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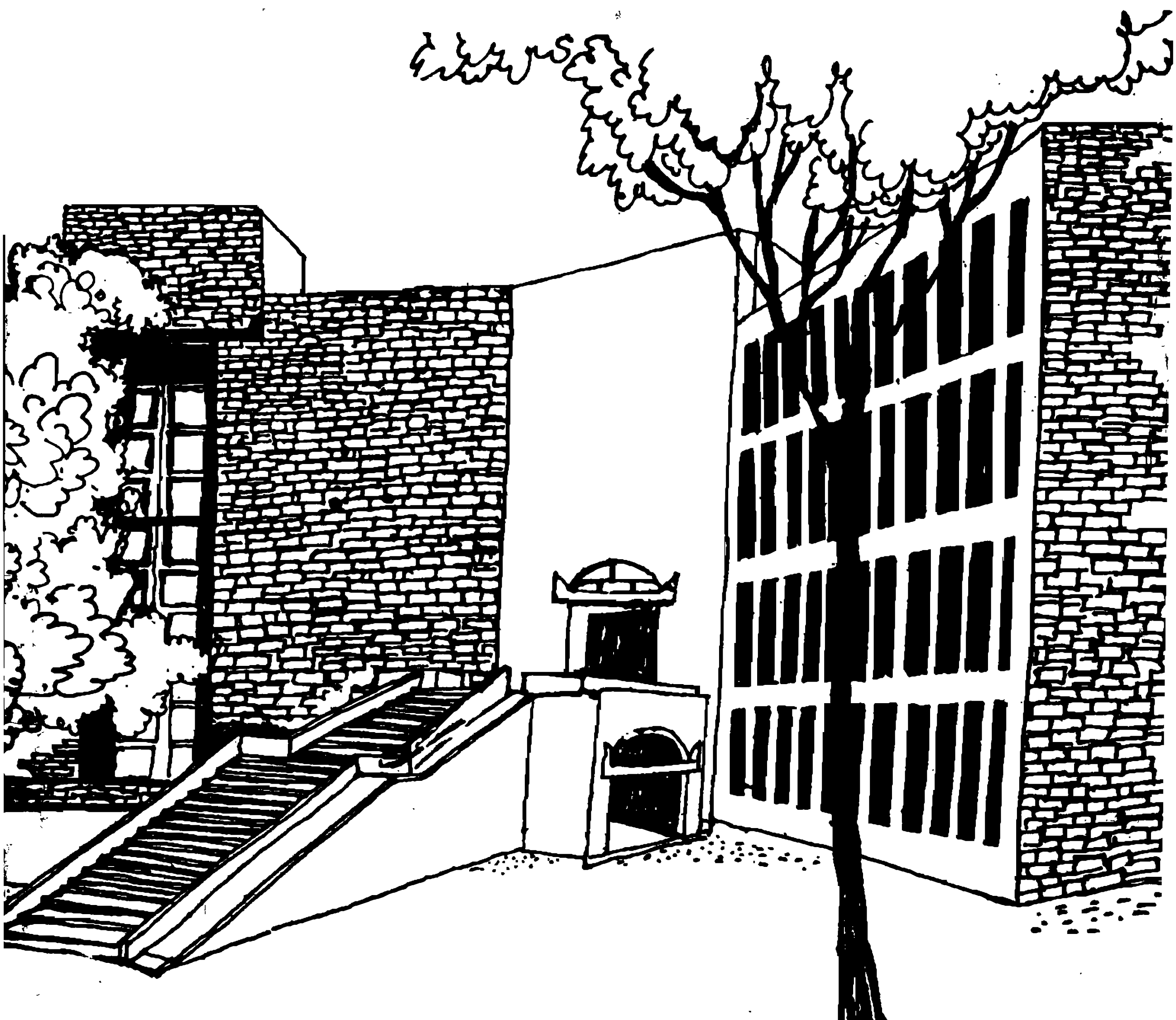

The Weighted Fair Division Problem

By

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ABSTRACT OF
The Weighted Fair Division Problem

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The exact problem we are concerned with in this paper is of the following nature. There are a finite number of producers each equipped with a utility function of the standard variety, which converts an input into a producer specific output. An allocation of the input among the producers is sought which is Pareto efficient i.e. there is no reallocation which increases the output of one producer without decreasing the output of any other. This, as is very widely known, corresponds to maximizing the weighted sum of the utility functions subject to a resource constraint. Alternatively, the weights can be interpreted as exogenously specified prices of the separate outputs and then the problem reduces to maximizing the aggregate revenue subject to a resource constraint. Our analysis focuses on the relations between the optimal solutions and the price and aggregate resource pair. Further, we also study the effect on the former of varying the latter pair.

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Introduction :- Formal graduate education of most economists begins by an exhaustive study of consumer choice theory. The paradigm that is generally favored is one where given a vector of prices and income, a rational agent equipped with a utility function, maximizes it subject to the budget constraint that the prices and income imply. The utility function is supposed to reflect the preferences of the consumer. An adequate analysis of the theory for our purposes can be found in Luenberger [1995]. If the utility function is interpreted as a rule which transforms inputs into a desirable output, the same model of consumer choice can be used to model the behaviour of an agent who seeks to maximize output (:or for that matter revenue at exogenously given prices for the output he produces), subject to the cost of production not exceeding a given investment (: which is irreversible). In this paper we investigate a closely related model which is meant to depict the problem of allocating a given amount of a single homogeneous resource among a finite number of producers i.e. the problem of fair division of a single commodity. A rather lucid introduction to the main concerns of this problem can be found in Moulin and Thomson [1997].

The exact problem we are concerned with in this paper is of the following nature. There are a finite number of producers each equipped with a utility function of the standard variety, which converts an input into a producer specific output. An allocation of the input among the producers is sought which is Pareto efficient i.e. there is no reallocation which increases the output of one producer without decreasing the output of any other. This, as is very widely known, corresponds to maximizing the weighted sum of the utility functions subject to a resource constraint. Alternatively, the weights can be interpreted as exogenously specified prices of the separate outputs and then the problem reduces to maximizing the aggregate revenue subject to a resource constraint. Our analysis focuses on the relations between the optimal solutions and the price and aggregate resource pair. Further, we also study the effect on the former of varying the latter pair.

The axiomatic study of resource allocation problems started off with the seminal work of Nash[1950]. Peters[1992] shows that a fair division problem such as what has been discussed above is representable as a kind of problem that Nash based his study on. In Lahiri[1996, 1998] we note that the converse is also true: a problem of the kind that Nash was concerned with is representable as a fair division problem.

The significance of this paper lies in adapting the methods used in the study of consumer choice to analyse problems of fair division. Like consumer choice theory we are able to establish the upper-hemicontinuity of both the weighted social choice rule and the dual weighted social choice rule and the continuity of the primal and dual value functions. Sensitivity properties of the value functions, similar to the sensitivity properties of the indirect utility function of consumer choice theory (i.e. the primal value function) and the expenditure function of consumer choice theory (i.e. the dual value function) are established in this framework. However, the primal value function for fair division problems seems to behave like the expenditure function of consumer choice theory and the dual value function (i.e. the expenditure framework) in our framework has a behaviour akin to the indirect utility function of consumer choice theory.

A consequence of this analysis is the observation that classical consumer choice theory in general and its associated analytical techniques in particular are very rich in scope and content.

2. The Model :- Let there be 'n' agents where 'n' is some positive integer greater than or equal to two. Let N= {1,..,n} denote the agent set. Let $u_i : \Re_+ \to \Re_+$ be a 'utility function' for agent 'i', denoting the amount of 'desirable' commodity that agent i can produce out of any given amount of a homogenous output. Throughout we assume that :
   (i) u_i is strictly increasing i.e. $[a,b \in \Re_+, a > b]$ implies $[u_i(a) > u_i(b)]$;
   (ii) u_i is continuous;
   (iii) u_i is concave i.e. $[a,b \in \Re_+, t \in [0,1]]$ implies $[u_i(ta+(1-t)b) \geq tu_i(a) + (1-t)u_i(b)]$;
   (iv) $u_i(0) = 0$

Let $X = \Re^n_+ \setminus \{0\}$. If $p \in X$, then p_i denotes the price of the commodity that agent 'i' produces.
Let w>0 be the total amount of the homogenous input that has to be allocated among the agent. Given w>0, a feasible allocation is a point $x \in \Re^n_+$ such that $\sum_{i \in N} x_i \leq w$. Let $S(w) = \{(u_1(x_1),...,u_n(x_n))/x$ is a feasible allocation$\}$. It is shown in Peters [1992] that S(w) is a non-empty, closed, bounded, convex subset of $\Re^n_+$ containing a strictly positive vector and allowing for free-disposability (i.e. $v, v^* \in \Re^n_+, v \geq v^*, v \in S(w)$ implies $v^* \in S(w)$). In Lahiri [1996] it is shown that given any non-empty, closed, bounded, convex subset S of $\Re^n_+$ containing a strictly positive vector and allowing for free disposability, there exists utility function $u_i: \Re_+ \to \Re_+$ as above and w>0, such that S=S(w).
Given w>0, a feasible allocation x is said to be Pareto efficient if there does not exist $v \in S(w)$ with $v_i \geq u_i(x_i) \forall i \in N$ and $v_i > u_i(x_i)$ for some $i \in N$. The following well known result is worth reproducing (see de la Fuente [2000]):

Proposition 1 :- Given w>0 a feasible allocation x is Pareto efficient if and only if there exists $p \in X$ such that whenever y is any other feasible allocation, we have $\sum_{i \in N} p_i u_i(x_i) \geq \sum_{i \in N} p_i u_i(y_i)$.

3. The Continuity of the Weighted Social Choice Rule :- Given $p \in X$ and $w \in \Re_{++}$(i.e. $\Re_{+}\backslash\{0\}$), let $F(p,w)= \{x \in \Re^n_+ / (a) \sum_{i \in N} x_i \leq w; (b)$ $\left[y \in \Re^n_+, \sum_{i \in N} y_i \leq w\right]$ implies $\left[\sum_{i \in N} p_i u_i(x_i) \geq \sum_{i \in N} p_i u_i(y_i)\right]\}$. Since $\left\{x \in \Re^n_+ / \sum_{i \in N} x_i \leq w\right\}$ is non-empty, closed and bounded and since the function $x \rightarrow \sum_{i \in N} p_i u_i(x_i) :: \Re^n_+ \rightarrow \Re$ is continuous, $F(p,w) \neq \phi$. Further since each u_i is concave, F(p,w) is a convex subset of $\Re^n_+$. Hence F $: X \times \Re_{++} \rightarrow\rightarrow \Re^n_+$ is a non-empty valued, convex valued correspondence. This correspondence is what we call the weighted social choice rule. Further for each $(p,w) \in X \times \Re_{++}$, F(p,w) is a closed and bounded subset of $\Re^n_+$.

Under our assumptions $(p,w) \in X \times \Re_{++}$:$[x \in F(p,w)]$ implies $[\sum_{i \in N} x_i = w]$.

Theorem 1 : F is a upper-hemicontinuous i.e. if $\langle (x^k, p^k, w^k) / k \in \aleph \rangle$ is a sequence in $\Re^n_+ \times X \times \Re_{++}$ with $x^k \in F(p^k, w^k)$ $\forall$ $k \in \aleph$ and $\lim_{k \rightarrow \infty} (x^k, p^k, w^k) = (x, p, w) \in \Re^n_+ \times X \times \Re_{++}$, then $x \in F(p,w)$.

Proof :- Let $\langle (x^k, p^k, w^k) / k \in \aleph \rangle$ and (x,p,w) be as required in the statement of Theorem 1. Towards a contradiction suppose $x \notin F(p,w)$. Since $\sum_{i \in N} x_i^k \leq w^k \forall k \in \aleph$, $\lim_{k \rightarrow \infty} x^k = x$, $\lim_{k \rightarrow \infty} w^k = w$, clearly, $\sum_{i \in N} x_i \leq w$. Hence there exists $y \in \Re^n_+$: $\sum_{i \in N} y_i \leq w$ and $\sum_{i \in N} p_i u_i(y_i) > \sum_{i \in N} p_i u_i(x_i)$. Clearly there exists $j \in N$: $p_j>0$ and $u_j(y_j)>u_j(x_j)$. Since u_j is strictly increasing $y_j>x_j>0$. Hence by the continuity of u_j, there exists $\delta>0$: $y_j-\delta>x_j>0$ and

$\left[\sum_{i\neq j} p_i u_i(y_i) + p_j u_j(y_j - \delta)\right] - \left[\sum_{i\in N} p_i u_i(x_i)\right] = \in > 0$. By the continuity of u_i, $i \in N$ and since $\lim_{k\to\alpha} p^k = p$, $\lim_{k\to\alpha} x^k$, we get,

$$\lim_{k\to\infty} \sum_{i\neq j} p_i^k \left[\left[u_i(y_i) - u_i(x_i^k)\right] + p_j^k\left[u_j(y_j - \delta) - u_j(x_j^k)\right]\right]$$

$$= \sum_{i\neq j} p_i\left[u_i(y_i) - u_i(x_i)\right] + p_j\left[u_j(y_j - \delta) - u_j(x_j)\right]$$

Hence $\exists\, M_1 \in \aleph : \forall\, k \geq M_1$

$$\sum_{i\neq j} p_i^k\left[u_i(y_i) - u_i(x_i^k) + p_j^k\left[u_j(y_j - \delta) - u_j(x_j^k)\right]\right] > \frac{\in}{2} > 0.$$

Now, $\lim_{k\to\alpha} w^k = w \rightarrow \exists\, M_2 \in \aleph : \forall\, k \geq M_2$, $\left|w^k - w\right| < \frac{\delta}{4}$.

$$\therefore \sum_{i\neq j} y_i + (y_j - \delta) = \sum_{i\in N} y_i - \delta \leq w - \delta < w - \frac{\delta}{4} < w^k$$

Let $M = \max(M_1, M_2)$. Thus for $k \geq M$, $x^k \notin F(p^k, w^k)$, contradicting our hypothesis.

Hence $x \in F(p,w)$.

Q.E.D.

Theorem 2 :- $\forall (p,w) \in X \times \Re_{++}$ and $t>0$: $F(tp,w) = F(p,w)$

Proof :- Let $x \in F(p,w)$. Thus $x \in \Re_+^n$ and $\sum_{i\in N} x_i \leq w$. Let $y \in F(tp,w)$.

Thus $\sum_{i\in N} tp_i u_i(y_i) \geq \sum_{i\in N} tp_i u_i(x_i)$. Thus $\sum_{i\in N} p_i u_i(y_i) \geq \sum_{i\in N} p_i u_i(x_i)$. Since $\sum_{i\in N} y_i \leq w$, $\sum_{i\in N} p_i u_i(y_i) = \sum_{i\in N} p_i u_i(x_i)$. Thus $x \in F(tp,w)$. Thus $F(p,w) \subset F(tp,w)$. Thus $F(tp,w) \subset F(\frac{1}{t}tp,w) = F(p,w)$ since $\frac{1}{t} > 0$.

Thus $F(p,w) = F(tp,w)$.

Q.E.D.

Suppose we assume in addition to our 'blanket' hypothesis about $u^i : \Re_+ \rightarrow \Re_+$ the following :

Strict Concavity :- $\forall i \in N, a, b \in \Re_+$ with $a \neq b$ and $t \in (0,1)$: $u_i(ta + (1-t)b) > tu_i(a) + (1-t)u_i(b)$.

Then for $p \in X$, the function $U_p: \Re^n_+ \to \Re_+$ defined by $U_p(x) = \sum_{i \in N} p_i u_i(x_i) \forall x \in \Re^n_+$ satisfies strict concavity i.e. $\forall a,b \in \Re^n_+$ with $a \neq b$ and $t \in (0,1)$: $u_i(ta+(1-t)b) > tu_i(a) + (1-t)u_i(b)$.

For $(p,w) \in X \times R_{++}$, $x \in F(p,w) \leftrightarrow$ [x is a feasible allocation, and whenever y is a feasible allocation, $U_p(x) \geq U_P(y)$]. Thus if $x,y \in F(p,w)$ with $x \neq y$, then $\frac{1}{2}x + \frac{1}{2}y$ is a feasible allocation with $U_p\left(\frac{1}{2}x + \frac{1}{2}y\right) > U_p(x) = U_p(y)$, contradicting x or for that matter y belongs to F(p,w). Hence we have the following:

Theorem 3: Let u_i satisfy strict concavity for all $i \in N$. Then $\forall (p,w) \in X \times \Re_{++}$: F(p,w) is a singleton.

When $F: X \times \Re_{++} \to\to \Re^n_+$ is a singleton $\forall (p,w) \in X \times \Re_{++}$, we write $F(p,w) = \{f(p,w)\} \forall (p,w) \in X \times \Re_{++}$.

Theorem 4 :- Let u_i satisfy strict concavity for all $i \in N$. Then the function $f: X \times \Re_{++} \to \Re^n_+$ is continuous.

Proof :- Let $\langle (p^k, w^k) / k \in \aleph \rangle$ be a sequence in $X \times \Re_{++}$ with $\lim_{k \to \infty} (p^k, w^k) = (p,w) \in X \times \Re_{++}$. Suppose towards a contradiction $\langle f(p^k, w^k) / k \in \aleph \rangle$ does not converge to f(p,w). Hence there exists $\epsilon > 0$: $\{k \in \aleph / |f(p^k, w^k) - f(p,w)| \geq \epsilon\} = \aleph_1$ is an infinite subset of $\aleph$. Since $\lim_{k \to \infty} w^k = w$ and $\sum f_i(p^k, w^k) = w^k \forall k \in \aleph$, the set $\{f(p^k, w^k) / k \in \aleph_1\}$ is a bounded subset of $\Re^n$. Hence there exists a subsequence $\langle f(p^{k(m)}, w^{k(m)}) / m \in \aleph \rangle$ with $k(m) \in \aleph_1$, $\forall m \in \aleph$, such that $\lim_{m \to \infty} f(p^{k(m)}, w^{k(m)}) = x$. By Theorem 1, $x \in F(p,w)$, since $\lim_{m \to \infty} p^{k(m)} = p$ and $\lim_{m \to \infty} w^{k(m)} = w$. But $|f(p^{k(m)} w^{k(m)}) - f(p,w)| \geq \epsilon \ \forall m \in \aleph$ implies $|x - f(p,w)| \geq \epsilon$. Thus $x \neq f(p,w)$. This contradicts F(p,w) is single valued and proves the theorem.

Q.E.D.

4. The Value Function :- Let $V: X \times \Re_{++} \to \Re_+$ be defined by $V(p,w) = \sum_{i \in N} p_i u_i(x_i)$ for $x \in F(p,w)$. Clearly V is well defined. V is called the value function.

Theorem 5 :- (i) V is continuous

(ii) $\forall p \in X$, the function $V(p,\cdot) : \Re_{++} \to \Re_+$ is strictly increasing;

(iii) $\forall w \in \Re_{++}$, the function $V(\cdot,w) : X \to \Re_+$ is convex i.e. $\forall p, p^* \in X$ and $t \in [0,1]$, $V(tp+(1-t)p^*, w) \le tV(p,w) + (1-t)V(p^*,w)$

(iv) $\forall (p,w) \in X \times \Re_{++}$ and $t>0$: $V(tp,w) = tV(p,w)$.

Proof :- (i) Let $\langle (p^k, w^k) / k \in \aleph \rangle$ be a sequence in $X \times \Re_{++}$ with $\lim_{k\to\infty} (p^k, w^k) = (p,w) \in X \times \Re_{++}$. Towards a contradiction suppose, $\langle V(p^k, w^k) / k \in \aleph \rangle$ does not converge to $V(p,w)$. Hence there exists $\epsilon > 0$: $\{k \in \aleph / |V(p^k, w^k) - V(p,w)| \ge \epsilon\} = \aleph_1$ is an infinite set. For $k \in \aleph$, let $x^k \in F(p^k, w^k)$. Since $\lim_{k\to\infty} w^k = w$ and $\sum_{i \in N} x_i^k = w^k$, $\{x^k / k \in \aleph_1\}$ is a bounded set. Hence there exists a subsequence $\langle x^{k(m)} / m \in \aleph \rangle$ of $\langle x^k / k \in \aleph \rangle$ such that $k(m) \in \aleph_1 \forall m \in \aleph$ and with $\lim_{m\to\infty} x^{k(m)} = x$.

Clearly, $x \in \Re_+^n$ with $\sum_{i \in N} x_i \le w$. By Theorem 1, $x \in F(p,w)$. Thus $V(p,w) = \sum_{i \in N} p_i u_i(x_i)$. Now, $V(p^k, w^k) = \sum_{i \in N} p_i^k u_i(x_i^k) \, \forall k \in \aleph$.

$\therefore \lim_{m\to\infty} V(p^{k(m)}, w^{k(m)}) = \lim_{m\to\infty} \sum_{i \in N} p_i^{k(m)} u_i(x_i^{k(m)}) = \sum_{i \in N} p_i u_i(x_i) = V(p,w)$,

contradicting $|V(p^{k(m)}, w^{k(m)}) - V(p,w)| \ge \epsilon \, \forall m \in \aleph$. Thus $\lim_{k\to\infty} V(p^k, w^k) = V(p,w)$. This proves (i).

(ii) Let $w^* > w > 0$ and let $V(p,w) = \sum_{i=1}^{n} p_i u_i(x_i)$ for some $x \in F(p,w)$.

Thus $\sum_{i=1}^{n} x_i \le w$. Let $y \in \Re_+^n$ with $y_i = x_i + \frac{w^* - w}{n}$, $i \in N$. Thus $\sum_{i \in N} y_i \le w^*$.

Since u_i is strictly increasing, $u_i(y_i) > u_i(x_i), i \in N$. $\therefore V(p,w^*) \ge \sum_{i \in N} p_i u_i(y_i) > \sum_{i \in N} p_i u_i(x_i) = V(p,w)$.

(iii) Let $p, p^* \in X$ and $t \in [0,1]$. Let $y \in \Re^n_+$ with $\sum_{i=1}^{n} y_i = w$. Thus $V(p,w) \geq \sum_{i \in N} p_i u_i(y_i)$ and $V(p^*,w) \geq \sum_{i \in N} p_i^* u_i(y_i)$. Thus $tV(p,w)+(1-t)$ $V(p^*,w) \geq \sum_{i \in N} [tp_i + (1-t)]u_i(y_i)$. Thus $tV(p,w)+(1-t)V(p^*,w) \geq V(tp+(1-t)p^*,w)$.

(iv) Follows directly from Theorem 2.

Q.E.D.

Now suppose that each $u_i: \Re_+ \to \Re_+$ is continuously differentiable with

(i) $Du_i(h) > 0 \; \forall h \in \Re_{++}$ and (ii) $\lim_{h \to 0} Du_i(h) = +\infty$.

Let $p \in \Re^n_{++} \equiv \{x \in \Re^n_+ / x_i > 0 \; \forall i \in N\}$. For $w>0$, consider the problem:

$$\sum_{i \in N} p_i u_i(x_i) \to \max$$

$$\text{s.t. } \sum_{i \in N} x_i \leq w, x \in \Re^n_+.$$

It is easy to see that if x^* solves the problem then $x^* \in \Re^n_{++}$. Hence by standard methods (discussed in Lahiri [2000]) there exists $\lambda > 0$: $p_i Du_i(x_i^*) = \lambda$, $i \in N$ and $\sum_{i \in N} x_i^* = w$. Suppose that each $u_i: \Re_+ \to \Re_+$ is strictly concave. Then for $(p,w) \in X x \Re_{++}$,
$p_i Du_i(f_i(p,w)) = \lambda$, $i \in N$ and $\sum_{i \in N} f_i(p,w) = w$.

Theorem 6:- Suppose in addition to above that $f: X \text{ x } \Re_{++} \to \Re^n_+$ is differentiable in $\Re^n_{++} x \Re_{++}$. Then, $\forall (p,w) \in \Re^n_{++} x \Re_{++}$:
$\frac{\partial}{\partial p_i} V(p,w) = u_i(f_i(p,w))$.

Proof:- $V(p,w) = \sum_{i \in N} p_i u_i(f_i(p,w)) \forall (p,w) \in \Re^n_{++} x \Re_{++}$. Thus

$$\frac{\partial V(p,w)}{\partial p_i} = u_i(f_i(p,w)) + \sum_{j \in N} p_j Du_j(f_j(p,w)) \frac{\partial f_j(p,w)}{\partial p_i} = u_i(f_i(p,w)) + \lambda \sum_{j \in N} \frac{\partial f_j(p,w)}{\partial p_i}.$$

Now, under our assumption $\sum_{j \in N} f_j(p,w) = w \; \forall (p,w) \in \Re^n_{++} x \Re_{++}$. Thus, $\sum_{j \in N} \frac{\partial f_j(p,w)}{\partial p_i} = 0$. This proves the theorem.

Q.E.D.

5. <u>The Expenditure Function</u> :- Let $(p,\nu)\in X \times \Re_+$ and $\forall i\in N$, $u_i:\Re_+\to\Re_+$ be strictly increasing, concave, continuous with $u_i(0)=0$ $\forall i\in N$. Suppose $\lim_{h\to\infty} u_i(h) = +\infty \forall i \in N$. Consider the problem:

$$\sum_{i\in N} x_i \to \min$$

$$\text{s.t. } \sum_{i\in N} p_i u_i(x_i) \geq \nu, x \in \Re^n_+.$$

Under our assumption the problem has a solution. Let $G(p,\nu)$ be the set of solutions for the above problem. Hence $G:X \times \Re_+ \to\to \Re^n_+$ is a non-empty valued correspondence. This correspondence is what we call the dual weighted social choice rule.

<u>Theorem 7</u> :- (i) G is a convex valued; (ii) G is upper-hemicontinous; i.e. if $\langle (x^k,p^k,\nu^k)/k\in\aleph\rangle$ is a sequence in $\Re^n_+ \times X \times \Re_{++}$ with $x^k\in G(p^k, \nu^k)$ $\forall$ $k \in\aleph$ and $\lim_{k\to\infty}(x^k,p^k,\nu^k)= (x, p, \nu)\in \Re^n_+ \times X \times \Re_{++}$, then $x\in G(p,\nu)$; (iii) $\forall(p,\nu) \in X\times\Re_+$ and $t>0$: $G(p,\nu)=G(tp,t\nu)$.

<u>Proof</u> :- (I) follows easily from the fact that $\forall i\in N$, u_i is concave.(iii) is easy to establish.Hence let us prove (ii). Let $\langle (x^k,p^k,\nu^k)/k\in\aleph\rangle$ and (x,p,ν) be as required in the statement of Theorem 7. Towards a contradiction suppose $x\notin G(p,\nu)$. Since $\sum_{i\in N} p_i u_i(x_i^k) \geq \nu^k \forall k\in\aleph$, $\lim_{k\to\infty} x^k = x$, $\lim_{k\to\infty} \nu^k = \nu$, clearly, $\sum_{i\in N} p_i u_i(x_i) \geq \nu$.

Hence there exists $y\in \Re^n_+$: $\sum_{i\in N} p_i u_i(y_i) \geq \nu$ and yet $\sum_{i\in N} y_i < \sum_{i\in N} x_i$. Thus, there exists $\delta > 0$ such that $\sum_{i\in N} y_i + \delta < \sum_{i\in N} y_i < \sum_{i\in N} x_i$. Let $\in$ = $\sum_{i\in N}(x_i - y_i) - \delta > 0$. Now, $p\in X$ implies that there exists $j\in N$:$p_j>0$. Thus, $\sum_{i\neq j} p_i u_i(y_i)+p_j u_j(y_j+\delta)>\sum_{i\in N} p_i u_i(y_i)\geq\nu$, since u_j is strictly increasing. Since $\lim_{k\to\infty} x^k = x$, and $\sum_{i\in N} y_i + \delta < \sum_{i\in N} x_i$ there exists $M_1\in\aleph$ such that $\forall k \geq M_1$: $\sum_{i\in N} y_i + \delta < \sum_{i\in N} x_i^k$ .Further, $\sum_{i\neq j} p_i u_i(y_i)+p_j u_j(y_j+\delta)>\nu$, $\lim_{k\to\infty} p^k = p$ and $\lim_{k\to\infty} \nu^k = \nu$ implies that there exists $M_2\in\aleph$ such that $\forall k \geq M_2$: $\sum_{i\neq j} p_i^k u_i(y_i)+p_j^k u_j(y_j+\delta)>\nu^k$.

Let $M=\max \{ M_1, M_2\}$.Thus,

$\sum_{i\in N} y_i + \delta < \sum_{i\in N} x_i^M$ and $\sum_{i\neq j} p_i^M u_i(y_i) + p_j^M u_j(y_j+\delta) > v^M$, contradicting $x^k \in G(p^k, v^k)$ $\forall$ $k \in \aleph$ and thus proving the theorem.
Q.E.D.

Theorem 8: Let u_i satisfy strict concavity for all $i\in N$. Then $\forall (p,v) \in X \times \Re_+$: $G(p,v)$ is a singleton.

Proof :- Suppose u_i satisfies strict concavity for all $i\in N$ and towards a contradiction suppose that there exists $(p,v)\in X \times \Re_+$: $G(p,v)$ is not a singleton. Let $x,y\in G(p,v)$ with $x \neq y$. Let $M = \{i\in N/\ p_i>0\}$. Suppose $x_j > 0$ for some $j\in N\backslash M$. Let $z \in \Re^n_+$ with $z_i = 0$ and $z_j = x_j$, for $j \neq i$. Thus, $\sum_{i\in N} z_i < \sum_{i\in N} x_i$ and since $p_j = 0$, $\sum_{i\in N} p_i u_i(z_i) = \sum_{i\in N} p_i u_i(x_i) \geq v$. This contradicts $x\in G(p,v)$. Thus, $x_i = 0$ for all $i\in N\backslash M$. Similarly, $y_i = 0$ for all $i\in N\backslash M$.
Now, $x \neq y$ implies that there exists $i\in M$ such that $x_i \neq y_i$. Thus, $u_i(\frac{1}{2}x_i + \frac{1}{2}y_i) > \frac{1}{2}u_i(x_i) + \frac{1}{2}u_i(y_i)$ and $u_j(\frac{1}{2}x_j + \frac{1}{2}y_j) \geq \frac{1}{2}u_j(x_j) + \frac{1}{2}u_j(y_j)$ $\forall j\in N\backslash\{i\}$. By continuity of u_i, there exists $\delta>0$, such that, $\frac{1}{2}x_i + \frac{1}{2}y_i - \delta > 0$ and $u_i(\frac{1}{2}x_i + \frac{1}{2}y_i - \delta) > \frac{1}{2}u_i(x_i) + \frac{1}{2}u_i(y_i)$. Thus,
$p_i u_i(\frac{1}{2}x_i + \frac{1}{2}y_i - \delta) + \sum_{j\neq i} p_j u_j(\frac{1}{2}x_j + \frac{1}{2}y_j) > \sum_{j\in N} p_j(\frac{1}{2}u_j(x_j) + \frac{1}{2}u_j(y_j)) \geq v$ and $\sum_{j\in N}(\frac{1}{2}[x_i + y_i] - \delta) < \sum_{i\in N}\frac{1}{2}[x_i + y_i] = \sum_{i\in N} x_i$, contradicting $x\in G(p,v)$ and proving the theorem.
Q.E.D.

Let $G(p,v) = \{g(p,v)\}$ for all (p,v) $X \times \Re_+$ in the event that u_i satisfies strict concavity for all $i\in N$.

Theorem 9: Let u_i satisfy strict concavity for all $i\in N$. Then the function g: $X \times \Re_+ \rightarrow \Re^n_+$ is continuous.
Proof :- Let $\langle (p^k, v^k) / k \in \aleph \rangle$ be a sequence in $X \times \Re_+$ with $\lim_{k\to\infty} (p^k, v^k) = (p,v) \in X \times \Re_+$. Towards a contradiction suppose

$\langle g(p^k, v^k) / k \in \aleph \rangle$ does not converge to g(p,v). Thus there exists $\in > 0$: $\aleph_1 = \{k \in \aleph / |g(p^k, v^k) - g(p,v)| \geq \in\}$ is infinite. Consider the allocation $z^k \in \Re^n_+$ such that $z_1^k = u_1^{-1}(v^k / p_1^k), z_i^k = 0 \; \forall i \in N \backslash \{1\}$. Thus, $\sum_{i \in N} p_i^k u_i(z_i^k) = v^k$. Thus, $\sum_{i \in N} g_i(p^k, v^k) \leq z_1^k \; \forall k \in \aleph$. If $\lim_{k \to \infty} z_1^k = +\infty$, then $\lim_{k \to \infty} v^k / p_1^k = +\infty$. Since $\lim_{k \to \infty} v^k \geq 0$, this means $\lim_{k \to \infty} p_1^k = 0$. Since $\lim_{k \to \infty} p^k = p \in X$, $\exists$ i∈N : $p_i > 0$. Without loss of generality assume $p_1 > 0$. Then clearly $\langle z_1^k / k \in \aleph \rangle$ is a bounded sequence. Thus $\langle g(p^k, v^k) / k \in \aleph \rangle$ is a bounded sequence. Let $\langle g(p^{k(m)}, v^{k(m)}) / m \in \aleph \rangle$ be a subsequence of $\langle g(p^k, v^k) / k \in \aleph \rangle$ such that $k(m) \in \aleph_1 \forall m \in \aleph$ and $\lim_{m \to \infty} g(p^{k(m)}, v^{k(m)}) = x \in \Re^n_+$. By theorem 7, x∈G(p,v). Thus x=g(p,v). This contradicts $|g(p^k, v^k) - g(p,v)| \geq \in \forall k \in \aleph_1$. Thus $\lim_{k \to \infty} g(p^k, v^k) = g(p,v)$. This proves the theorem.

Q.E.D.

Let $e: X \times \Re_+ \to \Re_+$ be defined thus : $e(p,v) = \sum_{i \in N} x_i$ where x∈G(p,v).

Clearly e is well defined. Further $\forall (p,v) \in X \times \Re_+$ and t>0 : e(p,v) = e(tp, tv). e is called the expenditure function.

Theorem 10 :- e is continuous.

Proof :- Let $\langle (p^k, v^k) / k \in \aleph \rangle$ be a sequence in $X \times \Re_+$ with $\lim_{k \to \infty} (p^k, v^k) = (p,v) \in X \times \Re_+$. Towards a contradiction suppose $\langle e(p^k, v^k) / k \in \aleph \rangle$ does not converge to e(p,v) Hence there exists $\in > 0$: $\aleph_1 = \{k \in \aleph / |e(p^k, v^k) - e(p,v)| \geq \in\}$ is infinite. Let $x^k \in G(p^k, v^k)$, $k \in \aleph$. Thus $e(p^k, v^k) = \sum_{i \in N} x_i^k$. For reasons similar to that discussed in the initial part of Theorem 9, $\langle x^k / k \in \aleph \rangle$ is a bounded sequence. Hence it has a subsequence $\langle x^{k(m)} / m \in \aleph \rangle$ such that $k(m) \in \aleph_1 \forall m \in \aleph$ and $\lim_{m \to \infty} x^{k(m)} = x$. Since G is upper hemicontinuous, x∈G(p,v).

$\therefore e(p,\nu)= \sum_{i\in N} x_i = \sum_{i\in N} x_i^{k(m)} = e(p^{k(m)},\nu^{k(m)})$ contradicting

$\left|e(p^k,\nu^k) - e(p,\nu)\right| \geq \epsilon\ \forall k \in \aleph_1$. Thus, e is continuous.

Q.E.D.

Assume each u_i is strictly concave and continuously differentiable with (i) $Du_i(h)>0\ \forall h\in \Re_{++}$; (ii) $\lim_{h\to 0} Du_i(h) = +\infty$.

Let $p\in \Re^n_{++} \equiv \{x\in \Re^n_+ / x_i>0 \forall i\in N\}$ and $\nu>0$. Thus there exists $\lambda>0$: $p_i Du_i(g_i(p,\nu)) = \lambda$, $i\in N$.

$e(p,\nu) = \sum_{i\in N} g_i(p,\nu)$, $g(p,\nu)\in \Re^n_{++}$. Suppose in addition that $g: X \times \Re_+ \to \Re^n_{++}$ is differentable on $\Re^n_{++} \times \Re_{++}$. We then have the following :

Theorem 11 :- $\forall\ (p,\nu)\in \Re^n_{++} \times \Re_{++}$:

$$\frac{\partial e}{\partial p_i}(p,\nu) \Big/ \frac{\partial e}{\partial \nu}(p,\nu) = -u_i(g_i(p,\nu))$$

Proof :- Since $e(p,\nu)= \sum_{j\in N} g_j(p,\nu)\ \forall\ (p,\nu)\in \Re^n_{++} \times \Re_{++}$, we have

$$\frac{\partial e}{\partial p_i}(p,\nu)= \sum_{j\in N}\frac{\partial g_j(p,\nu)}{\partial p_i}\ \forall\ (p,\nu)\in \Re^n_{++} \times \Re_{++}.$$

Now under our assumptions, $\sum_{j\in N} p_j u_j(g_j(p,\nu)) = \nu \forall (p,\nu)\in \Re^n_{++} \times \Re_{++}$.

$$\therefore u_i(g_i(p,\nu)) + \sum_{j\in N} p_j Du_j(g_j(p,\nu))\frac{\partial g_j}{\partial p_i}(p,\nu) = 0.$$

$$\therefore\ u_i(g_i(p,\nu)) + \lambda \sum_{j\in N} \frac{\partial g_j}{\partial p_i}(p,\nu) = 0.$$

$$\therefore \sum_{j\in N} \frac{\partial e}{\partial p_i}(p,\nu) = -\frac{u_i(g_i(p,\nu))}{\lambda}.$$

Now $\sum_{j\in N} p_j u_j(g_j(p,\nu)) = \nu \forall (p,\nu)\in \Re^n_{++} \times \Re_{++}$.

Hence, $\sum_{j\in N} p_j Du_j(g_j(p,\nu))\frac{\partial g_j}{\partial \nu}(p,\nu) = 1$.

$$\therefore \lambda \sum_{j\in N} \frac{\partial g_j}{\partial \nu}(p,\nu) = 1.$$

Finally, $e(p,v) = \sum_{j \in N} g_j(p,v) \ \forall \ (p,v) \in \Re^n_{++} \times \Re_{++}$

implies $\sum_{j \in N} \frac{\partial e}{\partial v}(p,v) = \sum_{j \in N} \frac{\partial g_j}{\partial v}(p,v) \ \forall \ (p,v) \in \Re^n_{++} \times \Re_{++}$.

$\therefore \frac{\partial e}{\partial v}(p,v) = \frac{1}{\lambda}$. This proves the theorem.

Q.E.D.

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