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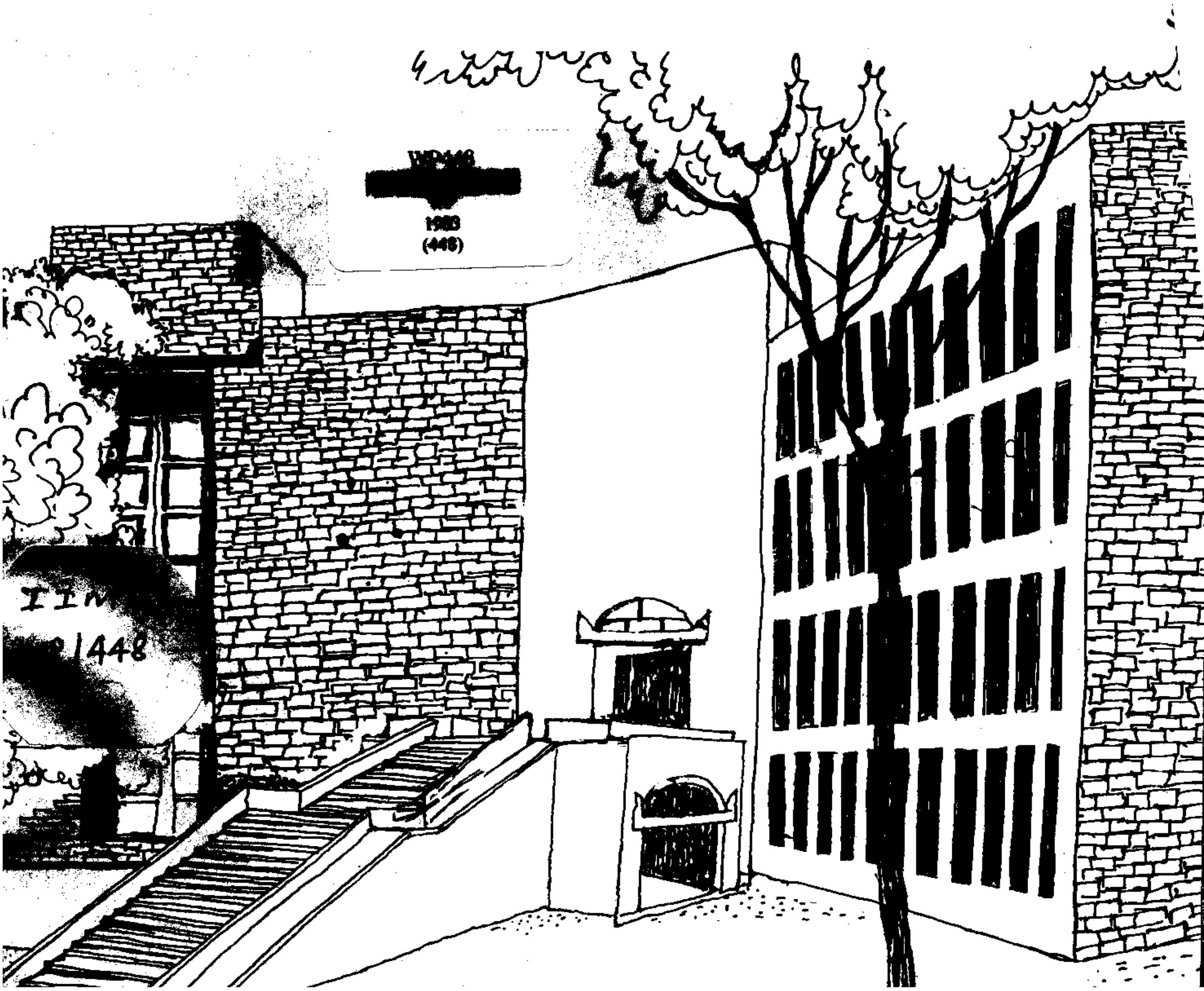

CORRELATION FUNCTIONS IN RELIABILITY THEORY

By

R. Subramanian

&

N. Ravichandran

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

Correlation Functions in Reliability Theory

R. Subramanian
Indian Institute of Technology
Madras - 600034

and

N. Ravichandran
Indian Institute of Management
Ahmedabad - 380015

ABSTRACT

The point process induced by the stochastic behaviour of a two-unit warm standby redundant repairable system is studied. Expressions for the product densities of the events corresponding to the entry into each of the states and the interval reliability are obtained. The reliability and availability are deduced as special cases.

1. INTRODUCTION:

Two-unit standby redundant repairable systems have been studied in th past extensively (Osaki and Nakagana (1976)). In most of the attempt the central quantity of interest is the Laplace transform of the availability or the reliability of the system. However these problems often give rise to some interesting stochastic processes which are important by themselves and have not been studied so far except in the stationary case (Srinivasan & Subramanian (1977)). These processes are essentially non-Markov and do not necessarily fall under the Semi-Markov type. In this contribution we study the multivariate stochastic point process induced by a given reliability problem. The layout of this paper is as follows. In Section 1 the model is described and Section 2 deals with interval reliability (Barlow, et.al. 1965). Section 3 is concerned with the correlation structure of the events induced by the reliability problem and obtains expressions for the first and second moments.

2. ASSUMPTIONS:

1. The System consists of two units, which are identical and statistically independent either unit performs the system operation satisfactorily.

2. At $t = 0$, one new unit is switched online and the other is kept as a warm standby. This initial condition will be denoted by $\mathcal{E}$.

3. The life time of a unit while online is a random variable with pdf $f_1(.)$.

4. A unit while in standby state has a constant failure rate ; for convenience its pdf is denoted by $f_2(.)$.

5. The repair time of a unit is a random variable with pdf $g(.)$.

6. Switch is perfect and switchover is instantaneous.

7. Each unit is 'new' after repair.

3. SOME PRELIMINARY RESULTS:

Let $Z(t)$ denote the state of the system at time t, representing the number of failed units at time t. Then $\{Z(t), \ t \geq 0\}$ is a discrete valued continuous time parameter stochastic process with state space $S = \{0, 1, 2\}$. The points of discontinuities of the process $\{Z(t)\}$ are the epochs corresponding to a failure or repair completion of the units. The value of $Z(t)$ has a negative jump when a repair completion takes place and a positive jump when a failure occurs, the magnitude of the jump being always unity.* Thus we identify the following events in this study.

E_i : Event that the process $\{Z(t)\}$ enters state i. $(i = 0, 1, 2)$.

In this contribution our aim is to study the above point vents. We note that the event E which is given to have occurred at = 0 cannot occur in the sequel. Further, some of the events E_i are regenerative while others are not. The events E_0 and E_2 are always non-regenerative. The event E_1 is regenerative if it occurs in the following ways:

i. Entry into State 1 from State 2.

ii. Entry into State 1 from State 0, due to the failure of the online unit.

It is not regenerative if it corresponds to the entry into State 1 from State 0, due to the failure of the standby unit. A regenerative E_1 event will be denoted by $\bar{E}_1$.

Let $\{t_i\}$ be the time epochs at which the system enters the various states. Then it is clear that $\{t_i\}$ is a stochastic point process (Srinivasan (1974)). To study this process it is convenient to use the following random variables.

X_t - random variable denoting the age of the online unit at time t if a unit happens to be operating online at t.

Y_t - random variable denoting the elapsed repair time of a unit at time t, if a unit happens to be undergoing repair at time t.

To start with we study the stochastic behaviour of the standby unit during the failure-free operation period of the online unit. In this interval the standby unit alternates between the operable (s) and repair (r) states successively. If S(t) denotes the state of the standby at any time t, then $\{S(t),\ t \geq 0\}$ is an alternating renewal process. This renewal process can be characterised by the following functions.

$$\pi_{is}(t) = \lim_{\Delta \to 0} \Pr\{S(t) = s \mid S(-\Delta) \neq S(0) = i\},$$

$$\pi_{ir}(t, x) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr\{S(t) = r;$$

$$x < Y_t < x + \Delta \mid S(-\Delta) \neq S(0) = i\},$$

$$i = s, r$$

We use the following notations

$*$ denotes the convolution operation.

$$C(t) = \int_0^t c(x)\, dx. \qquad c(.) \text{ any function}$$

$$\bar{c}(t) = 1 - c(t)$$

$c^{(n)}(t)$ = n fold convolution of $c(t)$ with itself in the (o, t).

We have by simple probabilistic arguments,

$$\pi_{ss}(t) = \bar{F}_2(t) + \gamma(t) * \bar{F}_2(t) \qquad \ldots. (1)$$

$$\pi_{rs}(t) = g(t) * \pi_{ss}(t) \qquad \ldots. (2)$$

$$\pi_{sr}(t, x) = f_2(t-x)\, \bar{G}(x) + [f_2(t-x) * \gamma(t-x)]\, \bar{G}(x) \qquad \ldots.. (3)$$

$$\pi_{rr}(t, x) = \bar{G}(x)\, \delta(t-x) + \gamma(t-x)\, \bar{G}(x) \qquad \ldots.. (4)$$

where

$$\gamma(t) = \sum_{n=1}^{\infty} h^{(n)}(t) = h(t) = f_2(t) * g(t)$$

We note that $\int_0^t \pi_{ir}(t, x)\, dx + \pi_{is}(t) = 1$, for $i = r, s$; this is as it should be since the LHS represents the probability that the standby is either operable or under repair at any time t. We also require the following functions which characterise the time interval between an E event and the next $\overline{E}_1$ event and the time interval between two successive $\overline{E}_1$ events.

Let $N_1(t)$ denote the number of $\overline{E}_1$ events in (o, t) and $f_{ij}(\cdot)$ be defined as follows:

$$f_{01}(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr\{ N_1(t + \Delta) - N_1(t) = 1; \; N_1(o, t) = 0 \mid E \text{ at } t = 0 \}.$$

$$f_{11}(t) = \lim_{\Delta, \Delta' \to 0} \frac{1}{\Delta} \Pr\{ N_1(t + \Delta) - N_1(t) = 1; \; N_1(o, t) = 0 \mid N_1(-\Delta', 0) = 1 \}$$

We have

$$f_{01}(t) = f_1(t)\,\pi_{ss}(t) + \int_0^t f_1(u)\,du \int_0^u \pi_{sr}(u, x) \frac{g(t-u+x)}{\overline{G}(x)}\,dx \quad \ldots (5)$$

and

$$f_{11}(t) = f_1(t)\,\pi_{rs}(t) + \int_0^t f_1(u)\,du \int_0^u \pi_{rr}(u, x) \frac{g(t-u+x)}{\overline{G}(x)}\,dx \quad \ldots\ldots$$

where $\pi_{ij}(\cdot)$ and $\pi_{ij}(\cdot, \cdot)$ are given by (1) – (4).

The expression for $f_{01}(t)$ is obtained by the following considerations. For an $\overline{E}_1$ event to occur in $(t, t + \Delta)$ given that E has occurred at t = 0, it is necessary that the unit switched online at t = 0 should fail before t. At the epoch of failure of the online unit, the standby unit can be either operable or under repair. In the former case $\overline{E}_1$ occurs and in the latter case the event $\overline{E}_1$ occurs at the epoch of repair completion of the standby.

We note that the epochs $\{t_i\}$ corresponding to the occurrences of $\overline{E}_1$ events form a renewal process with pdf $f_{11}(t)$. The corresponding renewal density $\alpha(t)$ is given by

$$\alpha(t) = \sum_{n=1}^{\infty} f_{11}^{(n)}(t) \quad \ldots\ldots (7)$$

The function $f_{11}(t)$ represents the pdf of the random variable denoting the time interval between two successive $\overline{E}_1$ events with or without a system failure in between. However, in our analysis we also require the pdf of the random variables representing the time interval between two successive $\overline{E}_1$ events there being no system failure in between. Denoting this pdf by $\tilde{f}_{11}(t)$ we have $\tilde{f}_{11}(t) = f_1(t)\,\overline{\Pi}_{rs}(t)$.

The corresponding renewal density $\beta(t)$ is given by

$$\beta(t) = \sum_{n=1}^{\infty} \tilde{f}_{11}^{(n)}(t) \quad \ldots\ldots (9)$$

4. OPERATING CHARACTERISTICS OF THE SYSTEM:

We next proceed to obtain the interval reliability $R(t, \tau)$, an important measure of the system. This is defined on the probability that the system is available at time t and is up in $(t, t+\tau)$ (Barlow and Proschan (1965)). It is clear that the reliability $R(\tau)$ and availability $A(t)$ of the system are obtained by setting $t = 0$ and $\tau = 0$ respectively in $R(t, \tau)$. Eventhough this is a very useful measure it has not received the attention it deserves. Only very recently the case of

Cold standby has been discussed (Subramanian and Ravichandran (1977)). To obtain an expression for $R(t, \tau)$ we proceed as follows. Since the system is available at t, it should be found then in one of the states 0 or 1 and for the future description of the system we need the age of the online unit, when the system is in sate 0 and in addition the elapsed repair time when it is in state 1. Thus we are led to define the following functions:

$$B_0(t, x) = \lim_{\Delta \to 0} Pr \{ Z(t) = 0, \; x < X_t < x + \Delta \mid E \text{ at } t = 0 \} / \Delta$$

$$B_1(t,x,y) = \lim_{\Delta, \Delta' \to 0} Pr \{ Z(t) = 1; \; x < X_t < x + \Delta; \; y < Y_t \leq y + \Delta' \mid E \text{ at } t = 0 \} / \Delta \Delta' \quad y \leq x.$$

$$B_2(t, y) = \lim_{\Delta, \Delta' \to 0} \{ E_2 \text{ in } (t, t + \Delta), \; y < Y_t \leq y + \Delta' \mid E \text{ at } t = 0 \} / \Delta, \Delta'$$

By probabilistic arguments we get the following expressions for the functions $B_i(\cdot\, , \cdot)$.

$$B_0(t, x) = \bar{F}_1(x) \delta(x-t) \pi_{oo}(x) + \psi(t-x) \bar{F}_1(x) \pi_{ro}(x) \quad \ldots\ldots (10)$$

$$B_1(t,x,y) = \bar{F}_1(x) \delta(x-t) \pi_{or}(t,y) + \psi(t-x) \bar{F}_1(x) \pi_{rr}(x, y) \quad \ldots\ldots (11)$$

$$B_2(t, y) = \int_0^{t-y} \psi(n) \; f_1(t-n) \; \pi_{rr}(t-n, y) \, dn + f_1(t) \pi_{or}(t,y) \quad \ldots\ldots (12)$$

where $$\psi(t) = f_{01}(t) + f_{01}(t) * \sum_{n=1}^{\infty} f_{11}^{(n)}(t) \quad \ldots\ldots\ldots\ldots (13)$$

and $\delta(.)$ is the Dirac delta function.

The functions $\pi_{ij}(t, x)$ and $\bar{\pi}_{ij}(t)$ used in equations above are given by (1) - (4).

To compute the interval reliability of the system $R(t, \tau)$ we make use of the following additional functions which represent the reliability of the system in $(t, t+\tau)$ under certain specified initial conditions.

$$R_0(t, \tau \mid y) = \Pr\left\{ \text{System up in } (t, t+\tau) \mid Z(t) = 0;\ X_t = y \right\}$$

$$R_1(t, \tau \mid y, z) = \Pr\left\{ \text{System up in } (t, t+\tau) \mid Z(t) = 1;\ X_t = y;\ Y_t = z \right\}$$

Using these functions and by considering the following mutually exclusive and exhaustive cases that:

(i) the online unit does not fail in $(t, t+\tau)$

(ii) the online unit fails in $(n, n+dn)$, $t < n < t+\tau$

We get the interval reliability $R(t, \tau)$ as

$$R(t, \tau) = \int_0^t R_0(t, \tau / y)\ B_0(t, y)\, dy + \int_0^t dy \int_0^y R_1(t, \tau \mid y, z)\ B_1(t, y, z)\, dz \quad \ldots\ldots (14)$$

where

$$R_0(t, \tau \mid y) = \frac{\bar{F}_1(y+\tau)}{\bar{F}_1(y)} + \int_0^{\tau} \frac{f_1(n+y)}{\bar{F}_1(y)}\ \bar{\pi}_{ss}(n)\ R_3(t-n)\, dn \quad \ldots(15)$$

$$R_1(t, \tau \mid y, z) = \frac{\bar{F}_1(y + \tau)}{\bar{F}_1(y)} + \int_0^{\tau} \frac{f_1(n + y)}{\bar{F}_1(y)} \, T_{ro}(n \mid z) \, R_3(\tau - n) \, dn \quad \ldots\ldots(16)$$

and

$$R_3(t) = \bar{F}_1(t) + \beta(t) * \bar{F}_1(t) \quad \ldots\ldots\ldots (17)$$

On simplification equation (14) becomes

$$R(t, \tau) = R_3(t + \tau) + \int_0^t \psi(t-y) \, R_3(\tau + y) \, dy \quad \ldots\ldots\ldots\ldots\ldots (18)$$

where $\psi(.)$ is given by (13).

The stationary value of $R(t, \tau)$ as $t \to \infty$, known as the limiting interval reliability and denoted by $R(\tau)$ is given by

$$R(\tau) = \frac{1}{\mu} \int_0^{\infty} R_3(x + y) \, dy \quad \ldots\ldots\ldots\ldots\ldots(19)$$

where $\mu = \int_0^{\infty} t \, \psi(t) \, dt.$

We can directly obtain the availability $A(t)$ of the system conditional upon an E event at $t = 0$ from the functions $B_0(t, x)$ and $B_1(t, x, y)$:

$$A(t) = \int_0^t B_0(t, x) \, dx + \int_0^t dx \int_0^x B_1(t, x, y) \, dy$$

On simplification this equation becomes

$$A(t) = \bar{F}_1(t) + \psi(t) * \bar{F}_1(t) \quad \ldots\ldots\ldots\ldots\ldots (20)$$

As an easy consequence of Key renewal theorem (Srinivasan 1974) we obtain the steady state availability of the system from equation (20).

$$\beta = \int_0^\infty \bar{F}_1(t)\, dt \Big/ \int_0^\infty t\, f_{11}(t)\, dt.$$

5. STOCHASTIC POINT PROCESS OF THE EVENTS E_i:

We next study the point process induced by the stochastic behaviour of the system. We have already observed that some of the E_i events are regenerative while others are not. In view of this, the point events are best studied by product densities (Rama Krishnan (1958), Srinivasan (1974)). We first derive expressions for the product densities of the first two orders and use them to get the first and second moments of the number of E_i events in (o, t). The product densities in this case are defined as follows:

$$h_i(t) = \lim_{\Delta, \Delta' \to 0} \frac{1}{\Delta} \Pr\left\{ N_{E_i}(t+\Delta) - N_{E_i}(t) = 1 \;\middle|\; N_E(-\Delta', 0) = 1 \right\}.$$

$$h_{ij}(t_1, t_2) = \lim_{\Delta, \Delta', \Delta'' \to 0} \frac{1}{\Delta'\Delta} \Pr\Big\{ N_{E_i}(t_1 + \Delta) - N_{E_i}(t_1) = 1;$$

$$N_{E_j}(t_2 + \Delta') - N_{E_j}(t_2) = 1 \;\Big|\; N_E(-\Delta'', o) = 1 \Big\}.$$

We first concentrate on the first order product densities. The function $h_o(t)$ which is the product density corresponding to the E_o event is given by

$$h_o(t) = \int_0^t C_o(t, x)\, dx$$

where

$$C_o(t, x) = \bar{F}_1(x)\, \delta(t-x)\, \nu(x)$$

$$+ \psi(t-x)\, \bar{F}_1(x) \left[g(x) + \nu(x) * g(x) \right] \qquad \dots (21)$$

The expression for C_0 (t, x) is obtained by considering the following mutually exclusive and exhaustive possibilities.

(i) The Online Unit does not fail in (0, t)

(ii) The Online Unit fails in $(u, u + d_n)_1$

u t

By similar arguments we get

$$h_1(t) = \int_0^t C_1 (t, x) dx$$

Where

$$C_1 (t_1 x) = \psi(t) \delta(x) + \bar{F}_1(x) \delta(t-x) \left[t_2(x) + t_2(x) * v(x) \right] \quad (22)$$
$$+ \psi(t-x) \bar{F}_1(x) v(x)$$

and

$$h_2(t) = \int_0^t C_2 (t, y) dy , \text{ where}$$

$C_2 (. , .)$ is the same as $B_2 (. , .)$ given by equation (12).

Next, we proceed to obtain the second order product densities of the E_i events. Since the E_i events are not regenerative in general, for the future description of the process we need the age of the online unit when the event E_0 or E_1 occurs.

The second order product density h_{ij} (. , .) is given by,

$$h_{ij} (t_1, t_2) = \int_0^{t_1} C_i (t_1, x) dx \left[\int_0^{t_2-t_1} C_{ij} (t_2-t_1, y|x) dy \right.$$
$$\left. + C_{ij} (t_2-t_1, y|x) \delta(y - (t_2-t_1+x) \quad (1 - \delta_{j2}) \right.$$
$$i, j = 0, 1, 2 \quad (23)$$

The expressions for the unknown functions appearing in equation (23) are given below:

$$C_{00}(t, y \mid x) = \frac{\bar{F}_1(x + t)}{\bar{F}_1(x)} \delta(y - (x + t)) \gamma(t)$$

$$+ \psi(t - y \mid x) \bar{F}_1(y) \left[g(y) + g(y) * \gamma(y) \right]$$

$$C_{01}(t, y \mid x) = \psi(t \mid x) \delta(y) + \psi(t - y \mid x) \bar{F}_1(y) \nu(y)$$

$$+ \frac{\bar{F}_1(x + t)}{\bar{F}_1(x)} \delta(y - (x + t)) \left[f_2(t) + f_2(t) * \nu(t) \right]$$

$$C_{02}(t, y \mid x) = \frac{f_1(x + t)}{\bar{F}_1(x)} \pi_{or}(t, y)$$

$$+ \int_0^{t-y} \psi(n \mid x) \, f_1(t - n) \, \pi_{rr}(t - n, y) \, dn$$

$$C_{10}(t, y \mid x) = \frac{\bar{F}_1(x + t)}{\bar{F}_1(x)} \delta(y - (x + t)) \left[g(t) + g(t) * \nu(t) \right]$$

$$+ \alpha(t - y \mid x) \bar{F}_1(y) \left[g(y) + g(y) * \gamma(y) \right]$$

$$C_{11}(t, y \mid x) = \frac{\bar{F}_1(x + t)}{\bar{F}_1(x)} \delta(y - (x + t)) \nu(t)$$

$$+ \alpha(t \mid x) \delta(y) + \alpha(t - y \mid x) \bar{F}_1(y) \gamma(y)$$

$$C_{12}(t, y \mid x) = \frac{f_1(x + t)}{\bar{F}_1(x)} \pi_{rr}(t, y)$$

$$+ \int_0^{t-y} \alpha(n \mid x) \, f_1(t - n) \, \pi_{rr}(t - n, y) \, dn$$

$$C_{20}(t, y \mid x) = \int_0^{t-y} \frac{g(x+n)}{\bar{G}(x)} C_{10}(t-n, y \mid n)\, dn$$

$$C_{21}(t, y \mid x) = \frac{g(x+t)}{\bar{G}(x)} \xi(y) + \int_0^{t-y} \frac{g(x+n)}{\bar{G}(x)} C_1(t-n, y\ o)\, dn$$

$$C_{22}(t, y \mid x) = \int_0^{t-y} \frac{g(x+n)}{\bar{G}(x)} C_{12}(t-n, y \mid o)\, dn.$$

The function $\psi(t \mid x)$, $\alpha(t \mid x)$ etc. used in the set of equations above are given by

$$\psi(t \mid x) = f_{01}(t \mid x) + f_{01}(t \mid x) * \gamma(t)$$

where

$$f_{01}(t \mid x) = \frac{f_1(t+x)}{\bar{F}_1(x)} \bar{\Lambda}_{00}(t)$$

$$+ \int_0^t \frac{f_1(z+x)}{\bar{F}_1(x)}\, dz \int_0^z \bar{\Lambda}_{0r}(z, n) \frac{g(t-z+n)}{\bar{G}(n)}\, dn$$

$$\alpha(t \mid x) = f_{11}(t \mid x) + \sum_{n=1}^{\infty} f_{11}(t \mid x) * f_{11}^{(n)}(t)$$

$$f_{11}(t \mid x) = \frac{f_1(t+x)}{\bar{F}(x)} \bar{\Lambda}_{r0}(t)$$

$$+ \int_0^t \frac{f_1(z+x)}{\bar{F}_1(x)}\, dz \int_0^z \bar{\Lambda}_{rr}(z, n) \frac{g(t-z+n)}{\bar{G}(n)}\, dn.$$

Thus the second order product densities are completely determined. The first and second moments of the number of events in the internal $(0, t)$ are obtained by integrating the first and second order product densities (Ramakrishnan, 1958). For example if N_d is the random variable representing the number of system downs then

$$E\left[N_d\right] = \int_0^t h_2(n)\,dn$$

and

$$E\left[N_d^2\right] = \int_0^t dt_2 \int_0^t h_{22}(t_1, t_2)\,dt_1 + \int_0^t h_1(n)\,dn$$

Concluding Remarks:

In this contribution we have analysed the multivariate stochastic point process induced by a given reliability problem. We have also obtained expressions for the first and second moments of the number of events of various types in an arbitrary time interval.

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