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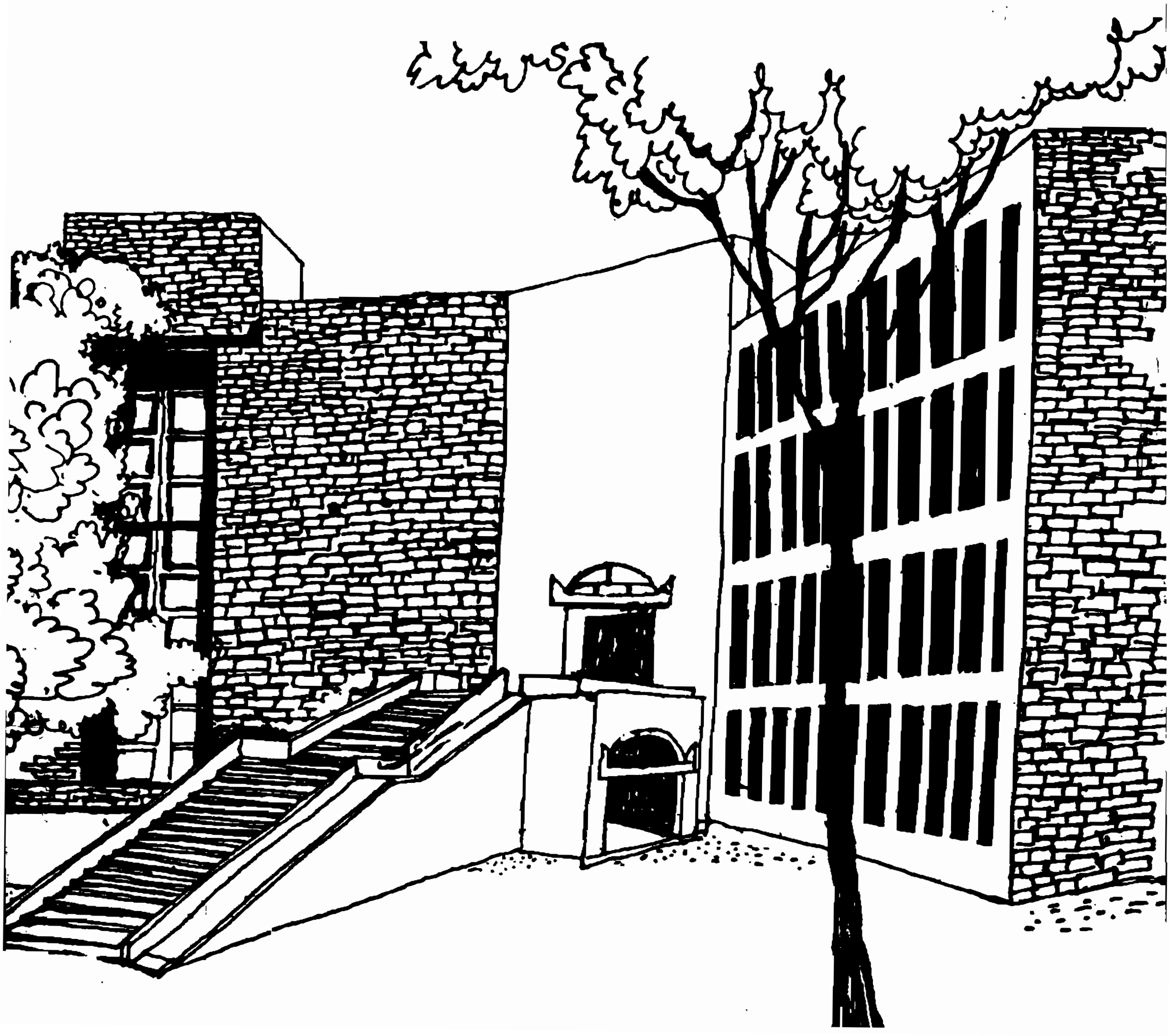

THE INDIRECT UTILITY EXTENSION: AXIOMATIC CHARACTERIZATIONS

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W.P.No. 2001-02-05
February 2001 /1640

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The Indirect Utility Extension: Axiomatic Characterizations

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Abstract

In this paper, we first provide several axiomatic characterizations of the indirect utility extension.

1. Introduction: The general problem we are interested in this paper is of the following variety: We are given a finite universal set and a linear ordering on it. What is a minimal axiomatic characterization of the indirect utility extension ? In keeping with the prevalent terminology in economic theory, an extension of a linear order from a set to the power set is called the indirect utility extension provided it is the case that one set is considered at least as desirable as a second if and only the largest element of the first set is no smaller than the largest element of the second .

In Kannai and Peleg [1984] we find the starting point of the related literature, wherein it is asserted that if the cardinality of the universal set is six or more, then there is no weak order on the power set which extends the linear order and satisfies two properties: one due to Gardenfors and the other known as Weak Independence. This result was followed by a quick succession of possibility results in Barbera, Barret and Pattanaik [1984], Barbera and Pattanaik [1984], Fishburn [1984], Heiner and Packard [1984], Holzman [1984], Nitzan and Pattanaik [1984] and Pattanaik and Peleg [1984]. Somewhat later, Bossert [1989] established a possibility result by dropping the completeness axiom for the binary relation on the power set and otherwise using the same axioms as in Kannai and Peleg [1984].

In recent times Malishevsky [1997] and Nehring and Puppe [1999] have addressed the problem of defining an "indirect utility preference". Malishevsky [1997] addresses the integrability problem: given a weak order on the power set, under what conditions is it an indirect utility preference? A similar question is also addressed in Nehring and Puppe [1999]. This problem is further extended and generalized in Lahiri [2001]. In Lahiri [2000], we provide axiomatic characterizations of several extensions including the indirect utility extension.

In this paper, we first provide several more axiomatic characterizations of the indirect utility extension.

2. <u>The Model</u>: Let n be a positive integer and let X be the set of first n positive integers. Let [X] denote the set of all non-empty subsets of X. Given $A \in [X]$, let $\#(A)$ denote the number of elements in A.

Let $\Re$ be a binary relation on [X]. It is said to be (i) <u>reflexive</u> if $\forall A \in [X], (A, A) \in \Re$; (ii) <u>complete</u> if $\forall A, B \in [X]$ with $A \neq B$, either $(A, B) \in \Re$ or $(B, A) \in \Re$; (iv) <u>transitive</u> if $\forall A, B, C \in [X]$, $[(A, B) \in \Re, (B, C) \in \Re]$ implies $(A, C) \in \Re$.

Let $I(\Re) = \{(A, B) \in \Re / (B, A) \in \Re\}$, $P(\Re) = \{(A, B) \in \Re / (B, A) \notin \Re\}$. $I(\Re)$ is called the symmetric part of $\Re$ and $P(\Re)$ is called the asymmetric part of $\Re$.

Given $A \in [X]$, let g(A) be the unique element of A satisfying $g(A) \geq x$ whenever $x \in A$ and let l (A) be the unique element of A satisfying $x \geq l(A)$ whenever $x \in A$.

A binary relation $\Re$ on [X] is said to satisfy

(a) <u>Gardenfor's Property(GP)</u> if $\forall A \in [X]$ and $x \in X \setminus A$, (i) $x > g(A)$ implies $(A \cup \{x\}, A) \in P(\Re)$; (ii) $l(A) > x$ implies $(A, A \cup \{x\}) \in P(\Re)$.

(b) <u>Weak Independence(W.IND)</u> if $\forall A, B \in [X]$ with $(A, B) \in P(\Re)$, if

$x \in X \setminus (A \cup B)$ then $(A \cup \{x\}, B \cup \{x\}) \in \Re$.

Kannai and Peleg [1984] show the following:

Theorem 1:- If $n > 5$, then there does not exist any binary relation on [X] which satisfies reflexivity, completeness, transitivity, GP and W.IND.

Bossert [1989] proves the existence of a unique binary relation on [X] which satisfies all the properties in Theorem 1 other than completeness.

3. The Indirect Utility Extension:Let $\overline{\Re} = \{(A,B) \in [X] \times [X] / g(A) \geq g(B)\}$. $\overline{\Re}$ is called the indirect utility extension. It is easy to see that $\overline{\Re}$ satisfies reflexivity, completeness, transitivity and W.IND, but does not satisfy GP. However, it satisfies the following property which modifies a similar one due to Barbera [1977]:

Property 1: $\forall$ x, y $\in$ X, [x > y implies $(\{x\}, \{x, y\}) \in \Re$ and $(\{x, y\}, \{y\}) \in P(\Re)$].

Further $\overline{\Re}$ satisfies the following modification of W.IND:

Property 2: $(A, B) \in \Re$ and $x \in X \setminus (A \cup B)$ implies $(A \cup \{x\}, B \cup \{x\}) \in \Re$.

Note: Property 2 implies W.IND.

A property found in Nehring and Puppe [1999] is the following:

Monotonicity (MON): $\forall$ A, B $\in$ [X], $B \subset A$ implies $(A, B) \in \Re$.

It is not difficult to see that $\overline{\Re}$ satisfies (MON). The following theorem is available in Lahiri [2000]:

Theorem 2: The only transitive binary relation on [X] to satisfy Property 1, Property 2 and MON is $\overline{\Re}$.

The next property which by rights should be attributed to Puppe[1996] is quite interesting, both in content and by way of implication:

<u>Puppe Property</u>: $(A, A\backslash\{x\}) \in P(\mathfrak{R})$ if and only if $g(A)= x$.

The following theorem has been established in Lahiri [2000]:

<u>Theorem 3:</u> The only reflexive, complete and transitive binary relation on [X] to satisfy the Puppe Property is $\overline{\mathfrak{R}}$.

It might be worth noting the following assumption due to Kreps [1979] and its immediate implication:

<u>Preference for flexibility</u> : $(A, A\backslash\{x\}) \in \mathfrak{R}$ and $(A, A\backslash\{x\}) \in P(\mathfrak{R})$ if and only if $g(A)= x$.

Preference for flexibity is a slight modification of GP. The next theorem has been proved in Lahiri [2000]:

<u>Theorem 4:</u> The only reflexive and transitive binary relation on [X] to satisfy the Preference for Flexibility is $\overline{\mathfrak{R}}$.

We shall now be concerned with the following axioms:

<u>Consistency (with respect to expansions)</u>: $\forall A,B \in [X]$ with $A \subset B$ and $x \in X\backslash(A \cup B)$: $[(A\cup\{x\},A) \in I(\mathfrak{R})]$ implies $[(B\cup\{x\},B) \in I(\mathfrak{R})]$.

<u>Weak Restricted Monoonicity</u>: $\forall x,y \in X$: $[x > y]$ implies $[(\{x\},\{x,y\}) \in I(\mathfrak{R})]$.

<u>Strict Restricted Monoonicity</u>: $\forall x,y \in X$: $[x > y]$ implies $[(\{x,y\},\{y\}) \in P(\mathfrak{R})]$.

<u>Restricted Monooonicity</u>: $\forall x,y \in X$: $[x > y]$ implies $[(\{x\},\{x,y\}) \in I(\mathfrak{R})$ and $(\{x,y\},\{y\}) \in P(\mathfrak{R})]$.

Clearly Restricted Monotonicity is equivalent to the simultaneous satisfaction of Weak Restricted Monotonicity and Strict Restricted Monotonicity.

<u>Robustness</u>: $\forall A \in [X]$ and $x \in X\backslash A$:$[g(A)>x]$ implies $(A\cup\{x\},A) \in I(\mathfrak{R})]$.

<u>Extension</u>: $\forall x,y \in X$: $[x > y]$ implies $[(\{x\},\{y\}) \in P(\mathfrak{R})]$.

Proposition 1: $\overline{\mathfrak{R}}$ satisfies Consistency, Restricted Monotonicity, Robustness and Extension.

Proposition 2: Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] which satisfies Weak Restricted Monotonicity and Consistency. Then, $\forall A \in [X]$: $(A,\{g(A)\}) \in I(\mathfrak{R})$.

Proof: Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] satisfying Consistency and Weak Restricted Monotonicity and let $A \in [X]$. If $A=\{g(A)\}$, then the proposition follows by the reflexivity of $\mathfrak{R}$.Hence suppose that $A = \{a_1,\ldots,a_m\}$ for some positive integer greater than or equal to two. Further suppose that $a_1>a_2>\ldots>a_m$.Thus, $g(A) = a_1$. By Weak Restricted Monotonicity, $(\{a_1,a_i\},\{a_1\}) \in I(\mathfrak{R})$ for $i \in \{2,\ldots,m\}$. If m=2, then the proposition stands established. Hence suppose that $m >2$. Since $\{a_1\} \subset \{a_1,a_2\}$ by Consistency $(\{a_1,a_2,a_i\},\{a_1,a_2\}) \in I(\mathfrak{R})$ for $i \in \{3,\ldots,m\}$. Suppose that $(\{a_1,\ldots,a_{k-1},a_i\}, \{a_1,\ldots,a_{k-1}\}) \in I(\mathfrak{R})$ for $i \in \{k,\ldots,m\}$ and $m-1 \geq k$. Since $\{a_1,\ldots,a_{k-1}\} \subset \{a_1,\ldots,a_{k-1},a_k\}$,it follows by Weak Restricted Monotonicity that $(\{a_1,\ldots,a_k,a_i\}, \{a_1,\ldots,a_k\}) \in I(\mathfrak{R})$ for $i \in \{k,\ldots,m\}$. By a standard induction argument we arrive at $(A, A\backslash\{a_m\}) \in I(\mathfrak{R})$. By transitivity of $\mathfrak{R}$, we may conclude that $(A,\{g(A)\}) \in I(\mathfrak{R})$.

Proposition 3 : Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] which satisfies Robustness. Then, $\forall A \in [X]$: $(A,\{g(A)\}) \in I(\mathfrak{R})$.

Proof: Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] satisfying Robustness and let $A \in [X]$. If $A=\{g(A)\}$, then the proposition follows by the reflexivity of $\mathfrak{R}$.Hence suppose that $A = \{a_1,\ldots,a_m\}$ for some positive integer greater than or equal to two. Further suppose that $a_1>a_2>\ldots>a_m$.Thus, $g(\{a_1,\ldots,a_k\}) = a_1>a_k$ for all $k \in \{2,\ldots,m\}$. By

Robustness, $(\{a_1,\ldots,a_{k-1}\}, \{a_1,\ldots,a_k\}) \in I(\mathfrak{R})$ for all $k \in \{2,\ldots,m\}$. By transitivity of $\mathfrak{R}$, we may conclude that $(A,\{g(A)\}) \in I(\mathfrak{R})$.

Proposition 4 : Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X].If $\mathfrak{R}$ satisfies Restricted Monotonicity then it satisfies Extension.

Proof: Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] which satisfies Restricted Monotonicity and let $x,y \in X$ with $x > y$. By Weak Restricted Monotonicity, $(\{x\},\{x,y\}) \in I(\mathfrak{R})$ and by Strict Restricted Monotonicity, $(\{x,y\},\{y\}) \in P(\mathfrak{R})$. By transitivity of $\mathfrak{R}$ we get, $(\{x\},\{y\}) \in P(\mathfrak{R})$.Thus, $\mathfrak{R}$ satisfies Extension.

Proposition 3 : Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] which satisfies Extension. Suppose that , $\forall A \in [X]$: $(A,\{g(A)\}) \in I(\mathfrak{R})$. Then, $\mathfrak{R} = \overline{\mathfrak{R}}$.

Proof: Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X] which satisfies Extension. Suppose that , $\forall A \in [X]$: $(A,\{g(A)\}) \in I(\mathfrak{R})$. Let $A,B \in [X]$. Suppose that, $g(A) = g(B)$. By reflexivity of $\mathfrak{R}$, $(\{g(A)\},\{g(B)\}) \in I(\mathfrak{R})$ and by hypothesis $(A,\{g(A)\})$, $(B,\{g(B)\}) \in I(\mathfrak{R})$. By transitivity of $\mathfrak{R}$ we get $(A,B) \in I(\mathfrak{R})$.

Now suppose that $g(A) > g(B)$. By Extension $(\{g(A)\},\{g(B)\}) \in P(\mathfrak{R})$ and by hypothesis $(A,\{g(A)\})$, $(B,\{g(B)\}) \in I(\mathfrak{R})$. By transitivity of $\mathfrak{R}$ we get $(A,B) \in P(\mathfrak{R})$. Thus, $\mathfrak{R} = \overline{\mathfrak{R}}$.

As a consequence of the above propositions the following theorem stands proved:

Theorem 5: Let $\mathfrak{R}$ be a reflexive and transitive binary relation on [X]. Then the following statements are equivalent:

(a) $\mathfrak{R}$ satisfies Weak Restricted Monotonicity,Consistency and Extension;

(b) $\mathfrak{R}$ satisfies Restricted Monotonicity and Consistency;

(c) $\mathfrak{R}$ satisfies Robustness and Consistency;

(d) $\mathfrak{R}$ satisfies Robustness and Restricted Monotonicity;

(e) $\mathfrak{R} = \overline{\mathfrak{R}}$.

Acknowledgement:-This paper has benefited from useful discussions with Prasanta Pattanaik. I would like to thank him for his encouragement and advise in this and related research.

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