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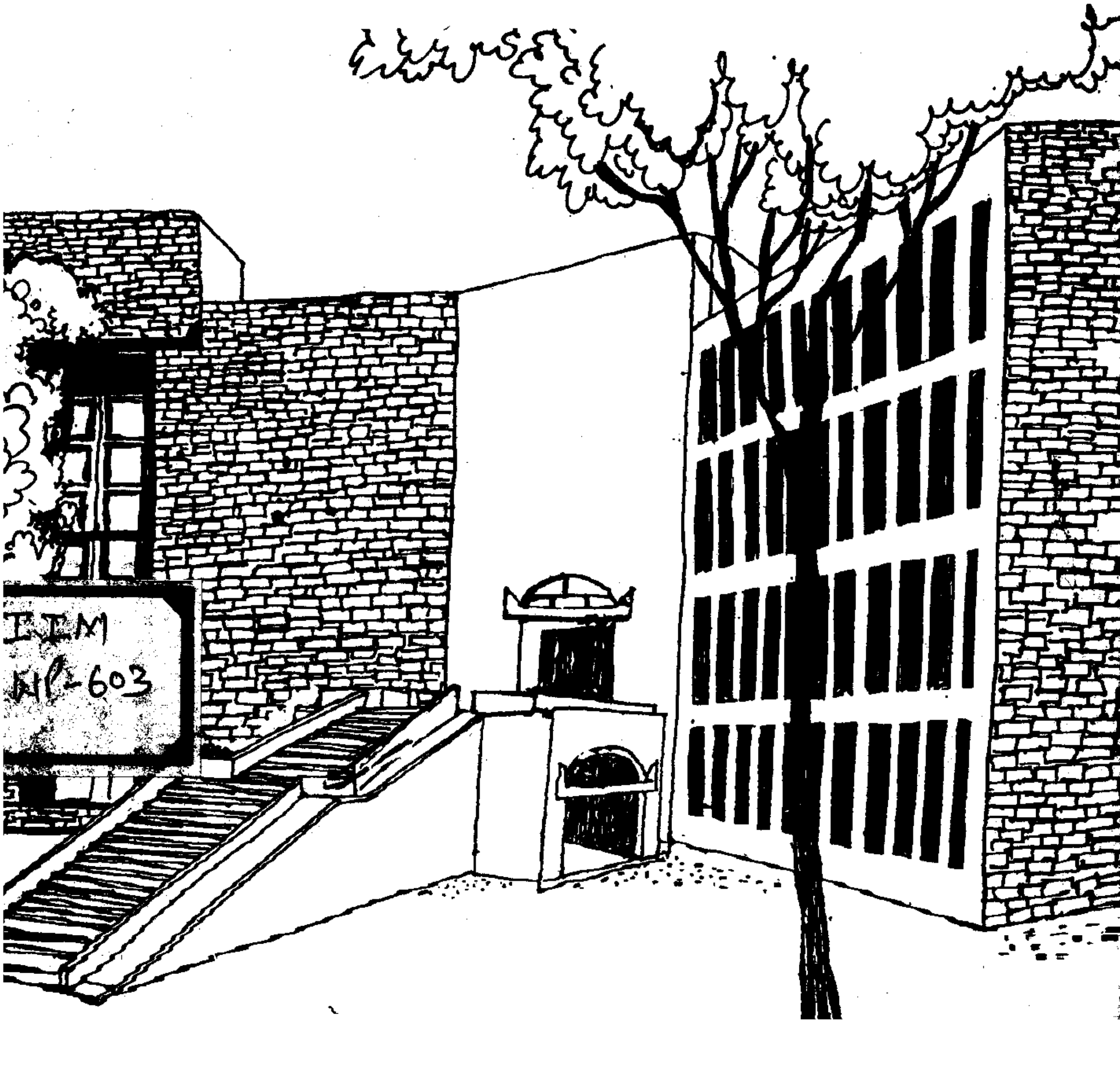

ANALYSIS OF (S,s) INVENTORY SYSTEM WITH DECAYING ITEMS

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ANALYSIS OF (S, s) INVENTORY SYSTEM WITH DECAYING ITEMS

S.K. Srinivasan* and N. Ravichandran**

ABSTRACT

This article obtains the stationary distribution of the inventory level of an (S, s) inventory model with decaying items. The demand to this inventory system is governed by a general renewal process. Items decay at a constant rate λ independently and identically. When the inventory reduces to a level $\leq s$, at a demand point an order for replenishment is placed and is received after a random duration whose distribution is arbitrary. The quantity realised is variable and is such that the resulting inventory level is equal to S. Both the cases of complete becklogging and lost sales are treated in the present analysis. The stochastic process L(t) representing the inventory level at any time t is analysed by

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identifying an imbedded MRP. The intensity of decay events and shortages are obtained in an explicit form. Based on these results a total cost expression is derived. Numerical results are presented to illustrate the use of cost expression in identifying the optimal reorder level.

INTRODUCTION

The object of this paper is to analyse the stochastic behaviour of an (S, s) ordering inventory model when the inventory items are subject to failure (or decay). The results are used to obtain a cost expression for the system, which may be an objective to optimize the parameters.

Inventory systems of (S, s) type with non-perishable or perishable items have been studied extensively in the past. The previous work can be generally classified into three major themes. There is a group of initial papers which extends the classical EOQ formula for the case of inventory systems with decaying items. Ghare and Shrader (1963) obtained the EOQ formula for exponentially decaying items considering the ordering cost and inventory cost. Shah and Jaiswal (1977) determined the optimal level S under the assumption of back orders and instantaneous lead time. The second set of papers available in the literature deal with the dynamic policy of arriving at an optimal order quantity for any period. Only in a few cases exact results have been reported. Because of the complexity of the problem, reported research is towards the use of various approximation procedures to obtain the optimal policy by the application of Dynamic programming methods.

Kaplan (1969) obtained the optimal sequential decision rules for the case of random demand considering the ordering, holding and shortage costs with the provision of discrete lead time and backlogging. Fries (1975) obtained the form and properties of the optimal policy for a perishable commodity with deterministic life time, in a finite horizon model and continuous demand and full backlogging. Subsequently, Weiss (1980) obtained the optimal ordering policy for continuous review model with instantaneous replenishment for items which have a fixed life time. The results are obtained for full backlogging as well as the lost sales case. Recently, the analysis of inventory systems with decaying items is triggered by the applications in the blood inventory model. References 1, 2, 4-7, 15 and 16 deal with such models.

There is a third set of papers published in the literature which is relevant for the purpose of this contribution. These papers discuss the stochastic process induced by the behaviour of inventory model of (S, s) type. SivaZlian (1975) established the uniform distribution of the inventory level in the stationary case, under the assumption of instantaneous lead time. Srinivasan (1979) provided an extensive analysis of such a system when the lead times and demand times form families of independent identically distributed random variables. Kalpakam etal (1983) analyse an (S, s) model with instantaneous lead time and Semi Markov demand. Dirickx etal (1977) obtain an expression for the cost of the system under the assumption of Compound Poisson demand and Stochastic lead time. Ravichandran (1984) provided a computationally feasible analysis to obtain the stationary distribution under the assumption of phase type lead time distributions.

Nahimas (1982) in his review points out the need for a Comprehensive analysis of (S, s) policy with decaying items under reasonable assumptions like positive lead time etc. This paper is to fill this gap and is complementary to the work of Srinivasan [13], in the case of decaying items.

The present paper analyses an inventory model for decaying items with a policy of (S, s) type. The maximum inventory level is S. The units in the inventory decay and are not usable after a random duration of time which is governed by a negative exponential distribution. The items in the inventory are assumed to be independent and identical with respect to the decay. The demand to the items stored in the inventory arise according to a general renewal process whose interarrival time is an arbitrary distribution. Unit quantity is demanded whenever there is a demand. The need to place an order is assessed after meeting the demand, and an order is placed when the stock level at the review point is $\leq s$. The quantity ordered is variable and can be decided at the time of delivery. The advantage of this assumption is that both the lost sales and complete backlogging situations can be accommodated in the same model. The actual quantity realised will be determined in such a way that when the stock arrives the inventory level is equal to S. The lead time is a positive random variable with arbitrary distribution.

The structure of this article is as follows: Section 1 developes the basic notation and concepts necessary for the analysis. While Section 2 contains the important relations and functions necessary to obtain the key results concerning the behaviour of the system. We then proceed to obtain an explicit expression for the Probability mass of the inventory level and the expected cost in maintaining this ordering policy. The next section relates several special cases with the present analysis.

1. FORMULATION OF THE PROBLEM:*

It is assumed that at time $t = 0$, the inventory position is $>s$ and there is a demand. The inventory level $L(t)$,

*From now on, we shall confine ourselves to the case of lost sales.

at any time t, is a random variable with values in $\{0, 1, 2, \ldots S\}$. Thus, the stochastic process $\{L(t), t \geq 0\}$ is an integer valued process and the behaviour of which is our primary concern in this paper. The points of discontinuity of the process are those epoch at which either there is a demand or a decay or a replenishment. In the case of a demand or decay the inventory level is reduced by one unit, whereas in the case of replenishment the inventory level is increased to S, because of our assumption on the variable reorder quantity. In view of the additional feature of decay of units, the behaviour of the process $\{L(t), t \geq 0\}$ during successive demand points when no order is pending, is governed by a state dependent death process. We shall designate in our analysis the demand events as a events, as they correspond to the arrival of a demand. Even though the inventory is depleted by the demand and decay, we assume that the inventory level is assessed for reordering only at those points when an a event occurs. A reorder results at those points corresponding to an a-event, where the inventory level <s. The reorder points are special events and are of significance in our analysis. The reorder events are denoted as r-events. It may be worthwhile to recollect that an r-event is simply an a-event with inventory level $\leq$s. When an order is placed and is pending the behaviour of the inventory is governed by a death process, which is the combination of the demands that arrive to the system

and the decay. Nevertheless, the inventory level raises to S when the order materialises. The points at which the inventory level reaches the level S are non-regenerative [18] because of the arbitrary distribution of the random variable governing two successive demands. However, the very first demand after replenishment is a regenerative event and such events are useful in tracing the behaviour of the process $\{L(t),\ t \geq 0\}$. These events are termed as b-events.

It is important to notice that at the time of a b-event the inventory level need not be equal to $S-1$, because of the decay factor. It may also happen that the b-event may coincide with an r-event or an a-event depending on whether $L(t) \leq s$ or $L(t) \geq s+1$. The demands that arise when the stock-level is zero are of special significance in terms of identifying a cost expression. We call a demand that arise when the stock level is zero as σ-event. Hence, a σ-event is also an a-event with the further requirement that the stock level is zero. Some of the b-events could be σ-events and no r-event can be a σ-event even though the stock level immediately after the r-event can be equal to zero. We use the stock level as a second subscript to augment the information on the inventory level. For example, the notation (r, i) will mean an r-event with inventory level after the r-event is equal to i, $0 \leq i \leq s$.

The following symbols and notations will be used in our distribution:

$L(t)$	-	Inventory level at any time t $L(t) \in \{0, 1, 2, \ldots S\}$
$f(.)$	-	pdf of the random variable representing the time interval between successive demand durations.
$b(.)$	-	pdf of the random variable representing the lead time.
λ	-	decay parameter of the individual item in the inventory
$N_\alpha(t)$	-	counting process associated with the α-events in the interval $(0, t]$, $\alpha = a, b, r, \sigma, d$
$R(t)$	-	number of reorders received in an interval of length t.
$*$	-	denotes convolution
$C^{(n)}(t)$	-	n fold convolution of an arbitrary function C(t) with itself.
$\bar{C}(t)$	-	$1 - c(t)$

2. <u>SOME USEFUL DERIVATIONS:</u>

We first study the behaviour of the process $L(t), t \geq 0\}$ during an interval of length t given that at time $t = 0$, there is an a-event. It is assumed that there is no a-event (o, t). If the initial a-event is an r-event, it is further assumed that this replenishment does not occur in (o, t). Let $P_{ij}(t) = pr\{L(t) = j \mid L(o) = i\}$ for such an interval. Then the changes in $L(t)$ during an interval of this type could be only due to decay. By using the independent and identical nature of the items stored in the inventory, it is easy to see, with $q = e^{-\lambda t}$, $p = 1-q$, that

$$P_{ij}(t) = \binom{i}{j} q^j p^{i-j}, \qquad 0 \le i \le s;\ j \le i$$
$$= 0 \qquad \text{Otherwise} \qquad \ldots.\ (2.1)$$

The functions

$$f^a_{ij}(t) = \lim_{\Delta,\Delta_1,\Delta_2 \to 0} \frac{1}{\Delta}\ \mathrm{pr}\left\{ \begin{array}{l|r} L(t) > L(t+\Delta_1) = j & L(0-) > L(0) = i \\ N_a(t+\Delta) - N_a(t) = 1 & N_a(-\Delta_2, 0) = 1 \\ & R(t) = 0 \end{array} \right\}$$

$$0 \le i \le s,\ 0 \le j \le \max(0, i-1)$$

representing the behaviour of the level process $L(.)$ during successive demands, with the further requirement that no re-order materialises in $(0, t)$ are given by the following expressions:

$$f^a_{ij}(t) = f(t) P_{ij+1}(t) \qquad 0 < j \le i-1,\quad i \ge 2$$
$$= f(t)[P_{i1}(t) + P_{io}(t)] \qquad j = 0,\ i \ge 1$$
$$= f(t) \qquad i = 0,\ j = 0 \qquad \ldots.(2.2)$$

The next result extends the previous result for the case of n successive demands. To do this, we define the functions $\bar{\pi}^n_{ij}(t)$ by (2.3)

$$\lim_{\Delta,\Delta_1,\Delta_2\to 0} \frac{1}{\Delta} \text{pr} \left\{ \begin{array}{l|l} L(t) > L(t+\Delta_1) = j & L(o-) > L(o) = i \\ N_a(t+\Delta) - N_a(t) = 1 & N_a(-\Delta_2, o) = 1 \\ N_a(t+\Delta) = n & R(t) = 0 \end{array} \right\} \quad \ldots (2.3)$$

satisfy the following recurrence relations:

$$\pi_{ij}^{n}(t) = 0 \qquad j > i - n$$

$$\pi_{ij}^{n}(t) = f_{ij}^{a}(t) \qquad \text{for all } i \text{ and } j$$

$$\pi_{ij}^{n}(t) = \sum_{k=j+1}^{i-(n-1)} \pi_{ik}^{n-1}(t) * f_{kj}^{a}(t) \qquad 0 < j \leq i - n$$

$$\pi_{io}^{n}(t) = \sum_{k=0}^{i-(n-1)} \pi_{ik}^{n-1}(t) * f_{ko}^{a}(t) \qquad \ldots (2.4)$$

Using (2,2) the functions $\pi_{ij}(t)$s can be explicitly computed.

Next, we obtain the pdf of the random variable representing the time interval from an a-event to the first reorder.

Define, for $i \geq s+1$, $0 \leq j \leq s$

$$f_{ij}^{ar}(t) = \lim_{\Delta,\Delta_1\to 0} \frac{1}{\Delta} \text{pr} \left\{ \begin{array}{l|l} N_r(t+\Delta) - N_r(t) = 1 & L(0-) > L(0) = i \\ L(t) > L(t+\Delta) = j & N_a(-\Delta_1, 0) = 1 \\ N_r(0, t) = 0 & \end{array} \right\}$$

It is worthwhile to notice that an r-event will result in an inventory level i, with $0 \leq i \leq s$. In the case of $i = 0$, it may correspond to a shortage (first) or it may be the lost demand that is satisfied from the available inventory. Noticing that an r-event can occur either at the very first demand or a subsequent demand we get the following explicit expression.

$$f_{ij}^{ar}(t) = f_{ij}^{a}(t) + \sum_{n=1}^{s+1-i} \sum_{k=s+1}^{i-n} \pi_{ik}^{n}(t) * f_{kj}^{a}(t) \qquad \ldots. \ (2.5)$$

As observed earlier, because of the renewal process characterising the demand process the replenishment points are non-regeneratives, but the first demand after a replenishment is regenerative and the inventory immediately after this can be anything from 0 to S-1. However, if the inventory is less than s+1, than the b-event under consideration is an r-event. The next result characterises the duration between an r-event and the b-event following it.

Let

$$f_{ij}^{rb}(t) = \lim_{\Delta,\Delta_1 \to 0} \frac{1}{\Delta} \, pr \left\{ \begin{array}{l|l} N_b(t+\Delta)-N_b(t)=1 & L(0-) > L(0) \\ N_b(o,t) \qquad =0 & N_r(-\Delta_1,0) = 1 \\ L(t+\Delta)=j < L(t) & \end{array} \right\},$$

$$0 \leq i \leq s, \quad j \geq s+1$$

An expression for $f_{ij}^{rb}(t)$ is obtained by reasoning as follows: For the next b-event to occur, the order placed at $t = 0$, has to materialise before t, say at u, $u \leq t$. Since, we are looking for the b-event at t, the demand following the replenishment should occur at t and the inventory level at this epoch should be $\geq s + 1$. Now to obtain the inventory level at t, we need to know the inventory level at the last demand before replenishment. The last demand before replenishment may be the first shortage or a subsequent shortage or a regular a-event. Alternatively the b-event is result of the very first demand that occurs after the reorder is placed. Using the obvious bounds on the total demand and the inventory level the following expression results.

$$f_{ij}^{rb}(t) = f(t) \int_0^t b(u) P_{S\,j+1}(t - u)du$$

$$+ \int_0^t \psi_i(u)du \int_u^t b(v) P_{S\,j+1}(t-v)\, f(t-u)dv \quad \ldots \ (2.6)$$

where

$$\psi_i(t) = \sum_{n=1}^{i-1} \sum_{k=1}^{i-n} \bar{\lambda}_{ik}^{n} (t)$$

$$+ \sum_{n=1}^{i-1} \sum_{k=1}^{i-n} \bar{\lambda}_{ik}^{n}(t) * [f(t)\, P_{k1}(t) + f(t) P_{ko}(t)]$$

$$+ \sum_{n=1}^{i-1} \sum_{k=1}^{i-n} \bar{\lambda}_{ik}^{n}(t) * [f(t) P_{k1}(t) + f(t) P_{ko}(t)] * \sum_{n=1}^{\infty} f^{(n)}(t)$$

$$\ldots\ldots \ (2.7)$$

The expression $\psi_i(t)$ represents the inventory level in an interval of length t, conditional on an r-event at the origin and the inventory level specified at $t = 0$. The interval under consideration has the additional property that the order placed at $t = 0$, does not materialise in $(0, t)$ and the depletion in the inventory is due to demand and decay. The expression takes care of the cases corresponding to no shortages, one shortage and several shortages.

The following remarks are in order:

1. For $0 < j \leq s$, the same expression (2.6) is true.
2. For $j = 0$, the $P_{ij}(.)$ terms in (2.6) are to be replaced by $P_{S1}(t) + P_{S0}(t)$, as the demand may or may not be shortage.
3. For the case $i = 1$, the structure of the equation is the same, but certain summation items will disappear.

For $i = 0, \quad 0 < j < s$

$$f_{ij}^{rb}(t) = f(t) \int_0^t b(u)\, P_{Sj+1}(t-u)\, du + \sum_{n=1}^{\infty} \int_0^t f^{(n)}(u)du \int_u^t b(v) P_{Sj+1}(t-v) f(t-u) dv \quad \ldots \text{(2.8)}$$

and

$$f_{oo}^{rb}(t) = f(t) \int_0^t b(u)\, [P_{S1}(t-u) + P_{So}(t-u)] du + \sum_{n=1}^{\infty} \int_0^t f^{(n)}(u)du \int_u^t b(v) [P_{S1}(t-v) + P_{So}(t-v)] f(t-u) dv \quad \ldots \text{(2.9)}$$

3. MAIN RESULTS:

The stochastic process $\{L(t),\ t \geq 0\}$ has an imbedded MRP whose transition probability functions are determined by Theorem-1.

THEOREM-1:

Let $$f_{ij}^{r}(t) = \lim_{\Delta,\Delta_1,\Delta_2 \to 0} \frac{1}{\Delta}\, pr \left\{ \begin{matrix} N_r(t+\Delta)-N_r(t)=1 \\ L(t)>L(t+\Delta_1)=j \end{matrix} \;\middle|\; \begin{matrix} L(0-)>L(0)=i \\ N_r(-\Delta_2,0)=1 \end{matrix} \right\}$$

then,

$$f_{ij}^{r}(t) = \sum_{k=s+1}^{S} f_{ik}^{rb}(t) * f_{kj}^{br}(t) + \sum_{j=1}^{s} f_{ij}^{rb}(t) \qquad i,\ j \leq s. \qquad \ldots\ldots (3.1)$$

PROOF:

We need to find the probability of a r-event in $(t,\ t+\Delta)$ conditional on a r-event at the time origin. For this to happen, the order placed at $t = 0$, has to materialise sometime before t and a demand should follow it. When this occurs this may be a b-event or an r-event. In the case of a b-event, we require an r-event in the remaining time duration conditional on a b-event. Hence the expression.

Using the functions $f_{ij}^{r}(t)$, we can determine the probability of an r-event with inventory level j at any other arbitrary time.

The $h_{ij}^{r}(t)$ functions defined as

$$\lim_{\Delta,\Delta_1,\Delta_2 \to 0} \frac{1}{\Delta}\, pr \left\{ \begin{matrix} L(t+\Delta_1)=j \\ N_r(t+\Delta)-N_r(t)=1 \end{matrix} \;\middle|\; \begin{matrix} L(0+)=i \\ N_r(-\Delta_2,0)=1 \end{matrix} \right\}$$

are related to the $f^r_{ij}(t)$ by means of the relation

$$h^r_{ij}(t) = f^r_{ij}(t) + \sum_{k=0}^{s} h^r_{ik}(t) * f^r_{kj}(t), \qquad i, j \leq s \qquad \ldots\ldots. (3.2)$$

Equation (3.2) in principle determines the functions $h_{ij}(t)$ at least in the Laplace transform form. However, when stationary behaviour is the concern, result (3.2) can be used to obtain the stationary distribution of the process $\{L(t), t \geq 0\}$. By using the previous results now we are in a position to obtain the inventory level at any time t. We first obtain the inventory level within a cycle of two successive r-events.

THEOREM-2:

For $o \leq j \leq s$, define $\psi_j(n, t)$ as

$$= \lim_{\Delta \to 0} pr\{L(t)=n, N_r(o,t) = 0 \mid N_r(-\Delta, 0) = 1, L(o+) = j\}$$

as representing the probability that the stock level at time t is, equal to n, conditional by an r-event at time t = 0, with inventory level equal to j, and the additional requirement of no r-event in (o, t).

In obtaining an expression for the probability $\psi_j(n, t)$, the following reasoning is applied. The major classification is based on whether there is a b-event during the interval (o, t) or not. In the case of a b-event before, care is taken to ensure that it is only a proper b-event by restricting the inventory level $\geq s+1$. Once a b-event occurs the further classifications correspond to: (a) no demand and hence only decay, (b) restricted number of demands and decay avoiding next r-event.

The case corresponding to the non-occurrence of b-event is classified further into the following sub-cases:

(a) no b-event because the order pending has not materialised in (o, t).

(b) order has materialised but no subsequent demand to cause a b-event, hence decay is the only cause for depletion.

$$\psi_j(n, t) = \int_0^t \psi_j(u)du \int_u^t b(v)\, \bar{F}\,(t-u)\, P_{Sn}\,(t-v)dv$$

$$+ \bar{F}\,(t) \int_0^t b(u)\, P_{Sn}\,(t-u)\, du$$

$$+ \sum_{k=s+1}^{S} f_{jk}^{rb}\,(t) * g_{kn}\,(t) \qquad 1 < j < s, \quad n > j. \qquad \ldots\ldots\ldots\ldots\ (3.3)$$

where $\psi_j(t)$ is given by (2.7) and

$$g_{kn}(t) = \bar{F}(t)P_{kn}(t) + \{ \sum_{n=1}^{k-(s+1)} \sum_{i=s+1}^{k-m} \pi_{ki}^{n}(t) \}* \bar{F}(t)\, P_{in}(t) \qquad \ldots\ldots(3.4)$$

$$\psi_j(n, t) = \text{All terms in } (3.3)$$

$$+ \bar{B}\,(t)\, \bar{F}(t)\, P_{jn}\,(t)$$

$$+ \bar{B}(t) \sum_{m=1}^{j-1} \sum_{k=1}^{j-m} \bar{\lambda}_{jk}^{m}(t) * [\bar{F}(t)\, P_{kn}(t)]$$

$$1 < j < s, \quad n \leq j$$

$$\ldots\ldots\ldots\ldots \quad (3.5)$$

$$\psi_1(n, t) = \bar{F}(t) \int_0^t b(u)\, P_{Sn}(t-u)\, du$$

$$+ \int_0^t f(u)du \int_u^t b(u)\, P_{Sn}(t-v)\, \bar{F}(t-u)dv$$

$$+ \int_0^t \{f(u) * \sum_{m=1}^{\infty} f^{(m)}(u)\}du \int_u^t b(v) P_{Sn}(t-v) \bar{F}(t-u)dv$$

$$+ \sum_{k=s+1}^{S} f_{jk}^{rb}(t) * g_{kn}(t) \quad \ldots\ldots (3.6)$$

$$1 \leq n \leq S$$

where $g_{kn}(t)$ is given by (3.4).

$\psi_j(o, t)$ = All the terms in (3.3) and (3.5)

$$+ h(t) * \bar{F}(t)\, \bar{B}(t) + h(t) * \sum_{m=1}^{\infty} f^{m}(t) * \bar{F}(t)\, \bar{B}(t)$$

$$\ldots\ldots (3.7)$$

where

$$h(t) = \sum_{n=1}^{j-1} \sum_{k=1}^{j-n} \bar{\lambda}_{jk}^{n}(t) * \{f(t)\, P_{k1}(t) + f(t)\, P_{ko}(t)\}$$

$$\ldots\ldots (3.8)$$

$$\psi_o(n, t) = \bar{F}(t) \int_0^t b(u)\, P_{Sn}(t-u)\, du$$

$$+ \sum_{m=1}^{\infty} f^{m}(u)du \int_u^t b(v)\, P_{Sn}(t-v)\, \bar{F}(t-u)dv$$

$$+ \sum_{k=s+1}^{S} f_{ok}^{rb}(t) * g_{kn}(t) \qquad n > 0$$

$$\ldots\ldots (3.9)$$

where again $g_{kn}(t)$ is given by (3.4).

$$\psi_o(o,\ t) = \text{All the terms in } (3.9) + \bar{B}(t) \qquad \ldots. \ (3.10)$$

$$\psi_1(o,\ t) = \text{All the terms in } (3.6) + \bar{B}(t)\ \bar{F}(t)\ P_{1o}(t)$$

$$+ \ \bar{B}(t)\ [\ f(t) * \bar{F}(t)\ ]$$

$$+ \ \bar{B}(t)\ [\ f(t) * \sum_{m=1}^{\infty} f^m(t)\ ] * \bar{F}(t) \qquad \ldots. \ (3.11)$$

Theorems 1 and 2 can be effectively used to obtain the stationary distribution of the process $\{L(t),\ t \geq 0\}$. We state this as,

THEOREM-3:

$$\bar{\lambda}_i\ (n,t) = pr\{L(t)=n \mid N_r(-\Delta_1,\ 0)=1,\ \ L(o\ +) = i\},$$

$$0 \leq i \leq s$$

representing the inventory distribution at any t, conditional on a r-event at the time origin is given by

$$\bar{\lambda}_i(n,t) \ = \ \psi_i(n,t) + \sum_{j=0}^{s} h^r_{ij}(t) * \psi_j(n,\ t)$$

where the functions $\psi_j(n,\ t)$ and $h^r_{ij}(t)$ are obtained from Theorems 1 and 2.

THEOREM-4:

The limiting distribution of the process $\{L(t),\ t \geq 0\}$ denoted as $\bar{\lambda}_n = \lim_{t\to\infty} \bar{\lambda}(n,\ t)$ is independent of i and is

$$\text{given by} \quad \bar{\lambda}_n \ = \ \Sigma\ \frac{1}{N_j} \int_o^{\infty} \psi_j(n,\ u)\ du, \qquad \ldots. \ (3.12)$$

$$\text{where} \quad N_j \ = \ \lim_{t\to\infty} h^r_{ij}\ (t). \qquad \ldots. \ (3.13)$$

We next attempt to characterize the shortages. Our primary interest is to obtain the rate of shortages, in the stationary case. This is best achieved by the first order product density associated with shortages.

Define

$$f_i^{r\sigma}(t) = \lim_{\Delta,\Delta_1 \to 0} \frac{1}{\Delta} \operatorname{pr} \left\{ \begin{array}{l} N_{\sigma}(t+\Delta) - N_{\sigma}(t) = 1 \\ \\ N_r\ (o,\ t] \quad = 0 \end{array} \middle| \begin{array}{l} N_r(-\Delta_1, 0) = 1 \\ \\ L(0-) > L(0) = i \end{array} \right\}$$

and

$$h_i^{\sigma}(t) = \lim_{\Delta,\Delta_1 \to 0} \frac{1}{\Delta} \operatorname{pr}\left\{ N_{\sigma}(t+\Delta) - N_{\sigma}(t) = 1 \middle| \begin{array}{l} N_r(-\Delta_1, 0) \ = 1 \\ \\ L(0-) > L(0) = i \end{array} \right\}$$

The next result characterizes the functions $f_i^{r\sigma}(t)$ and $h_i^{\sigma}(t)$ completely.

<u>THEOREM-5:</u>

$$h_i^{\sigma}(t) = f_i^{r\sigma}(t) + \sum_{j=0}^{s} h_{ij}^{r}(t) * f_j^{r\sigma}(t) \qquad \ldots (3.14)$$

and for $1 \le i \le \overset{*}{s}$.

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* The content of the expression for the case i = 1, will result in further simplifications in (.).

$$f_i^{r\sigma}(t) = f(t) \int_0^t b(u)\, P_{So}(t-u)du + \int_0^t \Psi_i(u)du \int_u^t b(v)dv\, P_{So}(t-v)\, f(t-u) + \bar{B}(t)\{\sum_{n=1}^{i-1} \sum_{j=1}^{i-n} \bar{\wedge}_{1j}^{n}(t) * [\sum_{k=0}^{1} f(t) P_{jk}(t)] * \sum_{n=1}^{\infty} f^{(n)}(t) \quad \ldots. (3.15)$$

where $\Psi_i(u)$ is given by (2.7).

The derivation of Result (3.15) is based on the usual classification of either a b-event in (o, t) or no b-event in (o, t). The stationary value of $h^{\sigma}(t)$ is obtained to be equal to $\sum_{j=0}^{s} \frac{1}{N_j} \int_0^{\infty} f_j^r(u)du$

where N_j' s are obtained from Theorem 4.

To obtain the cost expression, it is necessary to study the decay events. We use by now the familiar method to study the decay events. That is, we first study the decay events within a cycle of r-events and use the familiar Markov renewal argument to extend the results for the general case. Let d-denote the decay event and

$$h_i^{rd}(t) = \lim_{\Delta,\Delta_1 \to 0} \frac{1}{\Delta} \text{pr} \left\{ \begin{matrix} N_d(t+\Delta) - N_d(t) = 1 \\ N_r(o, t) = 0 \end{matrix} \middle| \begin{matrix} N_r(-\Delta_1, 0) = 1 \\ L(0-) > L(10) = i \end{matrix} \right\}$$

The following simple result will be needed to obtain the functions $h_i^{rd}(t)$.

The function $g_i^d(t)$ representing the pdf of decay in $(t, t+dt)$ given the information of no replenishment or demand in (o, t) is given by

$$g_i^d(t) = \lambda_i e^{-\lambda_i t} + \sum_{k=1}^{i} P_{ik}(t) * \lambda_k e^{-\lambda_k t}$$

with $\lambda_k = k\lambda$ and $P_{ik}(t)$ is given by expression (2.1).

THEOREM-6:

The functions $h_i^{rd}(t)$ is given by

$$h_i^{rd}(t) = \sum_{j=s+1}^{S} f_{ij}^{rb}(t) * [\,\overline{B}(t)\,\overline{F}(t)\; g_j^d(t)\,]$$

$$+ \overline{B}(t)\overline{F}(t)g_i^d(t) + \overline{F}(t)\int_o^t b(u)g_S^d(t-u)du$$

$$+ \overline{B}(t)\sum_{n=1}^{i-1}\sum_{k=1}^{i-n} \overline{\wedge}_{ik}^{n}(t) * \{\overline{F}(t)\; g_k^d(t)\}$$

$$+ \int_o^t \psi_i(u)\,du \int_u^t b(v)\,\overline{F}(t-v)\; g_S^d(t-v)dv \quad \ldots (3.16)$$

where $\psi_i(t)$ is determined by expression (2.7).

The stationary value of intensity of decays can be obtained as is done for the case of shortages and is given by

$$h^d = \sum_{j=1}^{S} \frac{1}{M_j} \int_o^\infty h_j^{rd}(u)du$$

In order to complete the cost expression in the stationary case for the present model, we only need to get the frequency of r-events, which is obtained by the following result:

THEOREM-7:

Let $h_i^r(t) = \lim_{\Delta,\Delta_1 \to 0} \frac{1}{\Delta} \, pr\left\{N_r(t+\Delta)-N_r(t) = 1 \,\middle|\, \begin{matrix} N_r(-\Delta_1,0) = 1 \\ L(0\,-) > L(0) \end{matrix} \right\}$

then

$h_i^r(t) = \sum_{j=0}^{s} h_{ij}^r(t)$, $h_{ij}^r(t)$ functions are determined by equation (3.2).

Defining $h^r = \lim_{t \to \infty} h_i^r(t)$ and $h_j^r = \lim_{t \to \infty} h_{ij}^r(t)$

the h_j^r's are determined by the system of equations

$h_{ij}^r = \sum_{k=0}^{s} h_k \int_0^\infty f_{kj}^r(u)du$. This leads to an expression for the frequency of r-events as $h^r = \sum_{j=0}^{s} h_j^r$.

THEOREM-8:

Let C_r be the reordering cost

C_b be the shortage cost/per item/per unit time.

C_s be the unit storage cost per unit time

C_d be the cost of decay per item.

An expression for the total expected cost per unit time in steady state is given by

$$E([C] = C_s \sum_{n=1}^{S} n \,\bar{\wedge}\,(n) + \frac{C_r}{h_r} + \frac{C_b}{h^{\sigma}} + \frac{C_d}{h^d}$$

where the intensity constants h^r, h^{σ} and h^d are computed from the previous results.

4. SPECIAL CASES

In this section we discuss several special cases of the model analysed in the earlier sections. Before doing that we note the following additional assumptions we have made in the classical (S, s) inventory model.

1. The inventory level is reviewed only at the demand locations governed by a renewal process.

2. The reorder quantity is variable and is equivalent to the inventory required to make the final inventory level equal to S.

Assumption 1 is essential for the present analysis in view of the non (negative) exponential demand durations and arbitrary lead times, and the decay of the items in the inventory. The reason for this review policy is as follows: The depletion in the inventory can occur due to failure or a demand. While demand epochs generate regeneration points, the epochs corresponding to the decay events are non-regenerative. Hence the continuous review policy of recordering whenever the inventory level is = s, by the nature of the system under consideration will require additional information to predict the future behaviour of the system.

Assumption 2 is not all restrictive. The analysis is true even when the reorder quantity is fixed and is a equal to a usual constant S-s. The logical derivation of the results and the associated probability arguments remain valid in this case as well.

4.1. RENEWAL DEMAND, MARKOVIAN LEAD TIME AND DECAY*

The stochastic behaviour of the inventory model under these assumptions is fairly simple. Infact, the demand locations induce an imbedded Semi-Markov Process, the analysis of which facilitates the characterisation of the process $L(t)$ representing the inventory level at any time t. The functions $f^{a}_{ij}(t)$ in the usual notation represent the kernal of the SMP and they are equivalent to,

$$f_{ij}(t) \quad = \quad P_{ij+1}(t)\, f(t) \qquad \ldots\ldots (4.1)$$

The functions $P_{ij}(t)$ used in (4.1) represents the transient behaviour during a demand interval of a birth and death process with state dependent death rates. The Chapman-Kolmogorov equations of the process are given by (4.2) under appropriate specified initial condition.
Set $\lambda_i = i\lambda$, $1 \leq i \leq S$, $\lambda_0 = 0$ μ^{-1} = mean lead time.

$$\dot{P}_{ij}(t) = -\lambda_j P_{ij}(t) + \lambda_{j+1} P_{ij+1}(t) + \mu P_{ij-(S-s)}(t) \quad ; \quad j \geq S-s$$

$$\dot{P}_{ij}(t) = -\lambda_j P_{ij}(t) + \lambda_{j+1} P_{ij+1}(t) \qquad 0 \leq j < S-s \qquad \ldots (4.2)$$

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* Throughout this paper, and in particular in this section it is assumed $(S-s) \geq s$, to ensure the continued evolution of the inventory system.

The distribution of the inventory level at any time $\bar{\Lambda}(n, t)$ is obtained by following the procedure in Section 3 and the equations (4.1) and (4.2).

When the items do not decay the $f_{ij}(t)$ functions have a much more simple expression, as they represent the behaviour of a pure death process due to demand pattern and birth due to replenishment. Further, the reorder points are uniquely determined and they form a renewal process. Let R represent the time duration between two successive reorder point and $f_R(t)$ the pdf. Then in terms of the usual notation $f_R(t) = f^r_{ss}(t)$. The following expression for $f_R(t)$ is obtained by classifying the events according to a shortage or not before replenishment. In the usual notation of Sections (1) - (3),

$$\begin{aligned} f_R(t) \quad = \quad & \int_0^t f(u)B(u)\, f^{(S-s-1)}(t-u)du \\ & + \sum_{n=1}^{s} \int_0^t f^{(n)}(u)du \int_u^t f(v-u)B(v-u)f^{(S-s-n-1)}(t-v)dv \\ & + \sum_{n=s+1}^{\infty} \int_0^t f^{(n)}(u)du \int_u^t f(v-u)B(v-u)f^{(S-s-1)}(t-v)dv \end{aligned} \qquad \ldots\ldots (4.3)$$

By using the imbedded renewal process it is possible obtain an expression for $\bar{\Lambda}(n, t)$ as

$$\bar{\Lambda}(n,t) = \psi(n, t) + \sum_{n=1}^{\infty} f_R^{(n)}(t) * \psi(n, t) \qquad \ldots\ldots (4.4)$$

where $\psi(n, t)$ are obtained as,

$$\psi(n, t) = \bar{B}(t) \int_0^t f^{(s-n)}(u)\, \bar{F}(t-u)du \qquad 1 \leq n \leq s$$

$$\psi(o, t) = \bar{B}(t) \int_0^t f^{(s)}(u)du$$

$$\psi(n, t) = \int_0^t f(u)\, B(u)\, P_{S-1n}(t-u)du$$

$$+ \sum_{m=1}^{s} \int_0^t f^{(n)}(u)du \int_u^t f(v-u)B(v-u)\, P_{S-m-1,n}(t-v)dv$$

$$+ \sum_{m=s+1}^{\infty} \int_0^t f^{(n)}(u)du \int_u^t f(v-u)B(v-u)P_{S-s-1n}(t-v)dv \qquad \ldots\ldots (4.5)$$

$$s < n < S$$

$$\psi(S, t) = \bar{F}(t)\, B(t)$$

and $P_{ij}(t)$ functions used in (4.5) are determined as

$$P_{ij}(t) = 0 \qquad j > i$$

$$= \bar{F}(t) \qquad j = i$$

$$= \int_0^t f^{j-i}(u)\, \bar{F}(t-u)du, \qquad j \leq i \quad \ldots (4.6)$$

When the lead time is instantaneous, the Stochastic process takes values in $\{s+1 \ldots\ldots S\}$. The pdf of the cycle time $f_R(t)$, in the absence of decay and lead time is equivalent to $f^{(S-s)}(t)$ and

$$\psi(n, t) = 0 \qquad 0 \leq n \leq s$$

$$= \overline{F}(t) \qquad n = S$$

$$= \int_0^t f^{(S-n)}(u)\overline{F}(t-u)du \qquad s < n \leq S \qquad \ldots (4.7)$$

$$\bar{\lambda}(n, t) = \psi(n, t) + \sum_{n=1}^{\infty} f_R^{(n)}(t) * (n, t) \qquad \ldots (4.8)$$

Further, $\bar{\lambda}(n) = \lim_{t \to \infty} \bar{\lambda}(n, t)$

$$= \frac{1}{(S-) \times E[\text{Demand Duration}} \times \int_0^\infty \psi(n, t)dt$$

$$= \frac{1}{(S-s)}. \qquad \ldots (4.9)$$

a result in agreement with Sivazlian [17].

4.2. POISSON DEMAND, EXPONENTIAL DECAY AND RANDOM LEAD TIME:

The analysis in this case is simple in view of a unique reorder point due to Poisson demand and exponential decay. The pdf $f_R(t)$ introduced in the earlier sub-section is equivalent to

$$f_R(t) = \sum_{j=0}^{\infty} [b(t)P_{sj}(t)] * P_{S-s+j,s+1}(t) \; \lambda_{s+1} \qquad \ldots (4.10)$$

where $\lambda_i = i\lambda + \lambda_d$, λ_d being the demand rate.

The $P_{ij}(t)$ functions used in (4.10) represent the transient behaviour of a death process due to decay and demand and explicit expression for them is provided below:

$$P_{ij}(t) = \lambda_i e^{-\lambda_i t} * \lambda_{i-1} e^{-\lambda_{i-1} t} * \ldots \lambda_{j+1} e^{-\lambda_{j+1} t} * e^{-\lambda_j t} \qquad 1 \leq j \leq i$$

$$= \int_0^t P_{i1}(u) * [\lambda_1 e^{-\lambda_1 u}] \, du \qquad j = 0$$

$$= e^{-\lambda_i t} \qquad j = i$$

$$= 0 \qquad \text{Otherwise} \ldots (4.11)$$

Using expressions (4.10) and (4.11) the procedure in Section 3 obtains the stationery distribution of $\{L(t), t \geq 0\}$.

It is worthwhile to observe the simple form of the Expression (4.10) aompared to (4.3). This simplification is due to the assumption of Poisson demand.

Table 1, provides the total cost in maintaining the system for unit time in the stationery case. The following specific relations and values of the parameters are considered. $S = 20$, unit variable cost = u, storage cost .2 x u, shortage cost = u + .2u, decay cost = u + .3u, fixed ordering cost = 100, demand rate $= \lambda$, decay rate $\lambda/10$, lead time distribution double exponential with parameters c and d with mean $\frac{c + d}{cd}$. The combinations of λ, u, c and d are presented below:

1. $\lambda = .1$, u = 5, c = .025, d = .1
2. $\lambda = .1$, u = 10, c = .025, d = .1
3. $\lambda = .5$, u = 10, c = .05, d = .1
4. $\lambda = .2$, u = 20, c = .4, d = .1.

Table 1

Value of S	Cost expression for set of values in 1	2	3	4
0	201.26	789.55	489.80	3727.34
1	203.81	799.43	478.58	3851.49
2	206.79	811.02	465.94	4001.56
3	210.13	823.96	452.03	4168.77
4	213.66	837.67	436.93	4345.68
5	217.24	851.56	420.66	4526.99
6	220.74	865.11	403.23	4709.13
7	224.05	877.87	384.60	4889.73
8	227.07	889.42	364.77	5067.22
9	229.69	899.38	343.68	5240.54
10	231.82	907.33	321.31	5408.96

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