T.R. NO.3

Technical Report

BEHAVIOUR OF FIRMS SUBJECT TO IMPERFECT MARKETS

by

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July 1973

Indian Institute of Management
Ahmedabad

No. ----------

To

Chairman (Research)
IIMA

Technical Report

Title of the report ..BEHAVIOUR OF FIRMS SUBJECT TO IMPERFECT MARKETS

Name of the Author ..Prof.P.N.Misra..............................

Under which area do you like to be classified? ..Economics.......

ABSTRACT (within 250 words)

A survey of available theories in economic literature would reveal that analysis of a firm's behaviour is performed, either, under the assumption of perfect competition in factor as well as output markets or by letting imperfections to prevail in these markets separately and then examining firm's decision making process in each case. At the same time it is widely accepted that perfect market conditions are simply a myth. In the present paper we analyse the behaviour of firm under imperfect market condition prevailing simultaneously in all the output and input markets. It is shown that the present approach enables us to solve for optimal magnitudes of size of the firm, factors of production, factor prices and the product price. We also discuss role of taxation on various aspects of the firm.

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Date 3/8/73....

P.N. Misra
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BEHAVIOUR OF FIRMS SUBJECT TO IMPERFECT MARKETS

by

P. N. MISRA[1]

1. Introduction

A survey of available theories in economic literature would reveal that analysis of a manufacturing firm's behaviour is performed, either, under the assumption of perfect competition in factor as well as output markets, or by letting imperfections to prevail in these markets separately and then examining the firm's decision making process in each case. At the same time, it is acknowledged that perfect market conditions are attained only in rare cases and, in actual practice, imperfections prevail in all the markets simultaneously [10, pp. 546-550]. We may visualize three situations, viz., (a) when all the markets are perfect, (b) when one or more markets are imperfect while the rest are perfect and (c) when all the markets are imperfect. Theory of a firm's behaviour under situation (a) is very well known and that under situation (b) has been discussed systematically in [3,7] by means of geometrical tools. A mathematical treatment of this theory may be found in Allen [2, pp.372-374] who, first of all, obtained optimality conditions under perfect market conditions and then, using these conditions, analysed the variation of the demands for the factors of production under imperfect factor markets. Very little, however, is known in case situation (c) prevails. In fact, situation (c) provides a general picture of markets in every day life and situations (a) and (b) are merely its particular cases when all or a few markets are perfect. For instance, the polar cases of perfect competition and monopoly [8] arise out of situation (c) when the demand for the output and the

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[1] This work was done while the author was at the University of Manchester, England.

supplies of the factors are perfectly elastic in the first case and the seller is able to exercise complete control on magnitude or price of the product being sold in the second. Therefore, if a comprehensive analysis in case of situation (c) were available, deductions regarding various other situations could be made in a straight forward manner.

An economic theory based upon realistic assumptions could prove more helpful in explaining emperical facts as well as increasing precision in actual decision making. Frequently, serious problems of interpretation of results arise when a firm's behaviour, say, production function, is quantified under the assumption of situations (a) or (b) while (c) is true. This is because firstly, theories corresponding to situations (a), (b) and (c) would be distinctly different from each other, secondly, maintained hypotheses for econometric analyses are usually guided by the available economic theory and finally, such a hypothesis would be grossly misspecified if theory itself were based upon unreaslistic assumptions. Under such circumstances, even though most sophisticated econometric methods were used, one may expect a confrontation between emperical results on the one hand and economic theory on the other. Further, since assumptions underlying an economic theory are not amenable to statistical testing, it appears essential to develop economic theory under most general situations.

The present article is an attempt towards achieving the aforesaid objective. Throughout this paper we shall use terms 'perfect competition' and 'perfect market conditions' interchangeably and their meaning should be understood same as in /8/, that is the elasticity coefficients are infinitely large both in context to demand and supply. Imperfection would merely mean lack of perfection. We consider imperfect market conditions in product and factor markets simultaneously and derive condition of equilibrium of a profit maximising firm. Deductions regarding market situations (a) and (b) are then made from this general condition. Next, we show that the present approach enables us to solve for optimal magnitudes of size of the firm, factors of production, product price and factor prices. We also provide some operational guidelines for firms equipped with adequate market research. Finally, we discuss the role of taxation in production and evolve appropriate tax functions regarding output as well as factor inputs which provide a basis for fixation of tax slabs in case of different products.

2. Equilibrium of a Firm

2.1 Equilibrium Under Imperfect Market Conditions

Let us consider a firm which operates under imperfect market conditions in respect of input factors as well as product. This implies that prices of output and factors are variables. Consequently, the decision making of the firm has to be based upon fluctuations in these prices because both sales revenue and cost are affected simultaneously. We suppose that the firm under consideration employs only two factors of production, namely, capital and labour. This assumption is not restrictive because our analysis could easily be extended to any number of factors of production without affecting our conclusions.

Denoting prices per unit of output (Q), labour (L) and capital (K) by p, w and r, respectively, we can write total sales revenue (R) as

(2.1) $$R = Qp$$

and total cost of production (C) as

(2) $C = Lw + Kr.$

Therefore, profit (Π) of the firm is given by

(2.3) $\Pi = R - C = Qp - Lw - Kr.$

Under imperfections prevailing in product market a firm would always look back to the shape of demand curve of its output before deciding how much to produce. Thus the demand function

(2.4) $Q = f(p)$

and price elasticity of demand

(2.5) $e = \frac{\partial Q}{\partial p} \frac{p}{Q}$

would enter the decision making process of the firm. On the other hand, cost of producing an amount Q would depend upon the supply functions of factors because a change in output implies changes in capital and labour which in turn influence prices through their supply functions. Therefore, the supply functions

(2.6) $L = f(w)$

of labour and

(2.7) $K = f(r)$

of capital as well as the corresponding supply elasticities

(2.8) $e_1 = \frac{\partial L}{\partial w} \frac{w}{L}$ and $e_2 = \frac{\partial K}{\partial r} \frac{r}{K}$

would have to be accounted for properly at every stage of decision making.

Besides being strained by these demand and supply conditions, a firm has to manufacture its product under the constraint of available technology which is reflected through the relationship between inputs on the one hand and output on the other, commonly known as production function. We may write it as

$$(2.9) \qquad Q = f(I, K)$$

where response in output to changes in L and K may be measured by the elasticity coefficients

$$(2.10) \qquad \alpha = \frac{\partial Q}{\partial L}\frac{L}{K} \quad \text{and} \quad \beta = \frac{\partial Q}{\partial K}\frac{K}{Q}$$

respectively.

The factors L and K are either substitutes or compliments. In the former case a decrease in L would lead to an increase in K and vice versa, whereas in case of latter both L and K increase or decrease together. The existence of such mutual response implies a functional relationship between L and K which we may express as

$$(2.11) \qquad K = f(L)$$

The relationship (2.11) is in fact a technological function because any technology could be precisely expressed through it. For a given technology the functional form of relation (2.11) could be specified or quantified as the case may be and changes in technology would lead to different functional forms. Response of capital to changes in labour may be measured by the elasticity coefficient

$$(2.12) \qquad \delta = \frac{\partial K}{\partial L}\frac{L}{K} = \frac{\alpha}{\beta}$$

which is positive if L and K are compliments and negative if they are substitutes. In case of compliments will range from 0 to ∞ and in case of substitutes the range of variation will extend from $-\infty$ to 0. A zero value of σ implies that the technology permits changes in labour employment for a given level of employed capital whereas $\sigma = \infty$ implies that capital can be varied for a constrant level of employed labour.

Thus profit II of the firm depends upon the manner in which the functions (2.4), 2.6), (2.7),(2.9) and (2.11) influence each other. Unlike perfect competition, each one of the symbols Q, p, L, w, K and r on the right hand side of (2.3) are variables and a change in any one of them will be transferred immediately to other variables through the chain of above mentioned functions. For instance, a change in Q would affect p through (2.4) and L and K through (2.9) or if only L is affected through (2.9) then K will be affected through (2.11). In turn changes in L and K would affect w and r through the functions (2.6) and (2.7), respectively. Accordingly, optimum point of function (2.3) may be sought by letting all six variables to influence II simultaneously and equilibrium of the firm would depend upon whether or not such an optimum exists.

Denoting total and partial changes in a variable x by dx and x, respectively and partial derivatives of II by f_i (i = 1, ..., 6) as below we can write

$$(2.13)\quad dII = \frac{\partial II}{\partial Q} dQ + \frac{\partial II}{\partial p} dp + \frac{\partial II}{\partial L} dL + \frac{\partial II}{\partial w} dw + \frac{\partial II}{\partial K} dK + \frac{\partial II}{\partial r} dr$$

$$= f_1 \, dQ + f_2 \, dp + f_3 \, dL + f_4 \, dw + f_5 \, dK + f_6 \, dr$$

At the optimum point of II, d II. must be equal to zero which implies that each one of f_i (i = 1, ..., 6) should be equal to zero. Considering f_1 and equating it to zero we obtain

(2.14) $f_1 = \frac{\partial \Pi}{\partial Q} = Q\frac{\partial p}{\partial Q} + p - L\frac{\partial w}{\partial Q} - w\frac{\partial L}{\partial Q} - K\frac{\partial r}{\partial Q} - r\frac{\partial K}{\partial Q} = 0.$

Finally, writing

(2.15) $\frac{\partial w}{\partial Q} = \frac{\partial w}{\partial L}\frac{\partial L}{\partial Q} \quad , \quad \frac{\partial r}{\partial Q} = \frac{\partial r}{\partial Q}\frac{\partial K}{\partial Q}$

and using (2.5), (2.8) and (2.10) we derive from (2.14) the equilibrium condition

(2.16) $(1 - e^{-1})\ Qp = \alpha^{-1}(1 + e_1^{-1})\ Lw + \beta^{-1}(1 + e_2^{-1})\ K r$

which can be rewritten as

(2.17) $Qp = a\ Lw + b\ Kr$

where

(2.18) $a = (1 - e^{-1})^{-1}(1 + e_1^{-1})\ \alpha^{-1}$ and $b = (1 - e^{-1})^{-1}(1 + e_2^{-1})\ \beta^{-1}$

The same condition as in (2.17) can be obtained if we equate any other f_i $(i = 2, \ldots, 6)$ to zero and make necessary simplification. This means that the equilibrium condition (2.17) implies optimum profit not only with respect to changes in Q but also with respect to changes in any one of the remaining five variables.

The optimality condition (2.17) does not necessarily imply a maximum. It may as well hold at a minimum point of the profit function (2.3). Therefore, we must seek for additional restriction under which the equilibrium condition (2.14) might guarantee maximum profit. From the notations adopted in (2.13) we observe that each f_i $(i = 1, \ldots, 6)$ corresponds to one variable. Retaining the same correspondence and denoting the second order partial derivatives by f_{ij} $(i,j = 1, \ldots, 6)$

we define a matrix

$$(2.19) \qquad H = \begin{bmatrix} f_{11} & \cdots & f_{16} \\ \vdots & & \\ f_{61} & \cdots & f_{66} \end{bmatrix}$$

It can then be seen that second order total derivative $d^2\Pi$ of Π is a quadratic form in terms of matrix H. Further, since a point in the present six dimensional space, satisfying condition $d\Pi = 0$, would be a maximum only if the point in its immediate neighbourhood slopes downwards, i.e., $d^2\Pi < 0$; therefore, the matrix H must be negative definite.[2] For actually obtaining a maximum one must quantify the equations in the first instance and then solve for the optimum levels of the variables with the help of condition (2.17). At the same time it should be verified that H is negative definite.

It will be observed that first order conditions of maximisation of profit through equation (2.3) as well imply that

$$(2.20) \qquad \frac{\partial R}{\partial Q} = \frac{\partial C}{\partial Q}, \ \ldots\ldots, \ \frac{\partial R}{\partial r} = \frac{\partial C}{\partial r},$$

that is, marginal revenue equals marginal cost in respect of all the six variables simultaneously. But computation of these marginal

[2] Some writers [I, p.478] have obtained second order condition under the constraint $d\Pi = 0$. But in that case $d^2\Pi = 0$ and therefore any other criterion requiring $d^2\Pi < 0$ subject to $d\Pi = 0$ would lead to [a] self contradiction.

magnitudes is not possible because, in pactice, one cannot hold five variables at constant levels and permit only one variable to change for examining changes in R or C. For example, $\frac{\partial R}{\partial Q}$ can be computed provided Q can be varied while L, K, v, r and p are held at constant levels which is clearly impractical. Accordingly, the marginal conditions as in (2.20) are of only theoretical value.

2.2 Equilibrium Under Perfect and Partially Imperfect Market Conditions

Conditions of equilibrium under perfect competition in all the markets as well as under partially imperfect market conditions, when only a few markets are perfect and the remaining ones are imperfect, can be derived directly from (2.17). Whatever may be the nature of imperfection, firms controlling the markets would have to function within the constraints imposed by ruling demand and supply functions such that degrees of imperfection would always find their expression through elasticity coefficients.[3] A specific market situation would, however, lead to specific values of the elasticity coefficients at the equilibrium point so that the coefficients a and b, defined in (2.18) would be constant so long as market situation remains unaltered. On the contrary, a change in market situation would affect a and b and therefore, for analysing equlibria conditions under different market situations, we may treat condition (2.17) as function of elasticity coefficients e, e_1 and e_2. Since we are concerned with fluctuations in three prices, namely, p, w and r the total number of ways in which one or more markets may be perfect is seven. We shall discuss some interesting situations only and skip over the rest.

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[3]See also Lerner [4].

The second order condition for equilibrium would require modification of matrix H, separately, for every special case of general imperfect market condition. For example, let perfect competition prevail in the product market so that price p is constant. Consequently, we are left with only five variables on the right hand side of (2.3) and therefore negative definitions of the modified matrix H*, obtained from H by deleting one column and one row corresponding to p, should be tested. Similar modifications of matrix H will be needed in respect of other types of perfections or partial perfections. We shall analyse changes in equilibrium condition (2.17) under various market situations and it would be understood that negative definiteness of an appropriate H matrix has been examined simultaneously in each case.

Let us consider a situation when perfect competition prevails in all the three markets simultaneously. In that case price variables p, w and r are at constant levels and elasticity coefficients e, e_1 and e_2 are equal to infinity. Consequently (2.18) reduces to

$$(2.21) \qquad a = \alpha^{-1}, \; b = \beta^{-1}$$

and equilibrium condition (2.17) is equivalent to

$$(2.22) \qquad Qp = \alpha^{-1} Lw + \beta^{-1} Kr.$$

It is obvious from (2.22) that entire revenue is distributed amongst the factors only if $\alpha = \beta = 1$, i.e.,

$$(2.23) \qquad \frac{\partial Q}{\partial L} = \frac{\partial Q}{\partial L} \text{ and } \frac{\partial Q}{\partial K} = \frac{\partial Q}{\partial K}$$

Working on the other points of the production curve where both α and β are less than unity, the entrepreneur would always make a profit irrespective of the form of production function.[4]

Next, we consider the situation when one market is perfect and others are imperfect. Allowing perfect competition to prevail in the product market, we obtain from (2.16)

$$(2.24) \qquad Qp = \alpha^{-1}(1+e_1^{-1})\ Lw + \beta^{-1}(1+e_2^{-1})\ Kr$$

where use has been made of the fact that $e = \infty$ under perfect competition. Similarly, perfect competitions in labour or capital market imply $e_1 = \infty$ and $e_2 = \infty$, respectively. The equilibria conditions under these situations may be derived from (2.17) as

$$(2.25) \qquad Qp = (1-e^{-1})^{-1}\ \alpha^{-1}Lw + b\ Kr$$

and

$$(2.26) \qquad Qp = a\ Lw + \beta^{-1}\ (1-e^{-1})^{-1}\ Kr.$$

The three remaining situations corresponding to imperfections in two markets and perfection in one may be obtained similarly by setting appropriate elasticity coefficients equal to infinity.

Considering an output having inelastic demand, that is the quantity demanded remains unaltered for changes in price, we note that $\frac{\partial Q}{\partial p} = 0$. Consequently, the production function (2.9) is reduced to a constant product curve such that $\frac{\partial Q}{\partial L} = \frac{\partial Q}{\partial K} = 0$. Therefore elasticity coefficients e, α and β are equal to zero simulatenously and the equilibrium condition (2.16) is satisfied because both sides

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[4]See also Joan Robinson /9/ where marginal analysis lead the author to a different conclusion.

are equal to infinity. But the coefficients a and b and in (2.18) appear to be indeterminate numbers. This is, however, not true because the limiting values exist. Using (2.5) and (2.10) we can write

$$(2.27)\qquad \alpha^{-1}(1-e^{-1})^{-1} = (\alpha - \alpha\, e^{-1})^{-1} = \left(\frac{\partial Q}{\partial L}\frac{L}{Q} + \frac{\partial Q}{\partial L}\frac{L}{Q}\frac{\partial p}{\partial Q}\frac{Q}{p}\right)^{-1} = \left(\frac{\partial Q}{\partial L}\frac{L}{Q} + \frac{\partial p}{\partial L}\frac{L}{p}\right)^{-1}$$

Therefore,

$$(2.28)\qquad \lim_{\partial Q=0} \left[\alpha^{-1}(1-e^{-1})^{-1}\right] = \frac{\partial L}{\partial p}\frac{p}{L} = \phi_1$$

which is elasticity of labour with respect to output price. Similarly, it can be shown that

$$(2.29)\qquad \lim_{\partial Q=0} \left[\beta^{-1}(1-e^{-1})^{-1}\right] = \frac{\partial K}{\partial p}\frac{p}{K} = \phi_2$$

where ϕ_2 is elasticity of capital with respect to output price. Combining (2.18), (2.28) and (2.29) with (2.17) we observe that limiting equilibrium condition can be written as

$$(2.30)\qquad Qp = \phi_1\,(1+e_1^{-1})\ Lw + \phi_2\,(1+e_2^{-1})\ Kr$$

which is, in fact, the equilibrium condition for the monopolist who chooses to keep output at a constant level.

On the contrary, if factor supplies are inelastic we obtain different sets of equilibrium conditions. For example, let supply of labour be inelastic, i.e., L is constant so that $\frac{\partial L}{\partial w} = \frac{\partial L}{\partial Q} = 0$.

This mean $\alpha^{-1} = 0$ and

(2.31) $\alpha^{-1}\ e_1^{-1} = \frac{Q}{L}\ \frac{\partial L}{\partial Q}\ \frac{L}{w}\ \frac{\partial w}{\partial L} = \frac{Q}{w}\ \frac{\partial w}{\partial Q} = \theta_1^{-1}$

where θ_1 is elasticity of output with respect to wages. Using these results we obtain from (2.17)

(2.32) $Qp = \theta_1^{-1}(1 - e^{-1})^{-1}\ Lw + b\ Kr$

Similarly, if supply of capital is inelastic we can derive

(2.33) $Qp = a\ Lw + \theta_2^{-1}\ (1-e^{-1})^{-1}\ Kr$

where

(2.34) $\theta_2 = \frac{\partial Q}{\partial r}\ \frac{r}{Q}$

is elasticity of output with respect to price of capital. In case supplies of both labour and capital are inelastic, the equilibrium condition can be written as

(2.35) $Qp = \theta_1^{-1}(1-e^{-1})^{-1}\ Lw + \theta_2^{-1}(\ 1 -e^{-1})^{-1}\ Kr$

So far we discussed extreme values of elasticity coefficients. For intermediate values the equilibrium condition (2.17) remains intact except when e = 1. In that case the equilibrium condition is given by

(2.36) $\alpha^{-1}(1+e_1^{-1})\ Lw + \beta^{-1}\ (1+e_2^{-1})\ Kr = 0.$

Since e_1 and e_2 are positive, the condition (2.36) would hold only if either α or β is negative. This means that equilibrium of the firm would be established only when the production process is under declining marginal returns with respect to one factor and increasing marginal returns with respect to the other.

For values of a below unity the equilibrium would be established when

$$(2.37) \qquad \alpha^{-1}(1+e_1^{-1})\, Lw + \beta^{-1}(1+e_2^{-1})\, Kr < 0$$

which is valied when decreasing marginal returns prevail with respect to both capital and labour, or, with respect to any one of the factors such that left hand side of (2.37) is negative.

3. Some Guidelines for a Profit Maximising Firm

The analyses of the preceding section indicate how a profit maximising firm would like to behave under different market situations. In day to day operation of business a firm may manage to be equipped with required information regarding market conditions with a view to enable itself to remain around theoretically optimal decision. Is it then possible to develop a few rules of thumb which might enable the firm to achieve its objective?

Combining (2.17) with (2.3) we obtain maximum profit II^* as

$$(3.1) \qquad II^* = (a - 1)\, Lw + (b - 1)\, K r$$

which tells that maximum profit is weighted average of factor costs Lw and Kr. Obviously, then, one may exploit the weights $(a - 1)$ and $(b - 1)$ to enhance further the magnitude of II^* because these weights would change with changes in market conditions.

Suppose now that a firm is capable of supplying its product to a number of markets having diff rent demand elasticities and can procure factors from a number of alternative markets having different supply elasticities. The firm is, then, confronted with a choice problem, Because, for each combination of ourput and factor markets, the condition (2.17) provides equilibrium of the firm but maximum profit II^* is different in each case in view of differences in the weights (a - 1) and (b - 1). This choice problem can be resolved as follows:

In the first instance, let us consider demand side, Maximum profit II^*, given in (3.1), would go up further if the firm can find market for its product corresponding to which the coefficients a and b are higher. From (2.18) we note that these coefficients are higher for lower value of e. Thus, choice of that product market which has lowest demand elasticity e would be most profitable. It may happen that in due course of time the initially chosen demand elasticity goes up. The firm's market would then consist of all markets, old as well as new, having almost equivalent elasticities. This process, in due course of time, may lead to disapparance of variations in different product markets and eventually a firm may reach a point of stagnation when demand side offers no further hope. The more severe the competition with other firms the quicker would the firm reach a stagnation point of this type. Careful examination of the supply side would still then provide helpful clues.

Choice of a new product market having lower demand elasticity implies moving on to a new demand curve which fetches relatively higher prices to the producer. Consequently, there will be a tendency to produce more. This, in turn, would lead to larger use of input factors. A decision problem, then, arises as to whether the firm should employ more of capital or of labour inputs. To examine this issue we must look at the form of supply curves. The supply curve of a factor will be steeper in economies where the factor is available in abundance as compared to those where it is scarce.

Let us examine the supply side of a firm in an economy where labour is scare. The obvious choice is then to use less labour and more capital to the extent permitted by technological relation (2.11). In case more than one labour markets are accessible, choice may be made in favour of cheapest one, that is, one corresponding to which supply elasticity is highest. As a result of lower employment of labour its marginal physical productivity would go up and accordingly α too would increase. Thus all the three components e_1^{-1}, α^{-1} and lw would be pushed downwards and the first term, namely, (a-1) Lw on the right hand side of (3.1) would touch its lowest point. On the contrary, employment of more capital would increase Kr and reduce β as well as e_2. This is because increase of capital intake would decrease its marginal physical productivity on the one hand, and compel the firm to procure capital even from costlier markets, on the other. As a result the component (b - 1) Kr of II^* would attain its largest possible value while the component (a-1) Lw would be at its minimum. Since sum of two numbers is maximum when the numbers are widely apart, II^* would attain largest value. Thus we conclude that, on the supply side, decisions in favour of capital intensification and drawing of labour from markets having highest supply elasticity would lead to greatest profit in a labour scarce economy.

Next, let us consider an economy where labour is available in abundance. The obvious choice in this case is to adopt labour intensive technique. Arguing on same lines as above, we find that purchasing capital from markets having highest supply elasticity would lead to highest value of II^*.

4. Optimum Size of Firm

It has been argued [7, p.97] that in real world of imperfect competition in markets a firm cannot attain its optimum size. We will show that such an apprehension is completely unfounded. According to present approach, determination of optimum size of firm means obtaining that value of Q which satisfies simultaneously equations (2.4), (2.9) and equilibrium condition (2.17). These equations involve five other variables, namely, p, L, w, K and r. Therefore solution of Q must be obtained simultaneously with these variables. Such a solution would not only provide optimum size of firm but also of equilibrium product and factor prices and of optimum levels of factor employment. We have six variables to be solved from six equations, viz., (2.4), (2.6), (2.7), (2.9), (2.11) and (2.17). Therefore, in actual practice unique solutions could be found provided exact mathematical forms of the functions were known. Whereas specific forms of these unknown functions could be obtained by econometricians for any given firm, we would specify them as follows with a view to illustrate the nature of problems involved in such exercises.

Let us write the functions (2.4), (2.6), (2.7), (2.9) and (2.11) in linear forms as

(4.1) $Q = q_0 - qp$,

(4.2) $L = l_0 + lw$,

(4.3) $K = k_0 + kr$,

(4.4) $$Q = \alpha_0 + \alpha_1 L + \alpha_2 K \ ,$$

and

(4.5) $$K = t_0 + tL \ ,$$

respectively and assume that the estimates of unknown coefficiants in these equations have been obtained by using T sample observations on the variables involved. Let $\bar{Q}$, $\bar{p}$, $\bar{L}$, $\bar{w}$, $\bar{K}$ and $\bar{r}$ be sample means and let

(4.6) $$Q^* = Q - \bar{Q} \ , \ p^* = p - \bar{p} \ , \ L^* = L - \bar{L}, \ w^* = w - \bar{w},$$

$$K^* = K - \bar{K} \text{ and } r = r - \bar{r}$$

denote deviations from the sample means.

Using (4.6), we can transform the equations (4.1) to (4.5) as

(4.7) $$Q^* = - qp^*,$$

(4.8) $$L^* = lw^*,$$

(4.9) $$K^* = kr^*,$$

(4.10) $$Q^* = \alpha_1 L^* + \alpha_2 K^* ,$$

and

(4.11) $$K^* = tL^*.$$

The equilibrium condition (2.17) can be expressed in terms of deviations around means as

(4.12) $$(Q^* + \bar{Q})(p^* + \bar{p}) = a(L^* + L)(w^* + \bar{w}) + b(K^* + \bar{K})(r^* + \bar{r}).$$

Now combining (4.7), (4.8) and (4.11) with (4.10) we can write

(4.13) $$p^* = - q^{-1} \, l \, (\alpha_1 + \alpha_2 t) \, w^*.$$

Next, using (4.13) and (4.7) we can deduce from the expression on the left hand side of (4.12) the following expression

$$(4.14)\quad (Q^* + \bar{Q})(p + \bar{p}) = -q^{-1}l^2(\alpha_1 + \alpha_2 t)^2 w^{*2} + [l\,\bar{p}(\alpha_1 + \alpha_2 t) - \bar{Q}\,q^{-1}l(\alpha_1 + \alpha_2 t)]\,w^* + \bar{Q}\,\bar{p}$$

which is a quadratic form in terms of w^*. Similarly, using (4.8), (4.9) and (4.11) we can rewrite the right hand side of (4.12) as

$$(4.15)\quad (al + bk^{-1}t^2l^2)\,w^{*2} + (al\bar{w} + a\bar{L} + btl\bar{r} + b\bar{K}k^{-1}tl)\,w^* + \bar{L}\bar{w} + \bar{K}\bar{r}.$$

Then combining (4.15) and (4.14) with (4.12) we obtain a quadratic equation in w^*

$$(4.16)\qquad d\,w^{*2} + d_1\,w^* + d_2 = 0$$

where

$$(4.17)\quad d = -q^{-1}l^2(\alpha_1 + \alpha_2 t)^2 - (al + b\,k^{-1}t^2l^2)$$

$$d_1 = l\,\bar{p}(\alpha_1 + \alpha_2 t) - \bar{Q}p^{-1}l(\alpha_1 + \alpha_2 t) - al\bar{w} + a\bar{L} + btl\bar{r} + b\bar{K}k^{-1}tl)$$

$$d_2 = \bar{Q}\,\bar{p} - a\,\bar{L}\,\bar{w} - b\,\bar{K}\,\bar{r}.$$

The equation (4.16) shall have two roots. Corresponding to each root we can find solutions for p^*, Q^*, L^*, K^* and r^* through equations (4.13), (4.7), (4.8), (4.11) and (4.9). In turn, using relation (4.6) we can derive solutions for original variables. Thus we have two sets of solutions. To decide which one of these is the desired one we can compute maximum profit Π^*, given in (3.1), by using alternative solutions and accept that one which yields greater Π^*. This procedure avoids verification of negative definiteness of H matrix which is relatively more laborious.

We may derive optimum values of Q, L and K explicitly in case perfect competition prevails in all the three markets. Let prices p, w and r be fixed at levels p_1, w_1 and r_1, respectively, then we are left with only three equations, namely (4.10), (4.11) and (4.12). Therefore, substituting the value of K^* from (4.11) into (4.10) and then substituting the values of K^* and Q^* in (4.12), we obtain

$$(4.18)\quad \hat{L}^* = [\alpha^{-1}w_1 + \beta^{-1}tr_1 - p_1(\alpha_1 + \alpha_2 t)]^{-1}(\bar{Q}p_1 - \alpha^{-1}\bar{L}w_1 - \beta^{-1}\bar{K}r_1)$$

Where $\hat{L}^*$ is solution value of L^* and use has been made of (2.21), (4.6) and the fact that mean value of a constant is the constant itself.

The solutions $\hat{K}^*$ and $\hat{Q}^*$ of K^* and Q^* can then be written immediately as

$$(4.19)\quad \hat{K}^* = t\hat{L}^*$$

and

$$(4.20)\quad \hat{Q}^* = (\alpha_1 + \alpha_2 t)\hat{L}^*.$$

Further, using (4.6) we obtain solutions of L, K and Q as

$$(4.21)\quad \hat{L} = \hat{L}^* + \bar{L},\ \hat{K} = \hat{K}^* + \bar{K} \text{ and } \hat{Q} = \hat{Q}^* + \bar{Q}.$$

5. Role of Taxation and Optimal Tax Functions.

Although theoretical deductions in the preceding sections provide adequate justification for firms to vizualise that their interests are served best if they educate themselves regarding supply and demand patterns throughout the country where they operate, yet, one cannot overlook the possibility and often reality of existenace of firms in whose caluclations holding control on means of production is more tempting than making fullest use of them. We shall, however, not go into possible reasons for this or any other motivation preventing a firm to work at full capacity. Instead, we shall examine such instruments in shape of taxes which might be used to regulate sluggish firms so that not only idle capacity is utilised but currently unused means of productions are passed on to deserving hands.

Let us consider a situation where taxes may be imposed upon output, labour and capital instead of traditional taxes on profit, unit cost and ad valorem tax etc.[5] Suppose further taxation scheme is determined according to following relationships

(5.1) $T_1 = (T(Q)\ ,\ T_2 = T(L)$ and $T_3 = T(K)$

where T_1, T_2, and T_3 denote volumes of taxes imposed when Q units of output is manufactured by using L Units of labour and K units of capital. The profit function to be maximised may, in this case, be written as

(5.2) $II_1 = Qp - T_1 - Lw - T_2 - Kr - T_3.$

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[5] See Musgrave /5/ and Prest /6/ for details regarding such taxes.

Denoting marginal taxes on output, labour and capital by t_1, t_2 and t_3, respectively, we can obtain from (5.1)

$$(5.3)\quad t_1 = \frac{\partial T_1}{\partial Q}, \; t_2 = \frac{\partial T_2}{\partial L} \text{ and } t_3 = \frac{\partial T_3}{\partial K}$$

and proceeding as we did in Section 2, we derive the equilibrium condition of the firm as given by

$$(5.4)\qquad Qp = aLw + bKr + (1 - e^{-1})^{-1}\,(t_1Q + t_2\alpha^{-1}L + t_3\beta^{-1}K).$$

Then, combining (5.4) with (5.2) we can write maximum profit Π_1^* in this case as

$$(5.5)\quad \Pi_1^* = \Pi^* + (1-e^{-1})^{-1}\,(t_1Q + t_2\,\alpha^{-1}L + t_3\beta^{-1}K) - T_1 - T_2 - T_3$$

where Π^* is functionally identical to maximum profit, given in (3.1), where there are no taxes.

Thus, we observe that maximum profit under taxes consists of two components. The first component is Π^* and the second is purely due to taxes imposed. Can we use taxes as merely instruments to motivate firms to move on to a new equilibrium position without feeling apparent bite of tax pressure? This can be done if we choose the functional forms of the tax functions, given in (5.1) such that the second component on the right hand side of (5.5) is identically zero. We shall call these implied functions as optimum tax functions. The second component on the right hand side can be grouped into three sub components and setting each sub component equal to zero we get three differential equations as follows

$$(5.6)\quad T_1 = (1-e^{-1})^{-1}\,t_1Q,\; T_2 = (1-e^{-1})^{-1}t_2\alpha^{-1}L,\; T_3 = (1-e^{-1})^{-1}t_3\beta K.$$

Using the notations in (5.3) we can solve equations in (5.6) to yield optimal tax functions

$$(5.7)\ T_1 = Q^{1-e^{-1}},\ T_2 = L^{\alpha(1-e^{-1})} \text{ and } T_3 = K^{\beta(1-e^{-1})}$$

corresponding to output, labour and capital, respectively.

On the contrary, one may like to impose such tax functions so that second component of II_1^* is a desired positive or negative quantity. Let δ_1, δ_2 and δ_3 be three constants such that

$$(5.8)\ t_1 Q = \delta_1 (1-e^{-1}) T_1,\ t_2 L = \delta_2 \alpha (1-e^{-1}) T_2 \text{ and}$$

$$t_3 L = \delta_3 \beta (1-e^{-1}) T_3$$

then, solving differential equations in (5.8) we obtain the following tax functions

$$(5.9)\ T_1 = (Q^{\delta 1(1-e^{-1})},\ T_2 = L^{\delta 2a(1-e^{-1})} \text{ and } T_3 = K^{\delta 3\beta(1-e^{-1})}.$$

It is obvious from (5.8) that second component of II_1^* will be positive for δ's greater than unity and negative for δ's being less than unity. In the former case there is a positive contribution to profit and therefore such a situation may be termed as under-taxation. Similarly, the latter situation may be called as over-taxation. The conditions of under-taxation or over-taxation may also be achieved by varying suitably any one of the δ's and holding others at constant levels. This provides greater manoeuvreability on the part of taxing authority to choose suitably taxation scheme for motivating an entrepreneur to adopt appropriate technology. For instance, over-taxation of capital would lead to higher employment of labour force in a labour surplus economy.

If taxes be imposed according to functions (5.9) or (5.7), firms would find it more profitable to adopt such policies which might prove beneficial to the interests of the society as a whole. For instance, in order to minimise payable taxes according to (5.9), a firm would like to sell its output in markets having minimum demand elasticity.[6] This implies that, instead of hoarding, goods shall be sold in those markets where they are needed most. In other words, the proposed taxation scheme would help in proper distribution of goods over space. Once an output market is chosen, the lower bounds of output level Q and input levels L and K are automatically determined in view of demand and production functions and furthe reduction in payable taxes on account of employed labour and capital may be affected only if the firm operates at that point of production curve where elasticity coefficients α and α are close to zero. This implies zero marginal physical productivity of labour as well as capital and therefore best utilisation of factors of production. Accordingly, fof firms working at unutilised capacity levels, taxation on capital and labour would provide a motive force to move on to their maximum productive potential. On the other hand firms, which might like to maintain ownership of means of production for only speculative or other reasons, will be penalised by way of payment of relatively more taxes. This will, naturally, prompt them to get rid of hoarded factors of production as soon as possible if the same cannot be employed for stepping up production due to extra-economic reasons.

[6] See also Prest [6, pp. 52 - 55] for similar remark through a different line of reasoning.

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