} \lambda_i g_i(x^t) \le \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{m} t\lambda_i [g_i(\bar{x}) - g(x)]$

$=(1-t) \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) < 0$ since $\lambda \in \mathbb{R}^m_+ \backslash \{0\}$ and $g_i(\bar{x})<0\ \forall i\in\{1,...,m\}$. By the continuity of u, for t sufficiently close to zero (: but greater than zero) $u(x^t) - u(x^*) \in C$. This is true since C is open in $\mathbb{R}^k$, $u(x)-u(x^*)\in C$ and $\lim_{t\to\infty} u(x^t) - u(x^*)$

$= u(x) - u(x^*)$. Thus $g(x^t)\in B$ with $\sum_{i=1}^{m} \lambda_i g_i(x^t) < 0$, contradicting $\lambda z \ge 0\ \forall z\in B$.

This proves the theorem.
Q.E.D.

A function $f:X\to\mathbb{R}$ is said to be quasi-concave if it is $\mathbb{R}_{++}$ - quasi-concave.
Corollary 1 (Luenberger [1968]): Suppose x^* solves the (vector) optimization problem $[u,g,\mathbb{R}_{++}]$. Then there exists non-negative real numbers $\lambda_1,...,\lambda_m$ not all zero such that x^* solves the following problem:

$u(x) \to \max$

s.t. $\sum_{i=1}^{m} \lambda_i g_i(x) \le 0$, $x\in X$.

Corollary 2 : Suppose x^* solves the vector optimization problem $[u,g,\mathbb{R}^k_{++}]$. Then there exists non-negative real numbers $\lambda_1,...,\lambda_m$ not all zero such

(a) $\sum_{i=1}^{m} \lambda_i g_i(x^*) \le 0$;

(b) there does not exist $x\in X$ with $\sum_{i=1}^{m} \lambda_i g_i(x) \le 0$ and $u_j(x)>u_j(x^*)\forall j\in\{1,...,k\}$.

3. Economic Applications :- This section is an adaptation of similar examples in Luenberger [1995].
   Example 1 :- Suppose that the preferences of a consumer is described by a multi objective utility function $u : \mathbb{R}^n_+ \to \mathbb{R}_k$ where $\mathbb{R}^n_+$ is the set of all possible consumption bundles of n-commodities and the consumer evaluates each consumption bundle by a set of k-criteria. Let $p_i > 0$ be the market price of the i^{th} commodity and let $w > 0$ be his/her disposable income. In addition suppose that the consumer has been issued a number of ration points that may be used along with money for the purchase of commodities. Suppose $d>0$ is the total number of ration points available with the consumer and let $v_i \ge 0$, be the point value assigned to the i^{th} commodity. Hence the constrained set faced by the consumer is given by

$$\sum_{i=1}^{n} p_i x_i \le w, \sum_{i=1}^{n} v_i x_i \le d, x\in\mathbb{R}^n_+$$

The objective of the consumer is to choose $x^* \in \mathbb{R}^n_+$ which satisfies the two inequalities such that there is no other x satisfying the constraints and yields $u_i(x)>u_i(x^*)$ $\forall$ $i \in \{1,\ldots,k\}$.

Example 2 : Enjoyment of some commodities (or activities) require time. Thus if in Example 1, we interpret $d>0$ as the total time available for consumption and $v_i \geq 0$ as the time required to consume one unit of the i^{th} commodity, then the problem transforms itself into one where time is a constraint and has to be allocated among the different commodities during consumption. Here we are assuming that no two different commodities can be consumed simultaneously.

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