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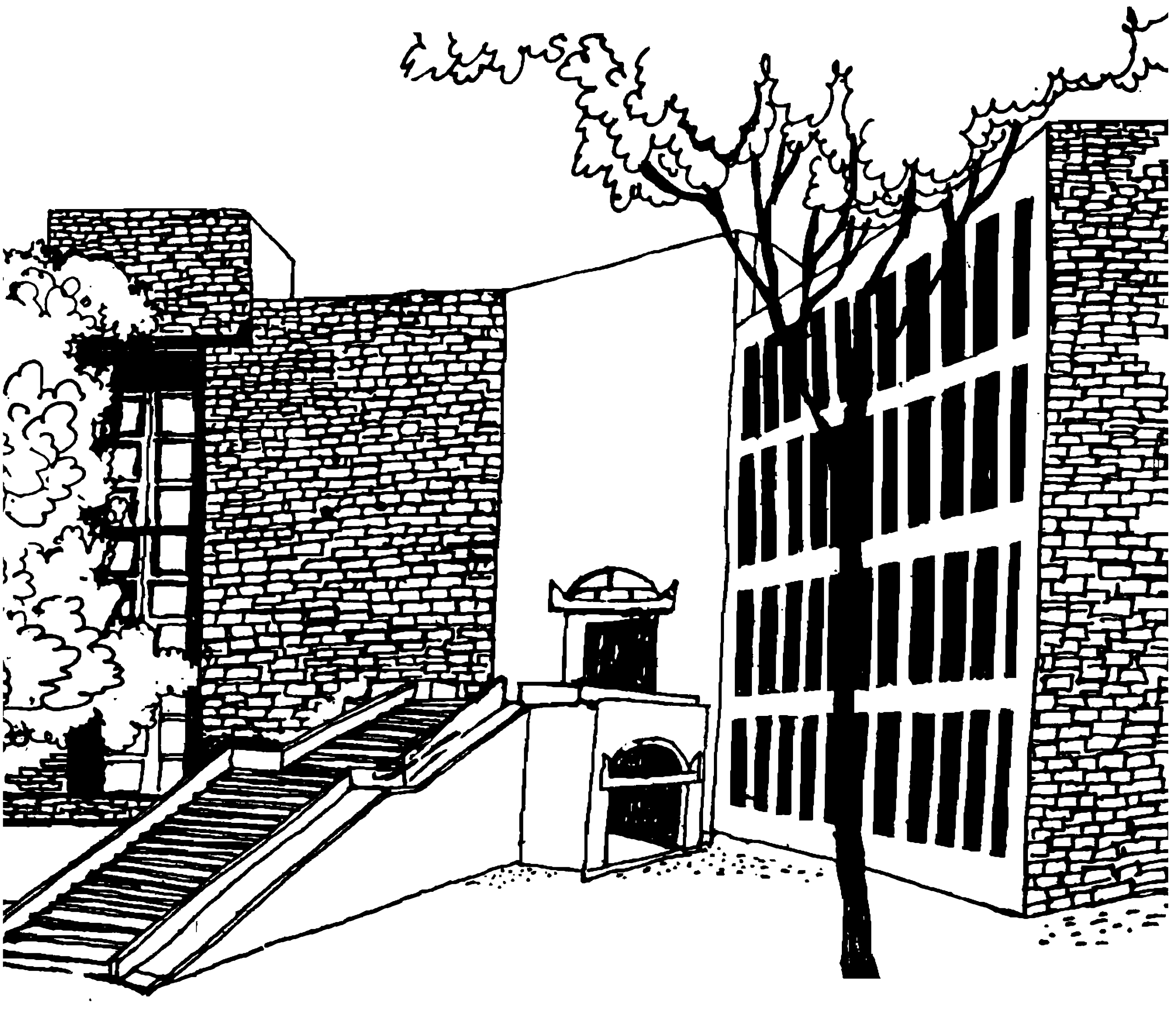

LINEAR AND NON-LINEAR BUDGET SETS

By

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Abstract

In this paper we show that every choice problem in a finite-dimensional Euclidean space can be viewed as the budget set corresponding to an economic environment (possibly non-linear) in consumer choice theory.

1. Introduction:- "To economize is to choose..." This is how Richter [1971] describes the fundamental problem of economics. In this sense, the problem that economics addresses is one and the same as the problem that choice theory deals with.

One of the most basic problems of economics is that of a consumer, confronted with a money income and a vector of prices for the commodities available in the market. He/She has to exercise his/her discretion and choose a consumption bundle from the available set of consumption bundles. Such is the problem of consumer choice theory - an age old problem, with some of the most revered names in the profession having expended their efforts and energies on it.

A possible generalization of the above framework has been indicated in Peters and Wakker [1991]: the case of non-linear budget sets. This occurs when the expenditure on a commodity is a non-linear function of the amount of the commodity purchased.

It is easy to visualize situations where this phenomena may arise. Consider a consumer who has available with him/her, some positive amount of an infinitely divisible consumption good today, which must be allocated either for consumption today or for production of consumption goods at a future date. If the rate of interest is assumed to vary both with time as well as with amount consumed, then we will be essentially confronted with a non-linear price schedule for each dated consumption good. The problem for the consumer is to choose a consumption level for today as well as for all future dates, within a finite time horizon.

Axiomatic choice theory, which originated in the seminal work of Nash [1950], has developed into a rich field concerned with picking a point given any non-empty, compact, convex, comprehensive subset of the non-negative orthant of a finite dimensional Euclidean space, each such subset admitting a strictly positive vector. This field has been surveyed in depth by Thomson [1995]. Such problems have been interpreted as games of fair division by Lahiri [1996]. Peters and Wakker [1991] indicate that the budget sets studied in consumer choice theory, whether linear (competitive) or non-linear are of the above type. Richter [1971] also draws a distinction between competitive and non-competitive budget sets, thereby implying that sets such as above could also be considered as budget sets.

In this paper we show that each set such as above is indeed a budget set with pricing rules being convex, non-decreasing functions. This would endow axiomatic choice theory with a much desired expression - as a meaningful theory of consumer choice.

2. The Model and Notations:- Consider a consumer who is confronted with the problem of allocating a positive amount of an infinitely divisible good for consumption over $l + 1$ periods; $t = 0, \ldots, l$. The consumer is endowed with $W > 0$ units of money. Confronted with an expenditure function $p_t : \mathbf{R}_+ \to \mathbf{R}_+$, for $t = 0, \ldots, l$, he must part with $p_t(c_t)$

units of money, if he wants $c_t \geq 0$ units of the good at date 't'. The consumer's budget set is given by

$$S(\langle p_1, \ldots, p_l; w\rangle) = \{(c_0, c_1, \ldots c_l) \,/\, c_t \geq 0,\ t = 0, \ldots, l \text{ and } \sum_{t=0}^{l} p_t(c_t) \leq w\}$$

We will assume that for $t = 0, \ldots, l$, $p_t : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies the following properties:

(i) $p_t(0) = 0$

(ii) p_t is a convex function which is non-decreasing

(iii) p_t is continuous and non-constant.

Under the above assumptions, S ($<p_0, \ldots, p_l; w>$) is a non-empty, compact, convex subset of $\mathbb{R}_+^{l+1}$ satisfying two other properties:

(a) $0 \leq (c_0, c_1, \ldots, c_l) \leq (d_0, d_1, \ldots, d_l) \in S(\langle p_0, \ldots, p_l; w \rangle)$

implies $(c_0, c_1, \ldots, c_l) \in S(\langle p_0, \ldots, p_l; w \rangle)$

(b) $\exists (c_0, c_1, \ldots, c_l) \in S(\langle p_0, \ldots, p_l; w \rangle)$ such that

$c_t > 0, \ \forall \ t = 0, \ldots, l.$

E = $< p_1, \ldots, p_l; w >$ satisfying the above properties is called the economic environment of the consumer. $\mathscr{E}$ denotes the class of all admissible economic environments. Given $E \in \mathscr{E}, S(E)$ is called a budget set. If E = $< p_0, \ldots, p_l; w > \in \mathscr{E}$ is such that p_t is a linear function for t = 0, ..., l, then E is called a competitive economic environment for the consumer and S (E) is called a competitive or linear budget set.

Now let us define a choice problem. A choice problem (of dimension (l+1)) is any non-empty subset S of $\mathbb{R}_+^{l+1}$ satisfying the following properties:

(i) S is compact, convex;

(ii) S is comprehensive i.e. $0 \leq x \leq y \in S \rightarrow x \in S$

(iii) $\exists\, x \in S,$ *with* $x \gg 0$.

Let B denote the class of all choice problems. Thus, for all $E \in \mathcal{E}$, $S(E) \in B$.

In the next section we shall show that given S ϵ B, there exists E ϵ $\mathcal{E}$ such that S = S (E). Thus, all budget sets are choice problems and all choice problems are budget sets.

3. <u>The Results</u>:- Given $x^1, \ldots, x^k \in \mathbb{R}_+^{l+1}$, let $co. \{x^1, \ldots, x^k\}$

denote the convex hull of $x^1, \ldots, x^k$

A choice problem $S \in B$ is said to be a polyhedral choice problem if there exists a finite set A such that if c, c' ϵ A with $c \neq c'$ then $c_t \neq c'_t$ whenever t = 0,...,1 and such that S

= co. $(A \cup \{0\})$

Now we state the main theorem.

Theorem 1 :- Let $S \in B$. Then there exists $E \in \mathscr{E}$ such that

S = S (E).

Proof:- Follows as a consequence of Theorem 2 which is proved below.

4. The Intertemporal Investment Planning Problem:- There is yet another interpretation of a choice problem, which goes to show that each choice problem in B can be obtained as the budget set corresponding to a economic environment belonging to a strict subset of $\mathscr{E}$ This corresponds to the situation where

the money available for expenditure on dated consumption goods is simply the consumption good available at period zero. Hence, the function $p_0 : \mathbb{R}_+ \to \mathbb{R}_+$ has the form

$$p_0(c_0) = c_0 \ \forall \ c_0 \in \mathbb{R}_+.$$

Let,

$$\overline{\mathscr{E}} = \{E \in \mathscr{E} / E = \langle p_0, p_1, \ldots, p_l; w \rangle \text{ implies } p_0(c_0) = c_0 \ \forall \ c_0 \in \mathbb{R}_+\}.$$

Given $E \in \overline{\mathscr{E}}$, if $E = \langle p_0, p_1, \ldots, p_l; w \rangle$, then let $\overline{E}$ denote $\langle p_1, \ldots, p_l; w \rangle$. Thus $\overline{E}$ conveniently represents a generic element in $\overline{\mathscr{E}}$.

Given $\overline{E} \in \overline{\mathscr{E}}$, let

$$S(\overline{E}) = \left\{(c_0, \ldots, c_l) \in \mathbb{R}_+^{l+1} / c_0 + \sum_{t=1}^{l} p_t(c_t) \le w\right\} \text{ where}$$

$$\overline{E} = \langle p_1, \ldots, p_l; w \rangle.$$

Clearly $S(\bar{E}) \in B$ whenever $\bar{E} \in \bar{\mathcal{E}}$.

<u>Lemma 1</u>:- Let S be a polyhedral choice problem. Then there exists $\bar{E} \in \bar{\mathcal{E}}$ such that $S = S(\bar{E})$.

<u>Proof</u>:- Let $u_t(S) = \max\{c_t / (c_0, \ldots, c_t, \ldots, c_l) \in S\}$ and let

$w = u_0(S)$.

Let $S = co.(A \cup \{0\})$, where it is being assumed that A is minimal i.e. there is no proper subset A' of A such that $S = co.(A' \cup \{0\})$,

For $(c_0, c_1, \ldots, c_t, \ldots c_l) \in S$ if $c_0 = w$, then define,

$p_t(c_t) = 0, t = 1, \ldots, l.$

Now let $c_0 < w$ with $(c_0, c_1, \ldots, c_t, \ldots, c_l) \in A$. Then there exists a unique $\lambda > 0$ such that

$$c_0 + \lambda \sum_{t=1}^{l} c_t = w.$$

Put $p_t\ (c_t) = \lambda\ c_t.$

Here λ may depend on the point in A that is chosen.

Let c ϵ w (S). Then, there exists $c^1,\ldots,c^k \in A$, such that $c \in co.\{c^1,\ldots,c^k\}$ and c does not belong to the convex hull of any proper subset of $\{c^1,\ldots,c^k\}$. Let

$$c = \sum_{i=1}^{k} \alpha^i c^i.\ \text{with}\ 0<\alpha^i<1, i=1,\ldots,k\ \text{and}\ \sum_{i=1}^{k} \alpha^i = 1.$$

Put $p_t(c_t) = \sum_{i=1}^{k} \alpha^i p_t(c_t^l),\ t=1,\ldots,l;$

$$\therefore\ c_0 + \sum_{t=1}^{l} p_t(c_t) = \sum_{i=1}^{k} \alpha^i c_0^i + \sum_{t=1}^{l}\sum_{i=1}^{k} \alpha^i p_t(c_t^i)$$

$$= \sum_{i=1}^{k} \alpha^i \left[c_0^i + \sum_{t=1}^{l} p_t(c_t^i) \right] = \sum_{i=1}^{k} \alpha^i w = w.$$

Thus $p_t: [0, u_t(S)] \to \mathbf{R}_+$ is convex, continuous, non-decreasing and non-constant with $p_t(0) = 0$ for all $t = 1, \ldots, l$.

Extend p_t beyond $u_t(S)$ so that $< p_1, \ldots, p_l; w > \in \bar{\mathscr{E}}$.

Thus $S = S(\bar{E})$ where $\bar{E} = \langle p_1, \ldots, p_l; w \rangle$.

Q.E.D.

Equipped with this lemma, we can now prove the following theorem:

Theorem 2:- Let $S \epsilon B$. Then there exists $\bar{E} \in \bar{\mathscr{E}}$ such that $S = S(\bar{E})$.

Proof:- Let $S \epsilon B$ and let $\{S^k\}$ be a sequence of polyhedral choice problems with

$S^k \subset S^{k+1}$ $\forall k \in \mathbf{N}$, $u_t(S^k) = u_t(S)$, $t = 0, \ldots, l$ and $k \in \mathbf{N}$ and

$\lim_{k \to \infty} S^k = S$ in the Hausdorff topology. Let $w = u_0(S)$ and as in

Lemma 1, let $p_t^k : [0, u_t(S)] \to \mathbf{R}_+$ be defined such that

$$S^k = \{(c_o, \ldots, c_l) \in \mathbf{R}_+^{l+1} / c_0 + \sum_{t=1}^{l} p_t^k(c_t) \leq w\}.$$

For $0 \leq c_t < u_t(S), let\ p_t(c_t) = \lim_{k \to \infty} p_t^k(c_t);$ let

$$p_t(u_t(S)) = \lim_{c_t \to u_t(S)} p_t(c_t).$$

Thus $p_t(0) = 0$, p_t is convex, continuous, non-decreasing and non-constant. Extent p_t beyond u_t (S) so that p_t continues to remain convex. Let $\widetilde{E} = \langle p_1, \ldots, p_l; w \rangle$. Clearly, S = S $(\widetilde{E})$.

<u>Q.E.D.</u>

<u>Note 1</u>:- In Lemma 1, we mention that $p_t: [0, u_t(S)] \to \mathbf{R}_+$ is a convex function (the other properties being easy to verify). This property is verified, by considering the set

$$T = \{(c_0, c_t) / (c_0, \ldots, c_k, \ldots, c_l) \in S, c_k = 0 \text{ for } k \neq 0, t\}$$

By construction,

$$T = \{ (c_0, c_t) / c_0 + p_t(c_t) \leq w \}.$$

The set T is a closed convex polygon and the function p_t is piecewise linear.

Let (c_0, c_t) and (c_0', c_t') belong to the Weakly Pareto Optimal set

of T.

Thus, $c_0 + p_t(c_t') = w$ and $c_0' + p_t(c_t') = w$.

Let us assume towards a contradiction that

$p_t(\alpha c_t + (1-\alpha) c_t') > \alpha p(c_t) + (1-\alpha) p(c_t')$ for some $\alpha \in (0,1)$.

Then $\alpha c_0 + (1-\alpha) c_0' + p_t(\alpha c_t + (1-\alpha) c_t')$

$$> \alpha \left[c_0 + p_t(c_t)\right] + (1-\alpha) \left[c_0' + p_t(c_t')\right] = w,$$

which is impossible, since,

$\left(\alpha c_0 + (1-\alpha) c_0', \alpha c_t + (1-\alpha) c_t'\right) \in T.$

Thus p_t (as defined in Lemma 1) is indeed a convex function.

Conclusion:- Our efforts in this paper go to show that axiomatic choice theory, developed in the tradition of the Nash [1950] tradition can now be applied to diagnose problems of consumer choice theory. In axiomatic choice theory a choice function is defined as a function $F : B \to \mathbb{R}_+^{l+1}$ such that $F(S) \in S \; \forall \; S \in B$.

Since each such problem can now be viewed as a budget set, a choice function is economically meaningful from the stand-point of consumer choice theory. As in axiomatic choice theory, we may now characterize choice functions for consumer choice theory, by imposing a set of minimal conditions that it should satisfy. The conditions should be decided on the basis of their intuitive and welfare theoretic appeal.

In the intertemporal setting we have invoked in the paper, it is commonplace to assume that consumption is appropriately discounted i.e. instead of c_t we consider $\delta^t c_t$, $\delta \in (0,1]$. A moment's reflection suggests that this presents no new analytical problems.

On the other hand, if we view each co-ordinate as representing the consumption of an infinitely divisible good in a region (or sector) of an economy and p_t (c_t) as the cost function for producing c_t units of the consumption good in region (or sector) t, then the above theory becomes applicable as a theory for games of fair division. In either case, its use cannot be overemphasized.

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Appendix

What implies SARP for two dimensional Choice Probelms?

In this appendix, let B denote the class of all (two-dimensional) choice problems.

A domain is any non-empty subset D of B.

Given $p \in \mathbb{R}^2_{++}$, let $S(p) = \left\{ x \in \mathbb{R}^2_{+} / p.x \leq 1 \right\}$.

Let $C = \left\{ S(p) / p \in \mathbb{R}^2_{++} \right\}$.

(is called the class of all (two-dimensional) linear (competitive) choice problems.

Given $\phi \neq D \subset B$, a choice function on D is a function

$F : D \to \mathbb{R}^2_{+}$ such that $F(S) \in S \; \forall \; S \in D$.

Given a choice function $F : D \to \mathbb{R}^2_{+}$, and $x, y \in \mathbb{R}^2_{+}, x \neq y$ write $x \, R_F \, y$

if and only if $\exists\, S \in D$ such that x = F (S) and y $\in$ S.

A choice function $F : D \to \mathbb{R}^2_+$, is said to satisfy the

(a) Weak Axiom of Revealed Preference (WARP) if R_F is assymetric;

(b) Strong Axiom of Revealed Preference (SARP) f R_F is acyclic.

A choice function $F : D \to \mathbb{R}^2_+$, is said to satisfy Nash's Independence of Irrelevant Alternatives Assumption (NIIA) if $\forall S, T \in D, S \subset T, F(T) \in S \Rightarrow F(S) = F(T)$.

Given $S \in D \subset B$, $D \neq \phi$, let

$P(S) = \{x \in S / y > x \to y \notin S\}$. P (S) is called the Pareto Optimal Set of S.

<u>Theorem 3</u> : (Rose [1958]): Let $F : C \to \mathbb{R}^2_+$ be a choice function

such that $F(S)^{*} \in P(S) \ \forall S \in C$. Then F satisfies WARP if and only if F satisfies SARP.

<u>Theorem 4</u> (Very Easily Proved): A choice function $F : B \rightarrow \mathbb{R}^2_+$ satisfies WARP if and only if it satisfies NIIA.

A choice function $F : D \rightarrow \mathbb{R}^2_+$ is said to satisfy Pareto Optimality (PO) if $\forall S \in D$, $F(S) \in P(S)$.

For the subsequent property we refer to Thomson [1981]:

Given, $D \subset B$, $D \neq \phi$ and $F : D \Rightarrow \mathbb{R}^2_+$,

let $p(F,S) = \{p \in \Delta \,/\, p.x \leq p.F(S) \ \forall x \in S\}$, where

$\Delta = \{(p_1, p_2) \in \mathbb{R}^2_+ / p_1 + p_2 = 1\}$.

A choice function $F : D \rightarrow B$ is said to satisfy <u>Independence of</u>

Irrelevant Expansions (IIE) if $\forall S \in D$, there exists $p \in p(F,S)$ such that whenever $S \subset T \in D, F(S) \in T$ and $p.x \leq p.F(S)$, $F(T) = F(S)$.

This property and a weaker version of the same has been studied in Lahiri [1997]. It is easy to see that whenever S satifies P O and I I E, and S ϵ D, then $p \in p(F,S) \Rightarrow p >> 0$.

We do not need I I E in its full generality, but something much weaker that P O and IIE implies:

A choice function $F : D \Rightarrow B$ with $C \subset D$ is said to satisfy Partial Independence of Irrelevant Expansions (PIIE) if $\forall S \in D$, there exists S (p) ϵ C such that $S \subset S(p)$, and $F(S(p)) = F(S)$.

If F satisfies P O and PIIE, then F (S) ϵ P (S (p)) for such an S (p) ϵ C.

Theorem 5:- Let $F : B \rightarrow \mathbf{R}^2_+$ be a choice function which satisfies P O, NIIA and PIIE. Then F satisfies SARP.

Proof:- Suppose not. Then there exists $x^0, x^1, \ldots, x^n$ with

$x^j\, R_F\, x^j\ \forall\, j \in \{1, \ldots, n\}$ and $x^n\, R_F\, x^0$. By NIIA and Theorem 4,n>1.

Infact it is possible to prove that NIIA implies n>2. However, suppose, $n \geq 2$. Then there exists $S^0, \ldots, S^n \in B$, such

that $x^j = F(S^j)\ \forall\, j=0, \ldots, n, x^j \neq x^{j+1} \in S^j\ \forall\, j=0, \ldots, n-1$

and $x^0 \in S^n$.

By PIIE. there exist $S(p^j) \in C$ such that $S^j \subset S(p^j)$ and

$F(S(p^j)) = x^j\ \forall\, j = 0, \ldots, n.$

By PO, $x^j \in P(S(p^j))\ \forall\, j = 0, \ldots, n.$

Now $x^{j+1} \in S^j\ \forall\, j = 0, \ldots, n-1$ implies

$x^{j+1} \in S(p^i)\ \forall\, j=0, \ldots, n-1\ x^0 \in S^n$ implies $x^0 \in S(p^n)$.

But by Theorem 3, this is not possible.

Hence F satisfies SARP.

Q.E.D.

This result becomes important in view of the interpretation of every choice problem as the budget set corresponding to an economic environment in consumer choice theory and Theorem 3, quoted in this appendix. It is easy to see that NIIA does not imply PIIE (and hence does not imply IIE).

Infact we can now make a much stronger statement. A choice function $F : B \to \mathbf{R}^2_+$ is said to be representable, if there exists a real valued function V on $\mathbf{R}^2_+$ such that

$$\forall S \in B,\ \{F(S)\} = \{x \in S / V(x) \geq V(y)\ \forall y \in S\}.$$

Given a choice function $F : B \to \mathbf{R}^2_+$, define $f : C \Rightarrow \mathbf{R}^2_+$ by

$$f(S(p) = F(S(p)) \text{ whenever } p \in \mathbf{R}^2_{++}.$$

Suppose F satisfies P O. Let $S = \{ y \in \mathbf{R}^2_+ / y \le x \}$, for x >> 0. Then F (S) = x. Thus $\mathbf{R}^2_{++} \subset$ range (F).

If F satisfies NIIA, then F satisfies WARP. Thus so does f. Hence by Theorem 3, f satisfies SARP.

Now by the theorem in Hurwicz and Richter (1971), there exists a function $V : \mathbf{R}^2_+ \Rightarrow \mathbf{R}$ which is uppersemicontinuous on $\mathbf{R}^2_+$, strictly monotonically increasing and strictly quasi-concave on $\mathbf{R}^2_{++}$, such that $\forall\, S \in C$, $\{F(S)\} = \{x \in S / V(x) \ge V(y)\}$. Thus we have the following theorem:

<u>Theorem 6</u>:- Let $F : B \Rightarrow \mathbf{R}^2_+$ be a choice function which satisfies PO, NIIA and PIIE. Then F is representable by a function $V : \mathbf{R}^2_+ \Rightarrow \mathbf{R}$ which is uppersemicontinuous on $\mathbf{R}^2_+$, strictly monotonically increasing, and strictly quasi-concave on $\mathbf{R}^2_{++}$.

If a choice function $F : B \rightarrow \mathbf{R}^2_+$ satisfies SARP, then we can say that there exists a complete and transitive binary relation R such that $\forall S \in B, \{ F(S) \} = \{ x \in S / xRy \ \forall y \in S \}$. This is what Theorem 5 implies. Theorem 6 goes a step further. It says that there exists a function $V : \mathbf{R}^2_+ \rightarrow \mathbf{R}$ such that the binary relation $\bar{R}$ *on* $\mathbf{R}^2_+$ defined by $x \bar{R} y \leftrightarrow V(x) \geq V(y)$ [: which is of necessity complete and transitive] helps us to define F i.e. $\{ F(S) \} = \{ x \in S / x \bar{R} y \ \forall y \in S \}$. These results should be contrasted with those in Peters and Wakker [1986].

A final point we would like to make in this appendix is that nowwhere in the proofs of Theorems 5 and 6 are we using everything that NIIA (or WARP for that matter) implies. Infact both our theorems remain intact if instead of NIIA we use a property called Restricted Weak Axiom of Revealed Preference (RWARP):

A choice function F on B is said to satisfy RWARP if whenever $x^\circ = F(S(p^\circ))$, $x^1 = F(S(p^1))$, $S(p^\circ) \in C$, $S(p^1) \in C$ and $x^\circ R_F x^1$ then it is not the case that $x^1 R_F x^\circ$.

Infact, for a choice function F on B, P O, PIIE and RWARP trivially implies NIIA. Thus, really the only requirement beyond those in Rose [1958], to extend SARP from C to B is the PIIE property.