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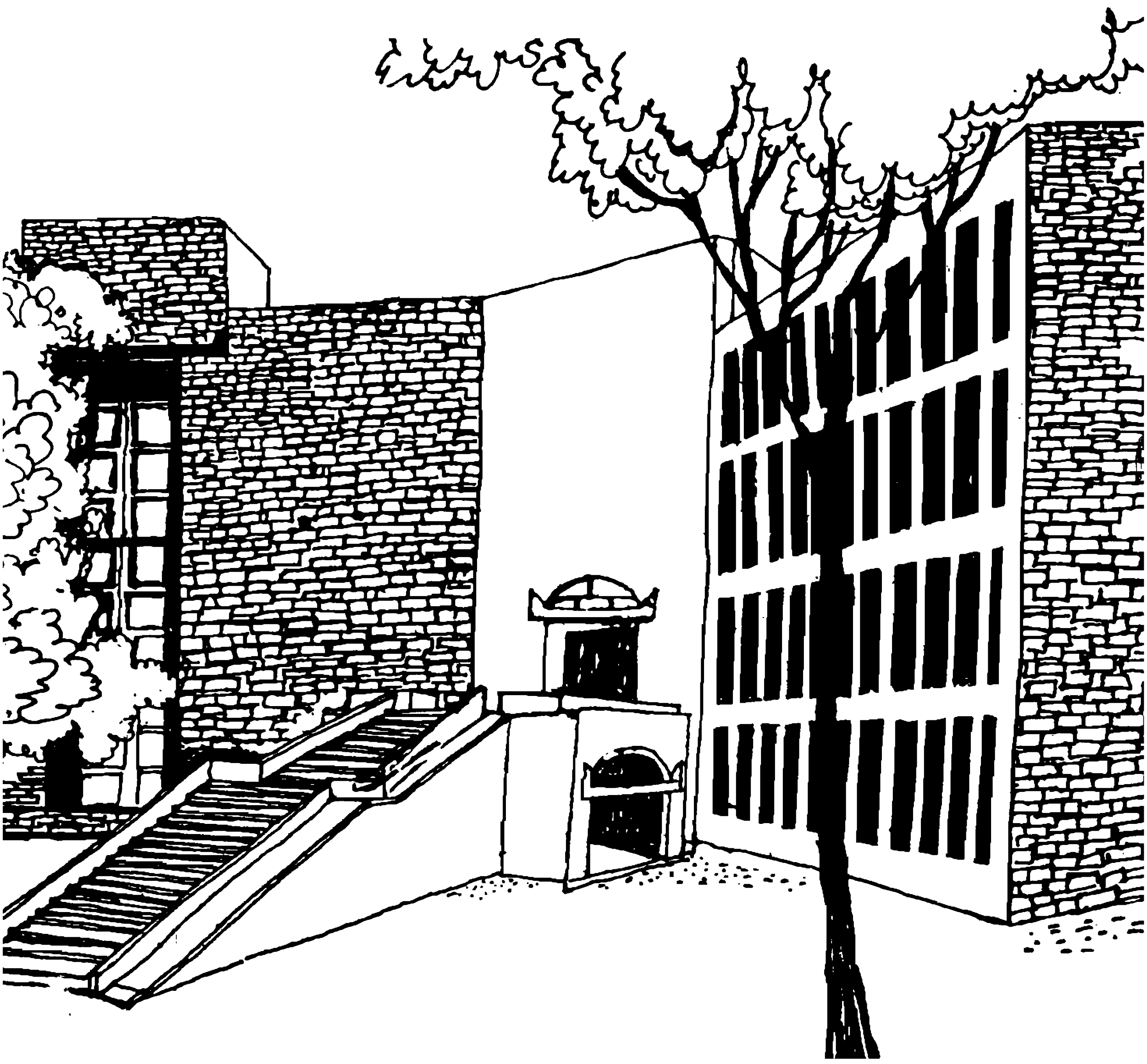

A COMMENT ON NASH'S INDEPENDENCE OF IRRELEVANT ALTERNATIVES ASSUMPTION FOR CHOICE PROBLEMS

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ABSTRACT

In a recent paper, Campbel (1994) shows that if a choice correspondence satisfies Arrow's choice axiom then it has a complete, reflexive and transitive rationalization, even if the domain doesnot include any set with fewer then m members, where m is a given positive integer. The purpose of thispaper isto provide a simpler proof (than the one provided by Campbell) of the same result when the choice correspondences are single - valued i.e, the case of choice functions. In such a situation Arrow's choice axiom is formally equivalent to Nash's Independence of Irrelevant Alternatives assumption.

1. Introduction: In a recent paper, Campbell (1994) shows that if a choice correspondence satisfies Arrow's choice axiom then it has a complete, reflexive and transitive rationalization, even if the domain does not include any set with fewer than m members, where m is a given positive integer. The purpose of this paper is to provide a simpler proof (than the one provided by Campbell) of the same result when the choice correspondences are single - valued i,e, the case of choice functions. In such a situation Arrow's choice axiom is formally equivalent to Nash's Independence of Irrelevant Alternatives assumption.

2. The Framework: Let X denote the universal set of alternatives. ζ represents the family of candidate feasible sets ie, a non-empty collection of non-empty subsets of X.(X, ζ) is called a choice space. A choice function on (X, ζ) is a function C : $\zeta \rightarrow X$ such that for all $S \in \zeta$, $C(S) \in S$. We set $X_C = \{C(S) : S \in \zeta\}$, the range of C. Let ζ_m be the collection of all m element subsets of X and all subsets with exactly (m+1) members.

A (binary) relation on X is any non-empty subset R of X x X. A choice function C on (X, ζ) is said to be rationalizable by a relation R if for all $S \in \zeta$, $\{C(S)\} = \{x \in S / (x, y) \in R$ for all $y \in S\}$. A relation R is said to be reflexive if $(x, x) \in R$ for all $x \in X$.A relation R is said to be complete if for all $x, y \in X$, $x \neq y$, either $(x,y) \in R$ or $(y, x) \in R$. A relation R is said to be transitive if for all $x, y, z \in X$, $(x, y) \in R$, $(y,z) \in R$ implies $(x, z) \in R$. A choice function C on (X, ζ) is said to be fully rational if there exists a binary relation R on X which is reflexive, complete and transitive such that C is rationalizable by R.

Theorem 1: Let C be a choice function on a choice space (X, ζ) such that ζ contains ζ_2. C is fully rationalizable if and only if C satisfies Nash's Independence of Irrelevant Alternatives Assumption (NIIA), where NIIA is defined as follows: $S, T \in \zeta$, $S \subset T$, $C(T) \in S \rightarrow C(S) = C(T)$.

Proof: Arrow (1959).

3. The Main Result:

Lemma 1: Let (X, ζ) be a choice space such that $\zeta_m \subset \zeta$ and suppose C is a choice function on (X, ζ) which satisfies NIIA. If $\phi \neq T \subset X$ and $T \subset S_1$, $T \subset S_2$ where $S_1, S_2 \in \zeta_m$ and $T \cap X_c \neq \phi$, then $C(S_1) = C(S_2)$, provided $C(S_1), C(S_2) \in T$, $|T| = m - 1$, $|S_1| = m$, $|S_2| = m$, where $|S|$= carditionality of the set S.

Proof: Let T be as in the hypothesis of the lemma. Clearly there exists at least one m element set S such that $T \subset S$. Let S_1 and S_2 be two such sets and let $S = S_1 \cup S_2$. Thus S $\in \zeta_m$. Clearly $C(S) \in S_1$ or S_2, so that either $C(S) = C(S_1)$ or $C(S_2) = C(S)$ by NIIA. Suppose $C(S) = C(S_1) \neq C(S_2)$. Thus $C(S) \in S_1 - S_2$. But then $C(S_1) \notin T$. This contradicts the hypothesis . Thus $C(S_1) = C(S_2)$. Q.E.D.

Theorem 2 (Campbell (1994)): Let $m \geq 2$ be a given integer, and suppose that (X, ζ) is a choice space such that ζ contains ζ_m but not any subsets of X with fewer than m members. A choice function C on (X, ζ) satisfies NIIA if and only if it is fully rationalizable.

Proof: If C is fully rationalizable then it satisfies NIIA without any restrictions. Hence let us prove the converse which we will do by induction. By theorem 1, the result is true if m = 2. Hence assume that the result is true for $m \leq n$ (>2) and suppose $m = n + 1$. Thus ζ_{n+1} contains all n+1 and n+2 member subsets of X but not any set with fewer than n+1 members. Let $\bar{\zeta}$ be the smallest superset of ζ consisting of all subsets of X with n members. We will extend C to $(X, \bar{\zeta})$ as follows: Let T be a n element subset of X. If $T \cap X_c \neq \phi$, then let $\bar{C}(T) = C(S)$ if S is any n+1 element set containing T such that $C(S) \in T$. Clearly $\bar{C}(T)$ is uniquely defined by Lemma 1.

Let P be any reflexive, complete, transitive binary relation on X such that the strict part of P is P itself. If T as above is such that $C(T \cup \{x\}) = x \ \forall x \in X$ then define $\{\bar{C}(T)\} = \{x \in T/(x,y) \in P \text{ for all } y \in T\}$. This set will be a singleton since the strict part is identical to the relation itself.

If $T \in \zeta$, then define $\bar{C}(T) = C(T)$. This extends C on (X, ζ) to $\bar{C}$ on $(X, \bar{\zeta})$. It is easy to observe that $\bar{C}$ satisfies NIIA and by the induction hypothesis there exists a complete, reflexive, transitive relation R on X which rationalizes $\bar{C}$. Since $\bar{C}$ extends C, R rationalizes C. This prove the theorem.

Q.E.D.

Remark: The framework of the above paper, particularly the concepts of a choice space and full rationality can be found in Suzumura (1983). The essential difference between the present paper and that of Campbell (1994), is in our use of choice functions. A detailed development of rational choice theory for choice functions and a lucid discussion of the same can be found in Richter (1971).

References:

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4. K. Suzumura (1983): "Rational Choice, Collective Decisions, and Social Welfare (Cambridge University Press, Cambridge).