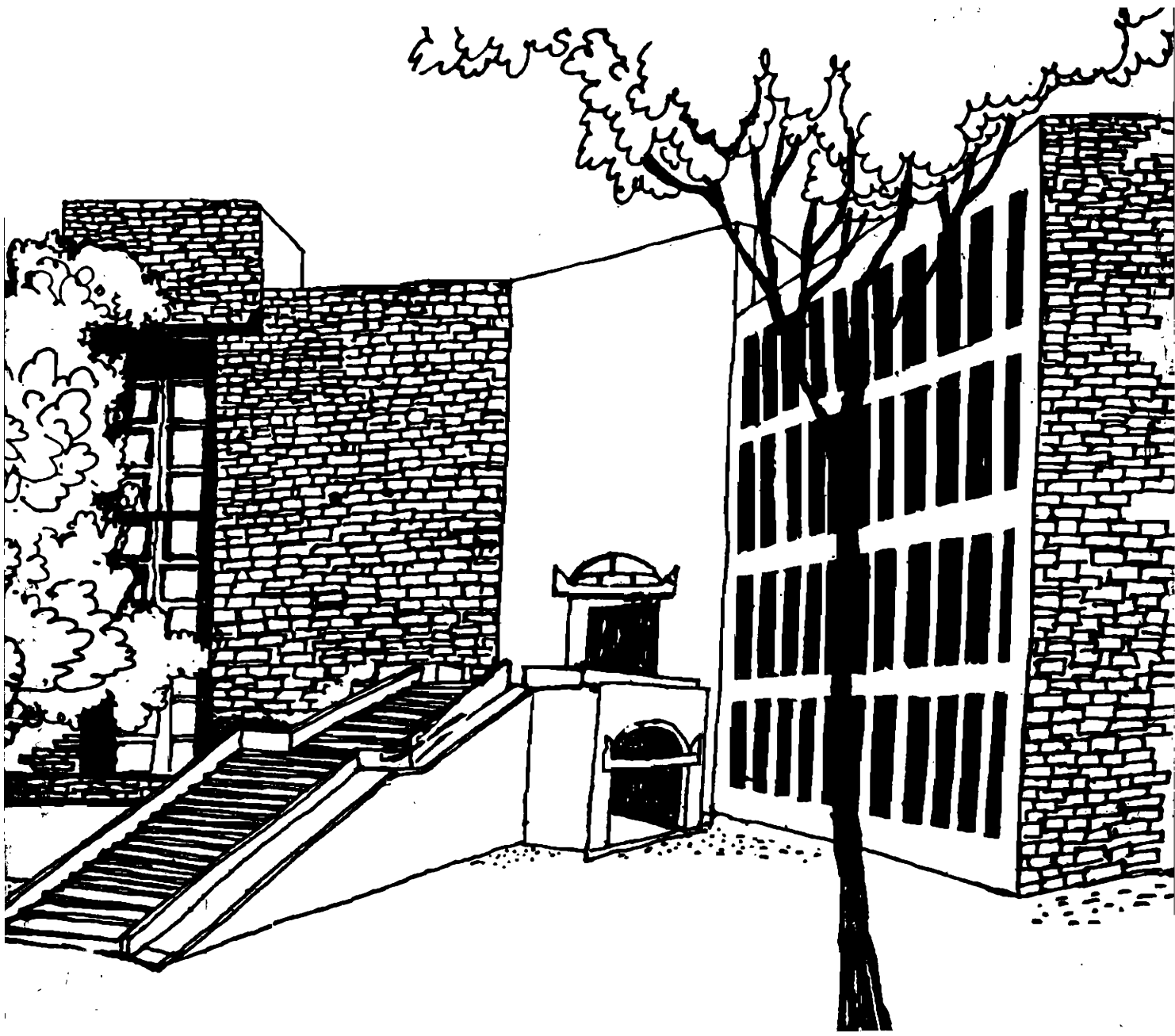




Working Paper



**AXIOMATIC CHARACTERISATION OF WEIGHTED
BOOLEAN VOTE AGGREGATORS**

By

Somdeb Lahiri

W.P.No. 2001-02-02
February 2001 /1637

WP1637

2001-02-02
(1637)

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

**INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA**

IIMA

WP-2001-02-02

250270

| |
|--------------------------------|
| PURCHASED |
| APPROVAL |
| GRATIS/EXCHANGE |
| PRICE |
| A/C NO. |
| VIKRAM SARABHAI LIBRARY |
| ANANDAPUR |

Axiomatic Characterisation of Weighted Boolean Vote Aggregators

Somdeb Lahiri

Email:lahiri@iimahd.ernet.in

Indian Institute of Management, Ahmedabad 380 015, India

February 2001.

ABSTRACT

A Federation Boolean Vote Aggregator allows a finite set of coalitions to unilaterally elect any candidate from a set containing exactly two candidates. There are several special types of Federation Boolean Vote Aggregators, all of which share a property :the candidates are assigned weights, and for a coalition to be decisive,it is necessary that the sum of the weights of its members exceed a pre-assigned quota. In this paper we address the following question: When is a Federation Boolean Vote Aggregator a Weighted Boolean Vote Aggregator ?

Axiomatic Characterisation of Weighted Boolean Vote Aggregators

Somdeb Lahiri

Email :lahiri@iimahd.ernet.in

Indian Institute of Management, Ahmedabad 380 015, India

February 2001.

1 Introduction

A model for analysing voting procedures where each individual in a society casts a ballot and a voting operator aggregates the ballots into elected outcomes has been modelled in Lahiri (1999,2001). A ballot is a set of alternatives chosen from a universal set of candidates. A ballot profile associates with each voter a ballot. A voting operator, selects a set of candidates from amongst those who have secured at least one vote. Further we assume that if there is at least one candidate who secures the vote of every individual, then at least one such candidate is definitely chosen. In Lahiri (2001), one is introduced to the idea of a vote aggregator which aggregates ballots which are singletons. This is definitely a more realistic model of democratic exercises as we see it in practise.

However, even though singleton ballots are a realistic premise for analysis, it is difficult to be theoretically sound and yet exclude the possibility of more than one elected outcome. Thus for instance, under plurality it is quite possible that two candidates receive the maximum number of votes. To accommodate such possibilities, the concept of a vote aggregator introduced in Lahiri (2001) is set valued. A vote aggregator is required to satisfy the rather innocuous assumption called unanimity; i.e. if every one votes for the same candidate then that is the only one who is elected. It is worth recalling in this context the seminal work of Arrow, where individuals are required to vote not for a single candidate, but for a preference ordering over the entire array of candidates. This and the related literature find a thorough discussion in Aleskerov (1999). Essentially what each voter votes for is a binary relation. These binary relations are aggregated into a single binary relation. Since a binary relation is nothing but a subset of the set of all ordered pairs of candidates, the classical framework of Arrow is more appropriately a special case of the scenario where ballots are sets instead of singletons. This observation can be found in Sholomov (2000).

In this paper we study vote aggregators, where each voter casts a vote for exactly one of two candidates. The two candidates are denoted 0 and 1 respectively. Ballot profiles in such a context are called Boolean ballot profiles. Further, such vote aggregators, which are a special case of the general model of vote aggregators, were once again introduced in Lahiri (2001) and are called Boolean Vote Aggregators. The Boolean Vote Aggregator we study in this paper, namely the Federation Boolean Vote Aggregator originates in the work of Aizerman and Aleskerov (1986,1995). Aleskerov (1999), contains an exhaustive discussion of the related literature. A Federation Boolean Vote Aggregator

allows a finite set of coalitions to unilaterally elect any outcome. Such coalitions are called minimal decisive coalitions. There are several special types of Federation Boolean Vote Aggregator, all of which share a property: the candidates are assigned weights, and for a coalition to be decisive, it is necessary that the sum of the weights of its members exceed a pre-assigned quota. First, there are those Federation Boolean Vote Aggregator where coalitions can unilaterally elect outcomes if and only if they have a requisite number of voters. A second type of Federation Boolean Vote Aggregator is an oligarchy, where the ability to unilaterally elect an outcome is invested in a single coalition. Finally, there is the type of Federation Boolean Vote Aggregator where the ability to unilaterally elect an outcome is invested in a single individual. Such Boolean Vote Aggregators are called Dictatorial Boolean Vote Aggregators. In this paper we address the following question: When is a Federation Boolean Vote Aggregator a Weighted Boolean Vote Aggregator? In the process of answering this question we exploit the formal similarity of a Federation Boolean Vote Aggregator, with a simple game due to Shapley (1962) and the formal similarity of a Weighted Boolean Vote Aggregator with a weighted voting game. The unique property which is necessary and sufficient for a Federation Boolean Vote Aggregator to be a Weighted Boolean Vote Aggregator is called robustness in this paper. This property is similar to the concept of trade robustness that was introduced by Taylor and Zwicker (1992), and which was shown by them to be necessary and sufficient for a simple game to be a weighted voting game. In our context what robustness implies is the following: Suppose we are given a collection of Boolean ballot profiles (each profile being possibly repeated several times) all of which lead to candidate one being elected. Suppose there is a second collection of Boolean ballot profiles (each profile being possibly repeated several times) such that each voter votes for candidate one the same number of times as before. Then there must be at least one profile in this new collection which elects candidate one.

In Sholomov (2000), a discussion of Weighted Vote Aggregators, restricted to Arrowian domains, can be found. Sholomov asserts that a social decision function (i.e. a Vote Aggregator which maps a profile of binary relations to a binary relation) with domain consisting of all profiles of binary relations which are semiorders has its range in the set of all acyclic binary relations, if and only if it can be expressed as the intersection of a social decision function which is a Weighted Vote Aggregator and a social decision function which satisfies binariness, neutrality to alternatives and non-imposition. The author further asserts that a monotone social decision function with domain consisting of all profiles of binary relations which are semiorders has its range in the set of all acyclic binary relations, if and only if it can be expressed as the intersection of a social decision function which is a weighted Vote Aggregator and a monotone social decision function which satisfies binariness, neutrality to alternatives and non-imposition.

The analytical framework in which aggregation rules are studied in this paper is similar to a device which is referred to in classical choice theory as a choice function. A comprehensive survey of rational choice theory (i.e. the theory concerned with specifying conditions on a choice function under which there exists a binary relation of a desired type whose "best" elements from a given set

allows a finite set of coalitions to unilaterally elect any outcome. Such coalitions are called minimal decisive coalitions. There are several special types of Federation Boolean Vote Aggregator, all of which share a property: the candidates are assigned weights, and for a coalition to be decisive, it is necessary that the sum of the weights of its members exceed a pre-assigned quota. First, there are those Federation Boolean Vote Aggregator where coalitions can unilaterally elect outcomes if and only if they have a requisite number of voters. A second type of Federation Boolean Vote Aggregator is an oligarchy, where the ability to unilaterally elect an outcome is invested in a single coalition. Finally, there is the type of Federation Boolean Vote Aggregator where the ability to unilaterally elect an outcome is invested in a single individual. Such Boolean Vote Aggregators are called Dictatorial Boolean Vote Aggregators. In this paper we address the following question: When is a Federation Boolean Vote Aggregator a Weighted Boolean Vote Aggregator? In the process of answering this question we exploit the formal similarity of a Federation Boolean Vote Aggregator, with a simple game due to Shapley (1962) and the formal similarity of a Weighted Boolean Vote Aggregator with a weighted voting game. The unique property which is necessary and sufficient for a Federation Boolean Vote Aggregator to be a Weighted Boolean Vote Aggregator is called robustness in this paper. This property is similar to the concept of trade robustness that was introduced by Taylor and Zwicker (1992), and which was shown by them to be necessary and sufficient for a simple game to be a weighted voting game. In our context what robustness implies is the following: Suppose we are given a collection of Boolean ballot profiles (each profile being possibly repeated several times) all of which lead to candidate one being elected. Suppose there is a second collection of Boolean ballot profiles (each profile being possibly repeated several times) such that each voter votes for candidate one the same number of times as before. Then there must be at least one profile in this new collection which elects candidate one.

In Sholomov (2000), a discussion of Weighted Voting Operators, restricted to Arrowian domains, can be found. Sholomov asserts that a social decision function (i.e. a Voting Operator which maps a profile of binary relations to a binary relation) with domain consisting of all profiles of binary relations which are semiorders has its range in the set of all acyclic binary relations, if and only if it can be expressed as the intersection of a social decision function which is a Weighted Vote Aggregator and a social decision function which satisfies binariness, neutrality to alternatives and non-imposition. The author further asserts that a monotone social decision function with domain consisting of all profiles of binary relations which are semiorders has its range in the set of all acyclic binary relations, if and only if it can be expressed as the intersection of a social decision function which is a weighted social voting operator and a monotone social decision function which satisfies binariness, neutrality to alternatives and non-imposition.

The analytical framework in which aggregation rules are studied in this paper is similar to a device which is referred to in classical choice theory as a choice function. A comprehensive survey of rational choice theory (i.e. the theory concerned with specifying conditions on a choice function under which there exists a binary relation of a desired type whose "best" elements from a given set

of alternatives, coincide with the elements chosen by the choice function) till the mid nineteen eighties is available in Moulin (1985).

2 The Model

Let n be a natural number. Let $N = \{1, \dots, n\}$ be the set of agents or voters. Let $X = \{0, 1\}$. Let $P(X)$ denote the power set of X , i.e. the set of all subsets of X . Let X^N denote the set of all functions from N to X . Any element $S = (S_1, \dots, S_n) \in X^N$, is called a (Boolean) ballot profile. A Boolean Vote Aggregator (BVA) is a function $C : X^N \rightarrow P(X)$ such that (1) $C(S) \subset \text{range}(S)$; (2) if there exists $x \in X$ such that if $\forall i \in N : S_i = x$, then $C(S) = \{x\}$.

Thus an element which appears on no ballot is never chosen and an element which appears on the ballot of every individual is invariably chosen. The latter property is known as unanimity. As a consequence of our unanimity it easily follows that given any $x \in X$, there exists $S \in X^N$ such that $\{x\} = C(S)$: simply take $\forall i \in N, S_i = x$.

Given $T \in X^N$ and $x \in X$, let $r(x; T) = |\{i \in N / x \in T_i\}|$ i.e. the cardinality of the set $\{i \in N / x \in T_i\}$. In the sequel we will be considering the following properties of vote aggregators:

Monotonicity : Let $x \in C(S)$ and let S and $T \in X^N$ with $\{i \in N / x = S_i\} \subset \{i \in N / x = T_i\}$. Then $x \in C(T)$.

Neutrality: For all $S \in X^N : C(E-S) = \{1-x / x \in C(S)\}$.

Robustness : Let m be a natural number and let S^1, \dots, S^m and $T^1, \dots, T^m \in X^N$ be such that $\forall i \in N : |\{k / S_i^k = 1\}| = |\{k / T_i^k = 1\}|$. Further, suppose that $\forall k \in \{1, \dots, m\} : 1 \in C(S^k)$. Then, there exists $k \in \{1, \dots, m\} : 1 \in C(T^k)$.

Given a collection $\Omega = \{w_1, \dots, w_q\}$ of nonempty subsets of N , let $W(\Omega) = \{w' \subset N / w \subset w', \text{ for some } w \in \Omega\}$ and $L(\Omega) = \{w' \subset N / w' \notin W(\Omega)\}$. Clearly, $L(\Omega)$ contains the empty set. Given $x \in S$ and $S \in X^N$, let $W(x, S) = \{i \in N / S_i = x\}$.

Definitions of Boolean Vote Aggregators :

- C is said to be a Federation BVA if there exists $\Omega = \{w_1, \dots, w_q\}$, a collection of nonempty subsets of N , such that $\forall S \in X^N : C(S) = \{x \in X / W(x, S) \in W(\Omega)\}$.
- C is said to be an oligarchy if C is a Federation BVA with $\Omega = \{w_1\}$.
- C is said to be a k -votes BVA (: where ' k ' is a positive integer with $k \leq n$) if C is a Federation BVA with $\Omega = \{w \subset N / w \text{ has exactly } k \text{ elements}\}$.
- C is said to be Dictatorial BVA if there exists $i \in N$ (: called a dictator) such that $\forall S \in X^N : C(S) = S_i$.
- C is said to be a weighted BVA (WBVA) if there exists a function $v : N \rightarrow \aleph \cup \{0\}$ (: where \aleph is the set of natural numbers) and a natural number κ (called the quota) such that $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b) $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$.

The following theorem has been proved in Lahiri (2001):

Theorem 1 : A BVA satisfies monotonicity and neutrality if and only if it is a Federation BVA.

The following observation is easy to verify:

Proposition 1 : Let C be a WBVA. Then C satisfies robustness.

Proof : Let C be a WBVA. Then there exists a function $v : N \rightarrow \mathbb{R} \cup \{0\}$ and a natural number κ such that $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b) $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$. Let m be a natural number and let S^1, \dots, S^m and $T^1, \dots, T^m \in X^N$ be such that $\forall k \in \{1, \dots, m\} : 1 \in C(S^k)$. Further, suppose that $\forall i \in N : |\{k / S_i^k = 1\}| = |\{k / T_i^k = 1\}|$. Towards a contradiction suppose that for all $k \in \{1, \dots, m\}$ it is true that $1 \notin C(T^k)$. Now, $\forall k \in \{1, \dots, m\} : 1 \in C(S^k)$ implies that $\forall k \in \{1, \dots, m\} : \sum_{i \in N} S_i^k v(i) \geq \kappa$. Further, $\forall k \in \{1, \dots, m\}$ it is true that $1 \notin C(T^k)$

implies that $\forall k \in \{1, \dots, m\} : \sum_{i \in N} T_i^k v(i) < \kappa$. Thus $\sum_{k=1}^m \sum_{i \in N} T_i^k v(i) < m\kappa$. Thus, $m\kappa >$

$\sum_{i \in N} v(i) \sum_{k=1}^m T_i^k = \sum_{i \in N} v(i) \sum_{k=1}^m S_i^k = \sum_{k=1}^m \sum_{i \in N} S_i^k v(i)$, contradicting that $\forall k \in \{1, \dots, m\} : \sum_{i \in N} S_i^k v(i) \geq \kappa$. Thus, there exists $k \in \{1, \dots, m\}$ such that $1 \in C(T^k)$.

Q.E.D.

3 The Characterization Theorem

The following lemma is crucial in what follows:

Lemma 1 :- Let $\sum_{j=1}^n a_{ij} x_j = b_i, i=1, \dots, k$ be a system of 'k' equation in 'n' unknowns

and suppose a_{ij}, b_i are all rational for $i=1, \dots, k; j=1, \dots, n$. Let (x_1^*, \dots, x_n^*) be a solution for the above system of equations. Then given $\varepsilon > 0$, there exists a solution $(\bar{x}_1, \dots, \bar{x}_n)$ with all co-ordinates rational such that $\|(x_1^*, \dots, x_n^*) - (\bar{x}_1, \dots, \bar{x}_n)\| < \varepsilon$.

Proof :- If $n=1$, then $a_{i1} x_1^* = b_i, i=1, \dots, k$ with all $a_{i1}, b_i, i=1, \dots, k$ rational implies

$x_1^* = \frac{b_i}{a_{i1}}$ whenever $a_{i1} \neq 0$. If $a_{i1} = 0 \forall i$, then $b_i = 0 \forall i$ and hence we can choose any

$\bar{x}_1 \in (x_1^* - \varepsilon, x_1^* + \varepsilon) \cap \mathbb{R}$ rational to solve the system. In either case the theorem is true

for $n=1$. Suppose the theorem is true for $1, 2, \dots, n-1$ where $n-1 \geq 0$. Let $\sum_{j=1}^n a_{ij} x_j = b_i, i=1, \dots, k$ be the system as desired and let (x_1^*, \dots, x_n^*) solve the system. Without

loss of generality suppose $a_{kn} \neq 0$. Let $x_n = \frac{1}{a_{kn}} \left[b_k - \sum_{j=1}^{n-1} a_{kj} x_j \right]$. Since the real valued

function $(y_1, \dots, y_{n-1}) \mapsto \frac{1}{a_k} \left[b_k - \sum_{j=1}^{n-1} a_{kj} y_j \right]$, with domain \mathbb{R}^{n-1} is continuous, there

exists $\delta > 0$: $\|(y_1, \dots, y_{n-1}) - (x_1^*, \dots, x_{n-1}^*)\| < \delta \rightarrow \left| \frac{1}{a_k} \left[b_k - \sum_{j=1}^{n-1} a_{kj} y_j \right] - x_n^* \right| < \frac{\varepsilon}{n}$.

Consider the system,

$\sum_{j=1}^{n-1} C_{ij} x_j = B_i, i=1, \dots, k$ where $C_{kj} = 0 = B_k$, for $j=1, \dots, n-1$ and $C_{ij} = a_{ij} - \frac{a_{in}}{a_{kn}} a_{kj}$,

$B_i = b_i - \frac{b_k}{a_{kn}}$ for $i=1, \dots, k-1, j=1, \dots, n-1$.

Now $\sum_{j=1}^{n-1} a_{ij} x_j^* + \frac{a_{in}}{a_{kn}} \left[b_k - \sum_{j=1}^{n-1} a_{kj} x_j^* \right] = b_i$, for $i=1, \dots, k-1$

$\therefore \sum \left(a_{ij} - \frac{a_{in}}{a_{kn}} a_{kj} \right) x_j^* = b_i - \frac{a_{in}}{a_{kn}} b_k$ for $i=1, \dots, k-1$.

$\therefore (x_1^*, \dots, x_{n-1}^*)$ satisfies the new system. By the induction hypothesis there exists $(\bar{x}_1, \dots, \bar{x}_{n-1})$ with all co-ordinates rational such that

$\|(\bar{x}_1, \dots, \bar{x}_{n-1}) - (x_1^*, \dots, x_{n-1}^*)\| < \min \left\{ \frac{\epsilon}{2}, \delta \right\}$. Let $\bar{x}_n = \frac{1}{a_{kn}} \left[b_k - \sum_{j=1}^{n-1} a_{kj} \bar{x}_j \right]$. Clearly \bar{x}_n is

rational since it is obtained from $\|(\bar{x}_1, \dots, \bar{x}_{n-1})$. Further, $|\bar{x}_n - x_n^*| < \frac{\epsilon}{n}$.

$\therefore \|(\bar{x}_1, \dots, \bar{x}_n) - (x_1^*, \dots, x_n^*)\|^2 = \sum_{j=1}^n (\bar{x}_j - x_j^*)^2 = \sum_{j=1}^{n-1} (\bar{x}_j - x_j^*)^2 < \frac{\epsilon^2}{4} + \frac{\epsilon^2}{n^2} \leq \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2}$.

$\therefore \|(\bar{x}_1, \dots, \bar{x}_n) - (x_1^*, \dots, x_n^*)\| < \frac{\epsilon}{\sqrt{2}} < \epsilon$. Now $\bar{x}_n = \frac{1}{a_{kn}} \left[b_k - \sum_{j=1}^{n-1} a_{kj} \bar{x}_j \right]$ implies

$\sum_{j=1}^n a_{kj} \bar{x}_j = b_k$. For $i < k$,

$\sum_{j=1}^{n-1} \left(a_{ij} - \frac{a_{in}}{a_{kn}} a_{kj} \right) \bar{x}_j = b_i - \frac{a_{in}}{a_{kn}} b_k$. $\sum_{j=1}^{n-1} a_{ij} \bar{x}_j - \sum_{j=1}^{n-1} \frac{a_{in}}{a_{kn}} a_{kj} \bar{x}_j = b_i - \frac{a_{in}}{a_{kn}} b_k$ or

$\sum_{j=1}^{n-1} a_{ij} \bar{x}_j + a_{in} \left[\frac{b_k}{a_{kn}} - \sum_{j=1}^{n-1} a_{kj} \bar{x}_j \right] = b_i$ i.e., $\sum_{j=1}^n a_{ij} \bar{x}_j = b_i$. Hence if the theorem is assumed

true for $1, \dots, n-1$ with $n-1 \geq 0$, then it is true for n . We have already shown that it is true for 1. Hence it is true for all n .

Q.E.D.

Proposition 2 : Let C be a Federation BVA which satisfies robustness. Then, C is a WBVA.

Proof :- We prove this proposition by induction on $n = \# N$. If N is a singleton, we have $N = \{1\}$. By unanimity, $\{1\} = \Omega$. Let $v: N \rightarrow \mathbb{R} \cup \{0\}$ be defined by $v(1) = 1$ and

let $\kappa = 1$. Then clearly, $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b)

$0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$.

Suppose the proposition is true for $\#N = 1, \dots, r-1$, where r is any natural number.

Let $\#N = r$. For $w \subset N$, let $e_w: N \rightarrow \{0, 1\}$ be defined by $e_w(i) = 1$ if $i \in w$, $e_w(i) = 0$ if $i \notin w$. Thus $1 \in C(e_w)$ whenever $w \in W(\Omega)$ and $1 \notin C(e_w)$, whenever $w \in L(\Omega)$.

Let $A = \left\{ \sum_{w \in P(\Omega)} t_w e_w / t_w \in [0, 1] \forall w \in W(\Omega) \text{ and } \sum_{w \in P(\Omega)} t_w = 1 \right\}$ and

$$B = \left\{ \sum_{w \in L(\Omega)} t_w e_w / t_w \in [0,1] \forall w \in L(\Omega) \text{ and } \sum_{w \in L(\Omega)} t_w = 1 \right\}$$

Both A and B are non-empty convex subsets of \mathfrak{R}^r . Let $\chi \in A$ and $\eta \in B$ with $\chi = \sum_{w \in \Omega} t_w e_w = \sum_{w \in L(\Omega)} t_w e_w = \eta$ and t_w being a rational number for all $w \in \Omega \cup L(\Omega)$. By

Taking the LCM of the denominators we may assume that $\forall w \in \Omega \cup L(\Omega)$,

$$t_w = \frac{n_w}{K} \text{ where } K \in \mathbb{N} \text{ and } n_w \in \mathbb{N} \cup \{0\}. \text{ Thus } \sum_{w \in \Omega} n_w = \sum_{w \in L(\Omega)} n_w = K. \text{ Taking } n_w$$

copies of w for each $w \in \Omega \cup L(\Omega)$ we get a violation of robustness.

Now suppose,

$$\sum_{w \in \Omega} t_w e_w - \sum_{w \in L(\Omega)} t_w e_w = 0, \text{ and}$$

$$\sum_{w \in \Omega} t_w = 1 = \sum_{w \in L(\Omega)} t_w$$

has a non-negative solution.

$$\therefore \sum_{\substack{w \in \Omega \\ t_w > 0}} t_w e_w - \sum_{\substack{w \in L(\Omega) \\ t_w > 0}} t_w e_w = 0, \text{ and}$$

$$\sum_{\substack{w \in \Omega \\ t_w > 0}} t_w = 1 = \sum_{\substack{w \in L(\Omega) \\ t_w > 0}} t_w$$

has a strictly positive solution.

Hence by Lemma 1, $\sum_{\substack{w \in \Omega \\ t_w > 0}} t_w e_w - \sum_{\substack{w \in L(\Omega) \\ t_w > 0}} t_w e_w = 0$, and

$$\sum_{\substack{w \in \Omega \\ t_w > 0}} t_w = 1 = \sum_{\substack{w \in L(\Omega) \\ t_w > 0}} t_w$$

has a strictly positive solution all whose co-ordinates are rational. Thus,

$$\sum_{w \in \Omega} t_w e_w - \sum_{w \in L(\Omega)} t_w e_w = 0, \text{ and}$$

$$\sum_{w \in \Omega} t_w = 1 = \sum_{w \in L(\Omega)} t_w$$

has a non-negative solution all whose co-ordinates are rational,

contradicting what we obtained earlier in the proof. Thus $A \cap B = \emptyset$. By the separating hyperplane theorem for non empty compact convex sets, there

exists $p \in \mathfrak{R}^N \setminus \{0\}$ such that $p \cdot \chi > p \cdot \eta \forall (\chi, \eta) \in A \times B$. Thus

$$p \cdot e_w > p \cdot e_{w'} \forall (w, w') \in W(\Omega) \times L(\Omega).$$

Suppose for some $j \in N: p_j < 0$.

Case 1 :- There exists $w \in \Omega$ such that $j \in w$. Let $w' \in w \setminus \{j\}$. Thus, $w' \in L(\Omega)$ and $p \cdot (e_w - e_{w'}) = p_j < 0$ contradicting what we obtained above.

Case 2 :- There does not exist $w \in \Omega$ such that $j \in w$. Hence, for all $w \in W(\Omega)$, $w \setminus \{j\} \in W(\Omega)$. Without loss of generality suppose $j = n$. Let $W(\bar{\Omega}) = \{w \setminus \{n\} / w \in W(\Omega)\}$.

Thus $\#(N \setminus \{n\}) = n-1$ and it is easily verified that the Federation BVA defined as: $\{\forall S \in \mathfrak{X}^{N \setminus \{n\}} : \bar{C}(S) = \{\chi \in X / W(\chi, S) \in W(\bar{\Omega})\}\}$ satisfies robustness. Then by the induction

hypothesis, there exists a function $v' : N \setminus \{n\} \rightarrow \mathbb{N} \cup \{0\}$ and a natural number κ such that such that $\forall S \in \mathfrak{X}^{N \setminus \{n\}} : (a) 1 \in \bar{C}(S)$ if and only if $\sum_{i \in N} S_i v'(i) \geq \kappa$; (b) $0 \in \bar{C}(S)$ if

and only if $\sum_{i \in N} v'(i) - \sum_{i \in N} S_i v'(i) \geq \kappa$.

Let $v: N \rightarrow \mathbb{N} \cup \{0\}$ be defined by setting, $v(i) = v'(i) \forall i \in N \setminus \{n\}$, and $v(n) = 0$. Then it is easily verified that $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b) $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$.

Hence $p \in \mathfrak{R}_+^N \setminus \{0\}$. Clearly there exists $\bar{p} \in \mathfrak{R}_+^N \setminus \{0\}$ with all co-ordinates rational such that $\min\{\bar{p} \cdot e_w / w \in \Omega\} > \max\{\bar{p} \cdot e_w / w \in L(\Omega)\}$. By multiplying the numerators of \bar{p} by the LCM of the denominators we get $v: N \rightarrow \mathbb{N} \cup \{0\}$ such that

$\min_{i \in w} \{\sum v(i) / w \in \Omega\} > \max_{i \in w} \{\sum v(i) / w \in L(\Omega)\}$. Let $\kappa = \min_{i \in w} \{\sum v(i) / w \in \Omega\}$. Thus

$\forall S \in X^N$: $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$. Since C is a Federation BVA, C satisfies neutrality. Hence, $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$. The proposition stands established by a standard induction argument.

Q.E.D.

Propositions 1 and 2 combined together, constitute a proof of the following theorem:

Theorem 2 : A Federation BVA C is a WBVA if and only if C satisfies robustness.

In view of Theorems 1 and 2 the following characterisation theorem for a WBVA is immediate.

Theorem 3 : A BVA C is a WBVA if and only if C satisfies monotonicity, neutrality and robustness.

Example 1: Let $C(S) = \{x \in X / w \subset W(x, S)\}, \forall S \in X^N$, for some non-empty subset w of N . Clearly C is an oligarchy. Define $v: N \rightarrow \mathbb{N} \cup \{0\}$ as follows: $v(i) = 1$ if $i \in w$, $v(i) = 0$ if $i \notin N \setminus w$. Let $\kappa = \#w$. Then, $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b) $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$.

Example 2: Let k be a positive integer less than or equal to n and let $C(S) = \{x \in X / |W(x, S)| \geq k\}, \forall S \in X^N$. Clearly C is a k -votes BVA. Define $v: N \rightarrow \mathbb{N} \cup \{0\}$ as follows: $v(i) = 1$ if $\forall i \in N$. Let $\kappa = k$. Then, $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b) $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$.

Example 3: Let $i \in N$ and let $C(S) = \{x \in X / i \in W(x, S)\}, \forall S \in X^N$. Clearly C is a Dictatorial BVA. Define $v: N \rightarrow \mathbb{N} \cup \{0\}$ as follows: $v(j) = 1$ if $j = i$, $v(j) = 0$ if $j \neq i$. Let $\kappa = 1$. Then, $\forall S \in X^N$: (a) $1 \in C(S)$ if and only if $\sum_{i \in N} S_i v(i) \geq \kappa$; (b) $0 \in C(S)$ if and only if $\sum_{i \in N} v(i) - \sum_{i \in N} S_i v(i) \geq \kappa$.

Note: In some senses, an oligarchy is a basic unit of any Federation BVA. For, let $C(S) = \{x \in X / W(x, S) \in W(\Omega)\}, \forall S \in X^N$ where $\Omega = \{w_1, \dots, w_q\}$ is a collection of nonempty subsets of N . For $i \in \{1, \dots, q\}$, let $v_i: N \rightarrow \mathbb{N} \cup \{0\}$ be defined as follows: $v_i(j) = 1$ if $j \in w_i$, $v_i(j) = 0$ if $j \in N \setminus w_i$. For $i \in \{1, \dots, q\}$, let $\kappa_i = \#w_i$. Thus, $[w \in W(\Omega)]$ if and

only if $[w \neq \emptyset, w \subset N \text{ and } [\exists i \in \{1, \dots, q\} : \sum_{j \in w} v_i(j) \geq \kappa_i]]$. In view of this observation, the following theorem stands established :

Theorem 4 : Let C be a Federation BVA. Then, there exists a natural number k and WBVA's C_1, \dots, C_k such that $\forall S \in X^N : C(S) = \cup \{ C_i(S) / i \in \{1, \dots, k\}$.

Let C be a Federation BVA. Then, $\min \{k / \forall S \in X^N C(S) = \cup \{ C_i(S) / \forall i \in \{1, \dots, k\} : C_i \text{ is a BVA} \}$ is called the dimension of C , and is denoted by $k(C)$. Clearly $k(C)$ is always greater than or equal to one, and is equal to one if and only if C is a Weighted voting operator. Thus the dimension of an oligarchy, a k - votes BVA and any Dictatorial BVA is one. However it is easy to provide examples of a Federation BVA for which $k(C)$ is greater than one.

Example 4: Let $n = 2k$ for some positive integer k . Let $\Omega = \{ w \subset N / w = \{2j-1, 2j\}, \text{ for some } j \in \{1, \dots, k\}\}$. Let C be a Federation BVA such that $\forall S \in X^N : C(S) = \{x \in X / W(x, S) \in \Omega\}$. Towards a contradiction suppose that C is a WBVA. Then, there exists a function $v : N \rightarrow \mathbb{N} \cup \{0\}$ and a natural number κ such that $[w \in W(\Omega)]$ if and only if $[w \neq \emptyset, w \subset N \text{ and } \sum_{i \in w} v(i) \geq \kappa]$. Thus, $v(1) + v(2) \geq \kappa, v(3) + v(4) \geq \kappa$, since both $\{1, 2\}$ and

$\{3, 4\}$ belong to Ω . Hence either $v(2) + v(3) \geq \kappa$ or $v(1) + v(4) \geq \kappa$. Thus, either $\{2, 3\}$ or $\{1, 4\}$ belongs to $W(\Omega)$, contradicting our definition of Ω . Thus, C is not a WBVA.

For $i \in \{1, \dots, k\}$, let $v_i : N \rightarrow \mathbb{N} \cup \{0\}$ be defined as follows: $v_i(2i-1) = v_i(2i) = 1, v_i(j) = 0$ if $j \in N \setminus \{2i-1, 2i\}$. For $i \in \{1, \dots, q\}$, let $\kappa_i = 2$. Thus, $[w \in W(\Omega)]$ if and only if $[w \neq \emptyset, w \subset N \text{ and } [\exists i \in \{1, \dots, k\} : \sum_{j \in w} v_i(j) \geq \kappa_i]]$. Infact, it is possible to establish via an induction

argument that $k(C) = k$. For $k = 1, 2$ it is easy to verify that $k(C) = k$. Assume that $k(C) = k$, for $k = 1, \dots, r-1 \geq 2$. Let, $k=r$ and towards a contradiction suppose that $k(C) < r$. Thus, for $i \in \{1, \dots, k(C)\}$ with $k(C) < r$, there exists functions $v_i : N \rightarrow \mathbb{N} \cup \{0\}$ and a natural number κ_i such that $[w \in W(\Omega)]$ if and only if $[w \neq \emptyset, w \subset N \text{ and } [\exists i \in \{1, \dots, k(C)\} : \sum_{j \in w} v_i(j) \geq \kappa_i]]$. Since, $\Omega = \{ w \subset N / w = \{2j-1, 2j\}, \text{ for some } j \in \{1, \dots, r\}\}$,

there exists $j_1, j_2 \in \{1, \dots, r\}$ with $j_1 \neq j_2$ and $h \in \{1, \dots, k(C)\}$, such that (a) $v_h(2j_1 - 1) + v_h(2j_1) \geq \kappa_h$; (b) $v_h(2j_2 - 1) + v_h(2j_2) \geq \kappa_h$; (c) $[\sum_{j \in w} v_h(j) \leq \kappa_h, \text{ if } w \notin P(\Omega)]$. By symmetry

of the problem under consideration, we may let $j_1 = 1$ and $j_2 = 2$. Thus for $k = 2, k(C) = 1$, which is not possible. Thus, $k(C) = r$. By a standard induction argument, it follows that $k(C) = k$ for every natural number k .

Acknowledgement : I would like to thank Fuad Aleskerov for his comments on this paper and his sustained help for my current research on social choice theory.

References

1. M. Aizerman, F. Aleskerov (1986): Voting Operators in the Space of Choice Functions. Math. Soc. Sci. Volume 11: 201-242.
2. M. Aizerman, F. Aleskerov (1995): Theory of Choice. North Holland, Amsterdam.

3. F.Aleskerov (1999) : Arrovian Aggregation Procedure. Kluwer Academic Publishers.Vol. 39, Series B. Theory and Decision Library.
4. S. Lahiri (1999) : Voting Operators on Ballot Profiles.In Harrie de Swart (ed.):Logic, Game Theory and Social Choice. Tilburg Univ.Press.
5. S.Lahiri (2001) : Axiomatic Characterizations of Voting Operators. *Mathematical Social Sciences*. Volume 41, Issue 2, pages 227-238.
6. S.Lahiri (2000) : Axiomatic Analysis of Vote Aggregators. mimeo
7. H.Moulin (1985):Choice Functions Over a Finite Set: A Summary. *Social Choice Welfare*. Volume 2,147-160.
8. L. Shapley (1962) :Simple Games: an outline of the descriptive theory. *Behavioral Science*. Volume 7,59-66.
9. L.A. Sholomov (2000) : Explicit form of neutral social decision rules for basic rationality conditions. *Mathematical Social Sciences*. Volume 39, 81-107.
10. A.Taylor and W.Zwicker (1992) :A Characterisation of Weighted Voting. *Proceedings of the American Mathematical Society*. Volume 115, 1089 - 1094.

PURCHASED
APPROVAL
GRATE/EXCHANGE
PRICE
ACC NO
VIJAYAM SARASWATI LIBRARY
110 AHMEDABAD