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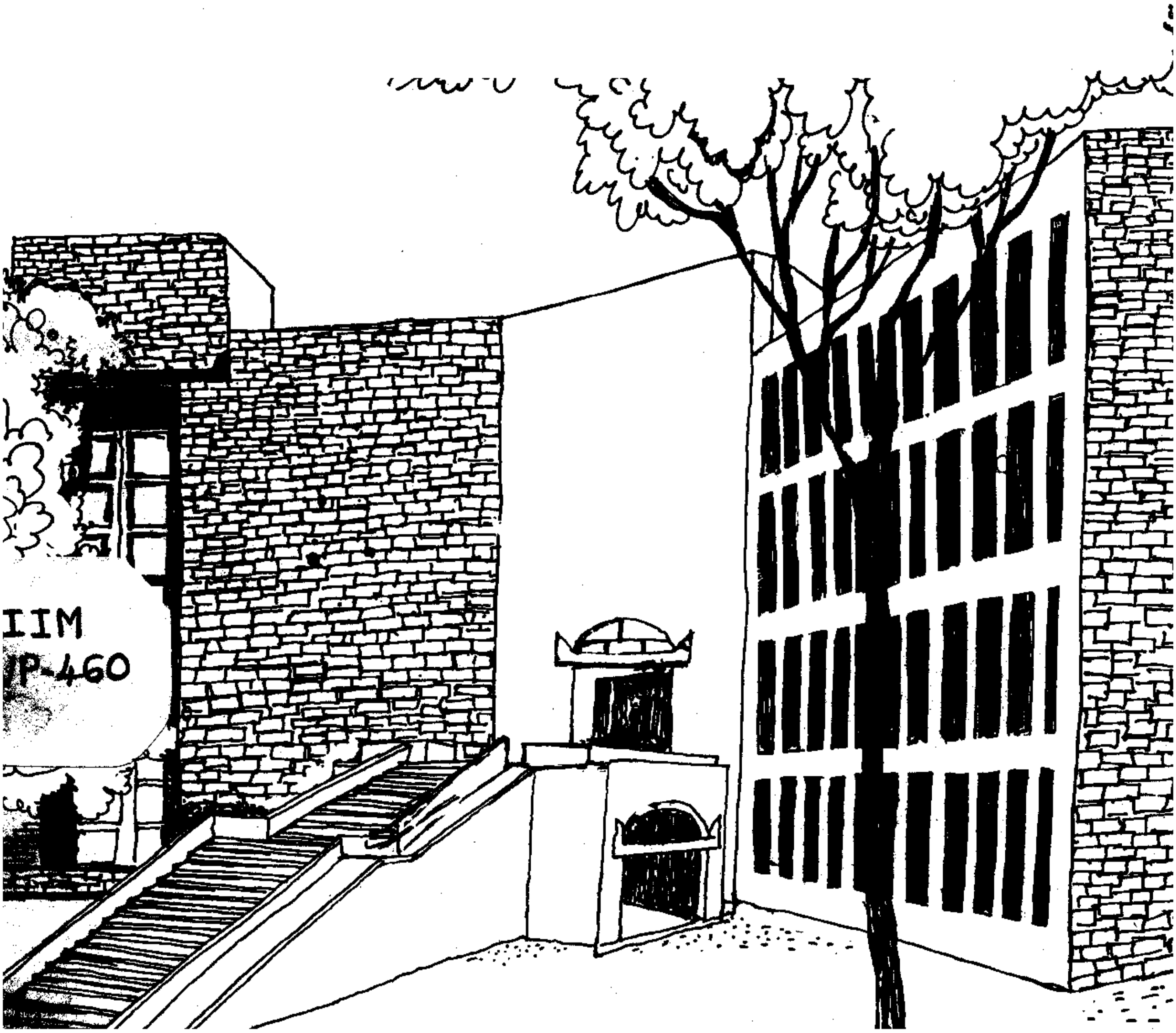

# SOME PROPERTIES OF OPTIMAL SCHEDULE OF JOBS WITH A COMMON DUE DATE

By

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# SOME PROPERTIES OF OPTIMAL SCHEDULE OF JOBS ABOUT A COMMON DUE DATE

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## INTRODUCTION

Kanet [2] considered the problem of minimizing the average deviation of job completion times about a common due date. Under the condition that the common due date exceeds the make-span of the job set, he obtained a procedure "SCHED" which yields optimal sequence for the problem. In this paper we consider the general problem without any condition on d. We prove some general properties of the optimal sequence. In particular, we prove that the optimal sequence is "V-shaped". We also prove that the SCHED algorithm of Kanet gives optimal sequence under less restrictive and a more practical condition on d. We also consider a few special cases and establish further properties of optimal sequences. These include the complete solution for 3 jobs.

## THE PROBLEM AND NOTATIONS:

Consider a single machine with n jobs immediately available for processing. All the jobs have a common due date d. For a sequence of jobs let $C_i$ denote the completion time for job i. The objective is to find a schedule S which minimizes

$$Z = \sum_{i=1}^{n} | C_i - d |$$

In this paper we will be using the following notations:

- n : number of jobs
- N : set of jobs $\{1, 2, \ldots n\}$.
- d : common due date.
- S : a schedule of n jobs. i.e. a permutation of (1, 2, .... n).
- $P_i$ : processing time of the job that is at i th position in S. $(i = 1, 2, \ldots n)$

$p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(n)}$ are the ordered processing times.

- $p_i$ : processing time of job i, $i = 1, \ldots n$.
- $t_o$ : starting time of sequence. $t_o \geq 0$.
- B : For a given sequence, B is the set of jobs which start before d and end at or before d.
- A : For a given sequence, A is the set of jobs which start at or after d
- $| U |$ : Number of elements in the set U.
- $\varepsilon > 0$ : Arbitrarily small positive number.

GENERAL PROPERTIES OF OPTIMAL SCHEDULES:

Kanet [2] considered the situation $d \geq \sum_{i=1}^{n} p_i$ and obtained a simple procedure called "SCHED" to obtain an optimal schedule. We shall consider the general case when no restrictions are imposed on d and obtain properties of the optimal schedules. We give below some of these properties possessed by optimal schedules.

<u>Property 1:</u>

For the general problem, we observe that it is not necessary to consider schedules that have idle time inserted between jobs in the sequence. The proof of this result given by Kanet [2] applies to this general case as well.

<u>Property 2:</u>

<u>Lemma 1:</u> The jobs in B are sequenced by longest processing time first (LPT) and the jobs in A are sequenced by shortest processing time first (SPT).

<u>Proof:</u> Same as the one given by Kanet [2, p. 647]. The proof given there does not depend on the condition $d \geq \sum_{i=1}^{n} p_i$.

<u>Property 3:</u>

When $d \geq \sum_{i=1}^{n} p_i$ Kanet [2] showed that one need not consider schedules which have a job begin processing before d and end processing after d. This result is not true without this condition $d \geq \sum_{i=1}^{n} p_i$ as the following example shows:

**Example 1:** Let there be $n = 3$ jobs with processing times 5, 10 and 100 units. Let $d = 101$. It can be verified that the optimal sequence is in the order 100, 5, 10 with starting time $t = 0$. See Figure 1 below

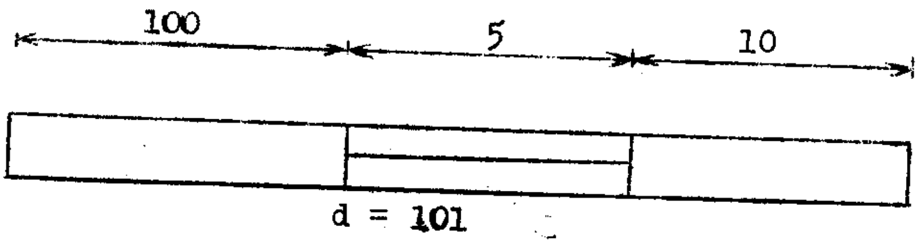


Figure 1

The objective function value is $1 + 4 + 14 = 19$ units. Here the job with the processing time 5 units starts strictly before d and ends strictly after d.

The result is, however, true if the optimal schedule starts at $t = t_o > 0$.

**Lemma 2:** Suppose there exists an optimal sequence starting at $t = t_o > 0$. Then some job ends at d. i.e. the last job in B ends at d.

**Proof:** Suppose that in every optimal schedule, no job ends at d. This means that, in every optimal schedule, the last job in set B ends strictly before d. If $B = \phi$, this means that the first job in the sequence starts at $t = t_o > 0$ and ends strictly after d. In either case, we see that there exists a job which starts strictly before d and ends strictly after d. Let k be this job and let $\ell$ be the last job in B. Let $C_\ell$ be the completion time of $\ell$. If $B = \phi$, take $C_\ell = t_o$. The positioning of the jobs is shown in the following Figure 2.

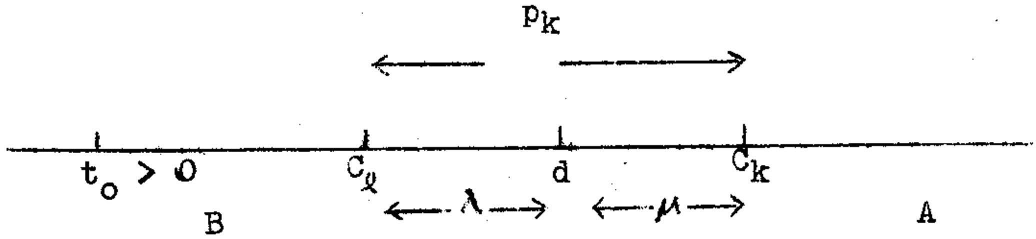


Figure 2

Let $d - C_\ell = \lambda$ and $C_k - d = \mu$. Let v denote the value of the objective function for the sequence.

Case 1: Suppose $|B| \leq |A|$, then start the process at $t = t_o - \varepsilon$, where $\varepsilon$ is an arbitrarily small positive number. Denote by $v_1$ the new value of the objective function. Then $v_1 - v = |B|\varepsilon - |A|\varepsilon - \varepsilon = (|B| - |A| - 1)\varepsilon < 0$ contradicting the optimality of the sequence.

Case 2: $|B| > |A|$. Shift all the jobs to the right by $\lambda$ units so that $C_k = d + p_k$ and let $v_2$ be the value of the new objective function. Then

$$v_2 - v = |A|\lambda - |B|\lambda + \lambda = (|A|-|B| + 1)\lambda \leq 0.$$

If $v_2 - v < 0$, it contradicts the optimality of the sequence. Henve $v_2 = v$ which means that we have found an optimal schedule in which some job ends at d.

Property 4:

The discussion of Case 1 in the above proof implies that for an optimal schedule in which $t_o > 0$, $|B| \not< |A|$. For, if $|B| < |A|$, we can start the sequence at $t_o - \varepsilon$ and reduce the objective function. Thus $|B| \geq |A|$. The next theorem states that either $|B| = |A|$ or $|B| = |A| + 1$.

Theorem 1: Suppose that an optimal schedule starts at some $t_o > 0$. Then there exists an optimal solution with starting point $> 0$ and $|B| = |A| = r$ if $n = 2r$ and $|B| = |A| + 1 = r$ if $n = 2r - 1$.

Proof: Since $t_o > 0$, by Lemma 2, the last job in B ends at d. $B \neq \phi$, since $t_o > 0$ implies that $|B| \geq |A|$. Let $|B| = s$ and let $P_1, P_2, \ldots\ldots P_s$ be the processing times of the first s jobs in B. $P_{s+1}, \ldots\ldots P_n$ are the processing times of the $(n - s)$ jobs in A. Then
$t_o = d - P_1 - P_2 - \ldots\ldots - P_s > 0$. See Figure 3 in next page.

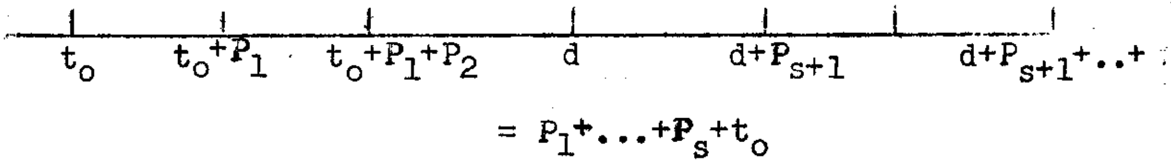


Figure 3

The objective function is seen to be

$P_2 + 2P_3 + ...+(s-1)P_s +(n-s)P_{s+1} +(n-s-1)P_{s+2} + ... + P_n.$

If the process starts at $t_o + \varepsilon$, where $\varepsilon$ is an arbitrarily small positive number, the increase in objective function is

$$-(s-1)\varepsilon + (n-s)\varepsilon + \varepsilon$$
$$= \varepsilon\ (n - 2s + 2).$$

This must be $\geq 0$ in view of optimality of the given sequ so that $n \geq 2(s-1)$.

If the process starts at $t_o - \varepsilon$, the increase is

$$s\varepsilon - (n - s)\ \varepsilon = \varepsilon(-n + 2s)$$

This is $\geq 0$ and this implies $n \leq 2s$. Thus $2(s-1) \leq n \leq 2s$ which implies $n = 2s-2$ or $2s-1$ or $2s$ will show that $n \neq 2s-2$. For if possible, let $n =$ i.e. $n - s = s - 2$. This implies

(2) $$|A| = |B| - 2.$$

Let $\ell$ be the last job in B. Start the process at $t_o + p_\ell$. Then the increase is

$$p_\ell \ |A| - p_\ell \ (|B| - 1) + p_\ell$$
$$= p_\ell \ (|A| - |B| + 2) = 0 \ \text{by (2)}.$$

Thus we get an equivalent optimal schedule with new sets $B^*$ and $A^*$ such that $|B^*| = s-1$, $|A^*| = n-s+1$ and $|B^*| - |A^*| = s-1-n+s-1 = 2s-n-2 = 0$ by (2). i.e. $|B^*| = |A^*|$. When $n = 2s-1$, $n-s = s-1$ so that $|B| - 1 = |A|$, when $n = 2s$, $n-s = s$ so that $|B| = |A|$. Thus we can find an optimal schedule with $|B| = |A|$ if $n = 2r$ and $|B| = |A| + 1$ if $n = 2r - 1$. This proves the theorem.

Property 5: (V-shape of Optimal schedule):

We prove an important property possessed by an optimal schedule for the general problem. It states that the optimal schedule is V-shaped. This means that in the optimal schedule the jobs are processed according to decreasing order of processing time until the job with the shortest processing time is completed and then the jobs are scheduled according to increasing order of processing times. The V-shaped property of optimal schedule also holds for the single machine sequencing problem if one wants to minimize the variance of the completion times. See, Eilon and Chowdhury [1].

Theorem 2: The optimal sequence is V-shaped.

Proof: Consider an optimal sequence and define sets B and A. Define the sets B and A corresponding to the optimal sequence.

We consider two cases: Case 1: $B \cup A = N$ and Case 2: $B \cup A = N - \{ s \}$, for some $s = 1, 2, \ldots\ldots\ldots n$.

Case 1: $B \cup A = N$: By Lemma 1, the jobs in B are processed according to LPT rule, and the jobs in A according to SPT rule. Since $B \cup A = N$, the sequence is V-shaped. Note that, in the Case 1 we are considering, if $B = \phi$, all the jobs in the sequence start at d and if $A = \phi$, all the jobs in the sequence are prooessed before d and the last job ends at d.

Case 2: $B \cup A = N - \{ s \}$: for some $s = 1, 2, \ldots. n$. It follows from the definitions of B and A that job s starts strictly before d and ends strictly after d. By Lemma 1, the jobs in B are processed according to LPT rule and the jobs in A according to SPT rule. Let $\ell$ be the last job in B and k be the first job in A. Let $p_\ell$, $p_s$, and $p_k$ be the processing times for job $\ell$, s, and k respectively.

We distinguish two sub-cases.

Sub-case 1: $p_s$ is an intermediate job in the sequence. i.e. $B \neq \phi$ and $A \neq \phi$. The following figure illustrates the positioning of jobs , s, and k.

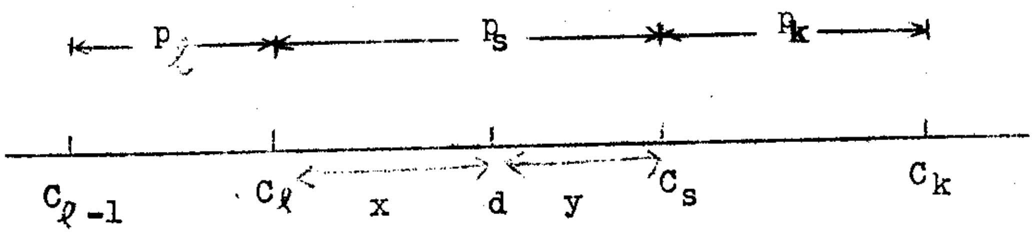


Figure 4

$C_\ell$ , $C_s$, $C_k$ are the completion times of the jobs $\ell$ , s, k respectively. $C_{\ell-1}$ is the completion time of the job immediately preceding $\ell$. Let $d - C_\ell = x$ and $C_s - d = y$, so that $x + y = p_s$. Let v denote the value of the objective function. We distinguish the following four possibilities:

(i) $p_\ell \geq p_s \geq p_k$

(ii) $p_\ell \geq p_s$ , $p_s < p_k$

(iii) $p_\ell < p_s \leq p_k$

(iv) $p_\ell < p_s$ , $p_s > p_k$.

The first three possibilities (i), (ii) and (iii) above show that the sequence is V-shaped (Recall that jobs in B follow LPT rule and jobs in A follow SPT rule). We have to establish the V-shaped property only for (iv). i.e. We want to show (iv) cannot happen in an optimal sequence. We prove this by effecting a reduction in the objective function by suitable interchanges in the given sequence.

Suppose now (iv) holds. We first show that $p_k < x$.

Suppose, if possible, $p_k \geq x$. Obtain a new sequence from the given optimal sequence by interchanging the positions of jobs s and k. Let $v_1$ be the new value of the objective function. The change in the objective function is seen to be

$$v-v_1 = (C_s-d+C_k-d) - (C_\ell + p_k - d + C_k - d)$$
$$= C_s - C_\ell - p_k = p_s - p_k > 0.$$

Thus the objective function is reduced, contradicting the optimality of the given sequence. Thus $p_k < x$.

Now, obtain a new sequence from the given optimal sequence by interchanging the positions of s and $\ell$. Let $v_2$ be the new value of the objective function. If, after this interchange, s ends between $C_\ell$ and d, then the change in the objective function is

$$v - v_2 = d - C_\ell - (d - (C_{\ell-1} + p_s))$$
$$= p_s - p_\ell > 0$$

which contradicts the optimality of the given sequence. If, after the interchange, s ends after d, then the change is

$$v - v_2 = (d - C_\ell) - C_{\ell-1} - p_s + d$$
$$= d - C_s - C_{\ell-1} + d$$
$$= d - C_s + (p_\ell + x)$$

(3) $$= x - y + p_\ell$$

We will now establish that $x > y$. From the given optimal sequence, we obtain a new sequence by interchanging the positions of s and k. Let $C_k^*$ be the new completion time for job k. Since $p_k < x$, $C_k^*$ lies between $C_\ell$ and d. Since the given sequence is optimal, we have

$$d - C_k^* \geq C_s - d$$

But $$d - C_\ell > d - C_k^*$$

so that $$d - C_\ell > C_s - d$$

i.e. $$x > y$$

Thus (3) is > 0 which again contradicts the optimality of the sequence. Thus (iv) cannot hold in Subcase 1, which establishes the V-shaped property in the subcase.

<u>Sub-case 2:</u> $p_s$ is not an intermediate job. i.e. either $B = \phi$ or $A = \phi$. If $B = \phi$, the optimal sequence will be as shown in Figure 5 below. Note that $t_o = 0$, since $B = \phi$.

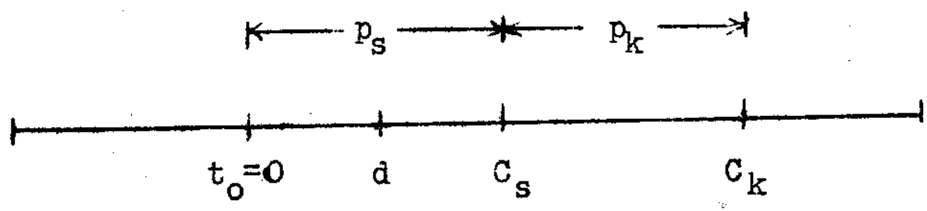


Figure 5

By Lemma 1, all the jobs in A are according to SPT rule. Thus whatever be the value of $p_s$, the optimal sequence is v-shaped. Similarly, if $A = \phi$, the jobs in B are according to LPT and the optimal sequence is again V-shaped.

This proves the theorem completely.

GENERALIZATION OF KANET's RESULT:

We show in this section that the SCHED algorithm proposed by Kanet [2] for the case $d \geq \sum_{i=1}^{n} p_i$ also holds under less restrictive condition on d. Let n = 2r-1 or 2r. The algorithm yields optimal schedule for the case

$$d \geq p_{(n)} + \ldots\ldots + p_{(n-r+1)}.$$

i.e. if the common due date is not less than the sum of the processing times of any r of the jobs 1, 2, ..... n where n = 2r or (2r - 1).

<u>Theorem 3:</u> "SCHED" algorithm yields optimal schedule if

(4) $$d \geq p_{(n)} + p_{(n-1)} + \ldots + p_{(n-r+1)}$$

where n =2r or (2r -1).

**Proof:** To prove the theorem we need only to establish that there is an optimal schedule satisfying the following properties:

(i) There exists no job which starts strictly before d and ends strictly after d.

(ii) Property (3) of Kanet's paper [2, p. 648]
Properties (1) and (2) follow from Lemma 1 and Theorem 1 of this paper).

We first prove (i) above. If $t_o > 0$ for the optimal schedule, Lemma 2 will establish (i). Suppose, if possible, $t_o = 0$ and that there exists a job which starts strictly before d and ends strictly after d. Start the process at $\varepsilon > 0$, where $\varepsilon$ is an arbitrarily small positive number. The increase in the objective function is

$$- |B| \varepsilon + |A| \varepsilon + \varepsilon$$
$$= \varepsilon ( |A| - |B| + 1).$$

Because of condition (4) and $t_o$ has been assumed to be zero, we have $|B| \geq r$. Further $|A| = n - |B| - 1$. Thus

$$|A| - |B| + 1 = n - 2 |B|$$

If $n = 2r$, $n-2|B| = 2r-2|B| = 2(r - |B|) \leq 0$

If $n = 2r-1$, $n-2|B| = 2r-1-2 |B| = 2(r-|B|) - 1 < 0$

Thus by starting the process at $\varepsilon > 0$, the objective function does not increase. If the increase is $< 0$ (reduction in objective function), it contradicts the optimality of the given schedule. If the increase is $= 0$, we get an equivalent optimal schedule where the starting time is $\varepsilon > 0$. Then by Lemma 2, there does not exist a job which starts strictly before d and ends strictly after d. This proves (i).

Property (3) of Kanet's paper holds under (4), since the proof given by Kanet for the case $d \geq \sum_{i=1}^{n} p_i$ is also applicable. By (i) above we can assume that in the optimal schedule some job ends at d. Then by Theorem 1, $|B| = r$ when $n = 2r$ or $(2r - 1)$. The condition (4) ensures that, in the proof of property (3) given by Kanet [2], $t_o$ is $\geq 0$, after the interchange of a job from set A to set B.

SOME SPECIAL CASES:

1. <u>When Some $p_i \geq d$</u>

   Suppose $k \geq 1$ is the first index such that $p_{(k)} \geq d$. If $k = 1$, the optimal schedule is clearly $p_{(1)}, p_{(2)}, \ldots\ldots p_{(n)}$ with $t_o = 0$. Assume therefore that $k > 1$. It is clear that among all schedules which start with a job whose processing time is $\geq d$, the best is $p_{(k)}, p_{(1)}, \ldots\ldots p_{(n)}$ with $t_o = 0$. We shall, therefore, consider only schedules where the first job is not the one with the processing time $\geq d$.

   By Lemma 1, all such jobs $(p_j \geq d)$ are scheduled in the end in SPT order and all jobs whose processing times are $< d$ precede them. If $t_o = 0$, the contribution to the objective function from jobs with processing time $\geq d$ is independent of the ordering of jobs $p_j$ with $p_j < d$. Thus the problem for $t_o = 0$ case reduces to sequencing of jobs each of whose processing times are $< d$.

Finally we consider the schedules with $t_o > 0$. Since, by Lemma 2, some job ends at d in the optimal schedule, all jobs with $p_j \geq d$ will be scheduled last in SPT order. Here, the contribution to the objective function from jobs with $p_j \geq d$ will depend on the ordering of jobs processed at and after d. If there are p jobs whose processing times are $\geq d$, the contribution to the objective function from these jobs will be $p(P_{r+1} + \ldots\ldots + P_{n-p})$ where as usual $|B| = r$. The problem is still reduced to the case of all $p_j < d$ but the objective function is different.

Suppose that the jobs indexed by 1, 2, ... (n-p) have processing times $< d$ and jobs indexed by (n-p+1), ..., n have processing times $\geq d$. The problem is to schedule the (n -p) jobs such that

(i) the starting time : $t_o > 0$.

(ii) some job ends at d.

(iii) The number of jobs in $B = r$. We have $r = \frac{n}{2}$ if n is even and $r = \frac{n+1}{2}$ if n is odd.

(iv) The objective function to be minimized is

$$\sum_{j=1}^{n-p} | C_j - d | + p\,(P_{r+1} + \ldots\ldots\ldots + P_{n-p}).$$

2. <u>When $p_{(n)} < d \leq p_{(1)} + p_{(2)}$:</u>

Suppose that $p_i < d$, $i = 1, \ldots n$ and $p_{(1)} + p_{(2)} \geq d$. In this case we can show easily that an optimal schedule is given by either 2 1 3 4 5 ...... n or 3 1 2 4 5 ...... n. With $t_o = 0$ in both the solutions. To prove this, note that the objective function is given by

$$Z = (n-2)P_1 + (n-1)P_2 + (n-2)P_3 + \ldots + 2P_{n-1} + P_n - (n-2)d$$

and this is minimum for either 2 1 3 4 5 ..... n or 3 1 2 4 5 ..... n with $t_o = 0$.

OPTIMUM SCHEDULE FOR n = 3:

Using the properties developed in the paper we give a complete solution for n = 3 jobs including the case of some jobs with processing time $\geq$ d.

n = 3 Case Optimal Sequences Under Different Conditions:

| Case Under Discussion: | | Condition(s) | Optimal sequence | Starting point $t_o$ |
|---|---|---|---|---|
| $p_{(3)} < d$ | (i) | $p_{(1)}+p_{(3)} > d$ | 312 | 0 |
| | (ii) | $p_{(1)}+p_{(3)} \leq d$ | 312 | $d-p_{(1)}-p_{(3)}$. |
| $p_{(1)} \geq d$ | | - | 123 | 0 |
| 2)$\geq d$, $p_{(1)}<d$ | (i) | $p_{(1)}+p_{(2)} > 2d$ | 123 | 0 |
| | (ii) | $p_{(1)}+p_{(2)} \leq 2d$ | 213 | 0 |
| $p_{(3)} \geq d$, $p_{(2)}<d$ | ( i) | $p_{(1)}+p_{(2)} > d$ | 213 | 0 |
| | (ii) | $\begin{cases} p_{(1)}+p_{(2)} \leq d \\ 2(p_{(3)}-d) > d-p_{(1)}-p_{(2)} \end{cases}$ | 213 | $d-p_{(1)}-p_{(2)}$ |
| | (iii) | $\begin{cases} p_{(1)} + p_{(2)} \leq d \\ 2(p_{(3)}-d) \leq d-p_{(1)}-p_{(2)} \end{cases}$ | 312 | 0 |

We see that even for n = 3, the description of the optimal sequence is quite complicated.

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